Finite Group Reshetikhin-Turaev

Kevin Donoghue

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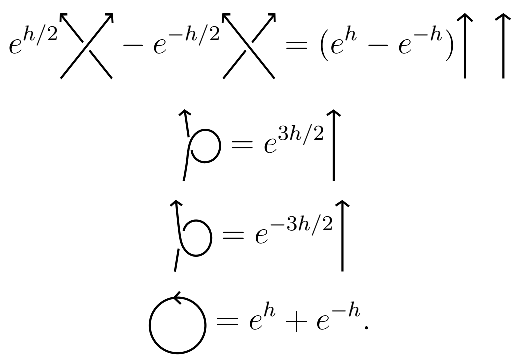
Here is (one version of) the definition of the Jones polynomial:



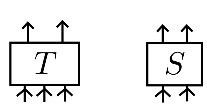
Unfortunately it has no intrinsically 3-dimensional definition. This is an expository note on two things

- The quantum group definition of the Jones polynomial and the related 3-manifold invariant [RT90], [RT91]
- A finite group analog [DW90], [DPR90], [AC92].

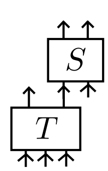
The latter construction, having a topological interpretation, hopefully sheds some light the 3-dimensional nature of the Jones polynomial.

1. Tensor Diagrammatics

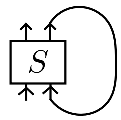
Let V be a finite dimensional complex vector space. By way of example, let two maps $T : V^{\otimes 3} \to V^{\otimes 2}$ and $S : V^{\otimes 2} \to V^{\otimes 2}$ be represented by the pictures



The following represents an obvious composition of T and S:

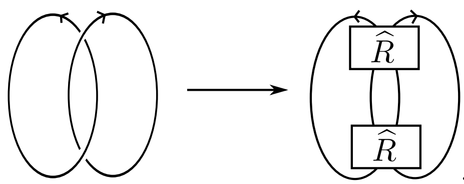


a map $V^{\otimes 4} \to V^{\otimes 3}$. The picture



means: represent S in $V^* \otimes V^* \otimes V \otimes V$ then contract along a map $V^* \otimes V \to \mathbb{C}$.

The idea behind the Reshetikhin-Turaev construction of the Jones polynomial is to convert a link diagram into a diagram of tensors:



Here the link has been turned into a contraction of a linear map $\widehat{R} \circ \widehat{R} : V \to V \to V \otimes V$. Such a contraction is simply a complex number. If the method of associating a tensor diagram to a link diagram is chosen well, then this number will be a link invariant.

2. The Quantum Group $U_h(\mathfrak{sl}_2)$

Start with the Hopf algebra $U_h = U_h(\mathfrak{sl}_2)$: it has generators E, F, H with relations

$$[H, E] = 2E, \ [H, F] = -2F, \ [E, F] = \frac{e^{hH} - e^{-hH}}{e^h - e^{-h}}.$$

Strictly speaking one has to allow for power series in this algebra, but this is no big deal. In the limit $h \to 0$, $U_h(\mathfrak{sl}_2)$ converges to the usual universal enveloping algebra $U(\mathfrak{sl}_2)$. Use the following notation: $\mathbb{C}_h := \mathbb{C}[[h]]$.

With respect to a basis $\{v_1, v_{-1}\}$, U_h acts on the right on $V_1 \cong \mathbb{C}_h^2$ as follows:

$$E \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ F \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ H \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This is the same as the action of $U(\mathfrak{sl}_2)$ on its defining representation. Like $U(\mathfrak{sl}_2)$, $U_h(\mathfrak{sl}_2)$ has an $n+1$-dimensional irreducible representation V_n for each $n \geq 0$. The higher irreps V_n will have matrix forms that look like those of $U(\mathfrak{sl}_2)$, but with certain entires replaced by quantum integers $[k]_{e^h}$:

$$[k]_{e^h} = \frac{e^{hk} - e^{-hk}}{e^h - e^{-h}}.$$

In $U(\mathfrak{sl}_2)$ there's a symmetric coproduct

$$\Delta(H) = H \otimes 1 + 1 \otimes H$$

$$\Delta(E) = E \otimes 1 + 1 \otimes E$$

$$\Delta(F) = F \otimes 1 + 1 \otimes F.$$

Through the coproduct, $U(\mathfrak{sl}_2)$ acts on $V \otimes W$. For example

$$(v \otimes w)H = vH \otimes w + v \otimes wH.$$

Since the coproduct is symmetric, the switch map

$$\sigma : V \otimes W \to W \otimes V$$

$$v \otimes w \mapsto w \otimes v$$

commutes with the action of $U(\mathfrak{sl}_2)$.

The coproduct in $U_h(\mathfrak{sl}_2)$ is not symmetric:

$$\Delta(H) = H \otimes 1 + 1 \otimes H$$

$$\Delta(E) = E \otimes e^{hH} + 1 \otimes E$$

$$\Delta(F) = F \otimes 1 + e^{-hH} \otimes F$$

so the switch map is not a U_h morphism. For example, let v_1 and v_{-1} be weight vectors in V_1:

$$v_{-1} \otimes v_1 \overset{F}{\mapsto} v_1 \otimes v_1 \overset{\sigma}{\mapsto} v_1 \otimes v_1.$$

$$v_{-1} \otimes v_1 \overset{\sigma}{\mapsto} v_1 \otimes v_{-1} \overset{F}{\mapsto} e^{-h} v_1 \otimes v_1.$$

One wants, given two U_h modules V and W, to find some isomorphism

$$V \otimes W \to W \otimes V.$$

The best one could hope for is that there exists an invertible element $R \in U_h \otimes U_h$ such that the following commutes for $a \in U_h$:

$$\begin{array}{ccccc} V \otimes W & \xrightarrow{R} & V \otimes W & \xrightarrow{\sigma} & W \otimes V \\ \downarrow a & & & & \downarrow a \\ V \otimes W & \xrightarrow{R} & V \otimes W & \xrightarrow{\sigma} & W \otimes V \end{array}.$$

For shorthand, let

$$\widehat{R}_{V,W} : V \otimes W \to W \otimes V$$

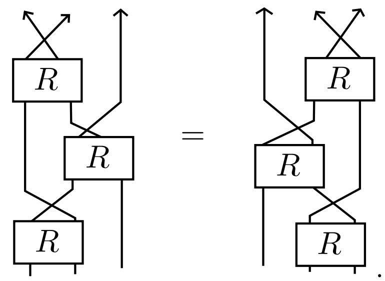
denote the top and bottom rows of this commutative diagram, so $\widehat{R}$ is R followed by the switch map. Of course such an R (if it exists) would need to satisfy some rules, one of which involves the the two ways of switching three tensor factors:

$$V \otimes W \otimes U \overset{\widehat{R}_{V,W}}{\to} W \otimes V \otimes U \overset{\widehat{R}_{V,U}}{\to} W \otimes U \otimes V \overset{\widehat{R}_{W,U}}{\to} U \otimes W \otimes V$$

needs to be the same as

$$V \otimes W \otimes U \overset{\widehat{R}_{W,U}}{\to} V \otimes U \otimes W \overset{\widehat{R}_{V,U}}{\to} U \otimes V \otimes W \overset{\widehat{R}_{V,W}}{\to} U \otimes W \otimes V.$$

In tensor diagram notation, this might be represented as



It's hard to look at this and not immediately think of a type 3 Reidemeister move. Such an R is called an "R-matrix."

Theorem 2.1 (Drinfeld)**.** *An R-matrix exists and is given by*

$$R = \sum_{n=0}^{\infty} \frac{e^{\frac{n(n+1)}{2}h}(1-e^{-2h})^n}{[n]_{e^h}!} e^{\frac{1}{2}hH\otimes H} E^n \otimes F^n \in U_h \otimes U_h.$$

For example, on $V_1 \otimes V_1$, all the terms $n \geq 2$ vanish so R acts as

$$e^{\frac{1}{2}hH\otimes H} + (e^h - e^{-h}) e^{\frac{1}{2}hH\otimes H} E \otimes F.$$

With respect to the basis $v_1 \otimes v_1, v_1 \otimes v_{-1}, v_{-1} \otimes v_1, v_{-1} \otimes v_{-1}$, R acts (on the right) as

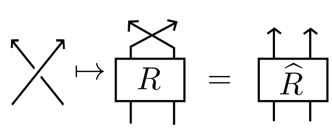
$$\begin{pmatrix} e^{h/2} & & & \\ & e^{-h/2} & e^{h/2} - e^{-3h/2} & \\ & & e^{-h/2} & \\ & & & e^{h/2} \end{pmatrix}$$

so that $\widehat{R}_{V_1,V_1}$ is

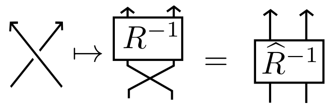
$$\begin{pmatrix} e^{h/2} & & & \\ & e^{h/2} - e^{-3h/2} & e^{-h/2} & \\ & e^{-h/2} & 0 & \\ & & & e^{h/2} \end{pmatrix}.$$

3. The Link Invariant From Quantum Groups

After looking at the braid relation satisfied by Drinfeld's R-matrix it is clear that one should define the map from knot diagrams to tensor diagrams as follows:

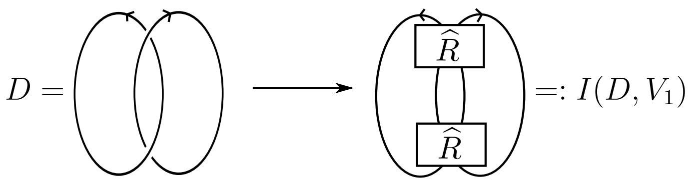


$\in \mathrm{End}(V_1 \otimes V_1)$



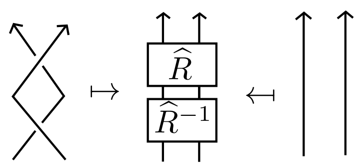
$\in \mathrm{End}(V_1 \otimes V_1).$

For example:



For this section I pick the strands to represent V_1 and the squares to represent tensorial objects in V_1. Accordingly, I call the number associated to the tensor diagram $I(D, V_1)$. The definition of $I(D, V)$ is obvious for an arbitrary U_h-module V. The invariant I constructed in this manner is due to Reshetikhin and Turaev.

Since R satisfies the braid relation, and the assignment of crossings is designed to play nice with the type 2 move:



it is clear that, for the "all arrows up" orientations, $I(D, V_1)$ is an invariant under type 2 and 3 Reidemeister moves. Invariance for the other orientations can be shown with some more work (essentially reversing the orientation corresponds to the change $V_1 \to V_1^*$). Hence:

Claim 3.1. *$I(D, V_1)$ is invariant of the framed link represented by D.*

One of things that makes the invariant I work, unmentioned to this point, is the trace on U_h modules is strange. If V is a right module the contraction map[1]

$$V \otimes V^* \to \mathbb{C}_h$$

$$v \otimes \xi \mapsto v\xi$$

is natural and accordingly is a U_h-morphism. The other contraction map

$$V^* \otimes V \to \mathbb{C}_h$$

$$\xi \otimes v \mapsto v\xi$$

is not a U_h-morphism. To remedy this, it turns out you can modify the contraction:

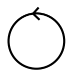
$$\xi \otimes v \mapsto v e^{hH} \xi$$

[1] V^* inherits a right U_h module structure from the antipode for U_h and $\mathbb{C}_h$ inherits a (trivial) right module structure from the counit for U_h.

to get a U_h-morphism. In the tensor diagrammatics it is this map that is used for the contraction $V^* \otimes V \to \mathbb{C}_h$. In V_1 e^{-hH} is represented by

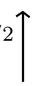
$$\begin{pmatrix} e^{-h} & \\ & e^{h} \end{pmatrix}.$$

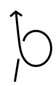
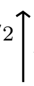
so that

 $\mapsto e^h + e^{-h}$

this of course being obtained by contracting the identity element $v_1^* \otimes v_1 + v_{-1}^* \otimes v_{-1}$ in $V^* \otimes V$.

Claim 3.2. *Under a type 1 move $I(D, V_1)$ scales by $e^{3h/2}$:*

 $= e^{3h/2}$ 

 $= e^{-3h/2}$ .

Proof. Write $\widehat{R}$ in $V^* \otimes V^* \otimes V \otimes V$ as

$$e^{h/2} v_1^* v_1 * v_1 v_1 + e^{-h/2} v_{-1}^* v_1^* v_1 v_{-1} + e^{-h/2} v_1^* v_{-1}^* v_{-1}^* v_1$$

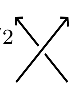
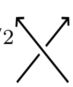
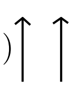
$$+(e^{h/2} - e^{-3h/2}) v_1^* v_{-1}^* v_1 v_{-1} + e^{h/2} v_{-1}^* v_{-1}^* v_{-1} v_{-1}$$

and contract the second and fourth factors using

$$v_1^* v_1 \mapsto e^{-h}, \;\; v_{-1}^* v_{-1} \mapsto e^h$$

as usual. This handles the first identity; the second is analogous but with $\widehat{R}^{-1}$. □

Claim 3.3.

$e^{h/2}$  $- e^{-h/2}$  $= (e^h - e^{-h})$ 

Proof. This is the identity of matrices:

$$e^{h/2} \widehat{R} - e^{-h/2} \widehat{R}^{-1} = (e^h - e^{-h}) \mathrm{id}$$

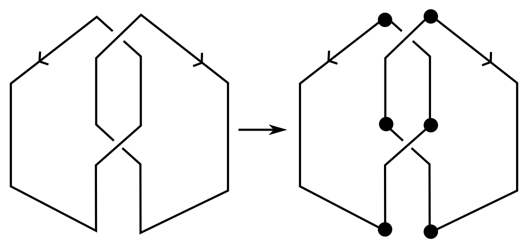
□

Corollary 3.4. *The invariant $I(D, V_1)$ is the Jones polynomial, as defined at the beginning.*

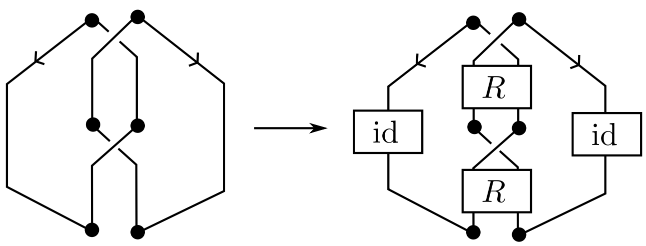
The invariants $I(D, V_n)$ (where the V_n are the irreps of U_h) are called the colored Jones polynomials.

4. The 3-Manifold Invariant

If you step back from the representation V_1 things clear up a little. You start by splitting a link diagram into crossings and individual strands:



Pictured here is a division into two crossings and two single strands. Actually there are six strands total (two in each of the crossings). To this picture you assign an element in $U_h^{\otimes \#\text{strands}}$

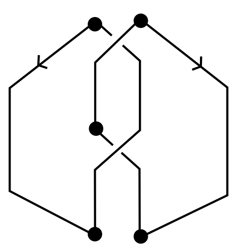


$= R \otimes R \otimes \mathrm{id} \otimes \mathrm{id} \in U_h^{\otimes 6}$.

On the diagrammatic side, fuse two of the separated strands; on the algebra side this corresponds to multiplication

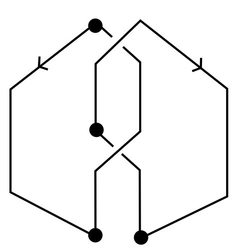
$$U_h \otimes U_h \to U_h$$

in the corresponding U_h factors:



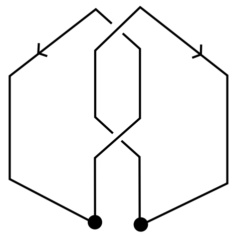
$\mapsto$ something in $U_h^{\otimes 5}$.

This element in $U_h^{\otimes 5}$ can't be written simply since it involves a multiplication of certain factors of the elements R. Continue multiplying the algebra elements in this way:



$\mapsto$ something in $U_h^{\otimes 4}$

etc, until you reach one strand in each link:

 $\mapsto$ something in $U_h^{\otimes 2}$.

This element in $U_h^{\otimes 2}$ is not canonically associated with the link, but if you hit it with $\mathrm{tr}_{V_1} \otimes \mathrm{tr}_{V_1}$ you get the invariant $I(D, V_1)$ applied to the original diagram.

In general this process assigns to a link a (noncanonical) element in $U_h^{\otimes \#\text{components}}$. Further assign a representation to each component and take traces of the algebra elements in the corresponding representation. This is a link invariant. Assigning V_1 to each component gives the Jones polynomial.

It makes sense to assign a formal linear combination of representations

$$a_1 V^{(1)} + \cdots + a_n V^{(n)}, \ a_i \in \mathbb{C}_h$$

to a link component: just extend the usual invariant multilinearly.

Theorem 4.1 (Reshetikhin-Turaev). *Let $h = \frac{2\pi i}{k+2}$ for k some positive integer. Let*

$$\omega = \sum_{i=0}^{k} [i+1]_{e^h} V_i.$$

Then, up to a factor involving the linking matrix of L, $I(L, \omega)$ is an invariant of $S^3(L)$, the manifold obtained by surgery on L.

The proof of the theorem is by checking that $I(L, \omega)$ is invariant under Kirby moves. It is hard to get an intuitive feel for why it should be a 3-manifold invariant. Also note that ω is suspiciously similar to the regular representations of a compact Lie group:

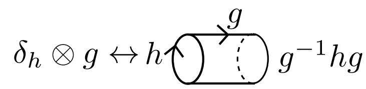
$$L^2(G) \cong \bigoplus_{V_i \text{ irreps of } G} V_i^{\dim V_i}.$$

Hopefully the conclusion of this note will, by way of analogy, shed some light on this 3-manifold invariant.

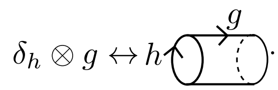
5. The Drinfeld Double of the Group Algebra

Fix a finite group G. Write $\mathcal{F}G$ for the algebra of complex valued functions on G. Write $\mathbb{C}G$ for the group algebra of G.

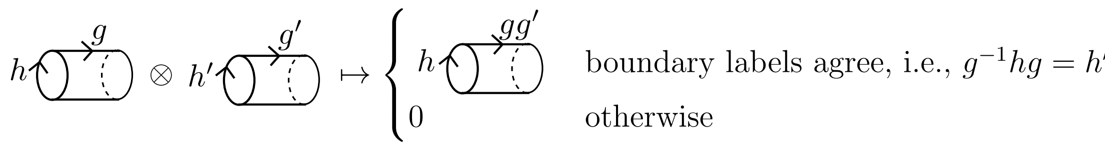
Define an algebra A as follows. As a vector space, A is $\mathcal{F}G \otimes \mathbb{C}G$, but it has a funny multiplication described as follows. Write an element of A as a cylinder with boundary components and a transverse arc labeled by group elements[2]:



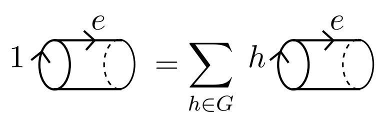
The right boundary is implied by the other data, so this can be written



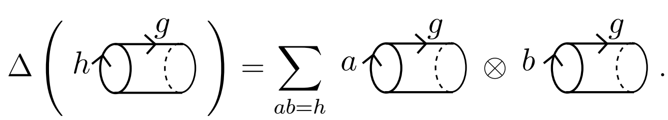
The multiplication in A is given by



The identity element for this multiplication is



where here $1 = \sum_h \delta_h$ is the constant 1 function in $\mathcal{F}G$. In fact A is a Hopf algebra. I mention here only that the coproducts of $\mathcal{F}G$ and $\mathbb{C}G$ combine to give a coproduct for A:



If G is nonabelian, this is not a symmetric coproduct. One can ask about the existence of an R for the algebra A. Indeed there does exist an R-matrix.

Before I write down the R-matrix, it is worth telling a little of the story behind Drinfeld's R-matrix for U_h. He constructs a map

$$U_h(\mathfrak{b}_+) \otimes U_h(\mathfrak{b}_-) \to U_h(\mathfrak{sl}_2)$$

and some kind of pairing

$$U_h(\mathfrak{b}_+) \otimes U_h(\mathfrak{b}_-) \to \mathbb{C}_h.$$

Here $\mathfrak{b}_\pm$ are the subalgebras spanned by H, E and H, F respectively. Drinfeld finds two bases, "dual" with respect to the pairing, of $U_h(\mathfrak{b}_\pm)$. Call them $\{e_i\}$ and $\{e^i\}$. He then shows that the element

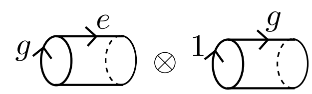
$$\sum_i (e_i \otimes 1) \otimes (1 \otimes e^i)$$

[2]The labeled cylinder can be thought as G-bundle on the cylinder trivialized at two points on the boudaries. Alternatively, a representation into G of the fundamental groupoid with two basepoints.

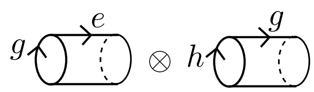
is an R-matrix for $U_h(\mathfrak{b}_+) \otimes U_h(\mathfrak{b}_-)$. This descends to the R-matrix for $U_h(\mathfrak{sl}_2)$. The whole process more or less falls under the name "Drinfeld double."

Repeat this procedure for $\mathcal{F}G$ and $\mathbb{C}G$ in place of $U_h(\mathfrak{b}_\pm)$. Dual bases are given by $\{\delta_g\}$ and $\{g\}$. Then the R-matrix is

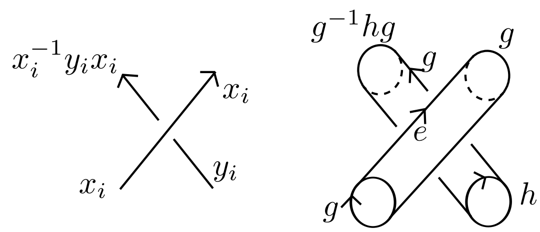
$$R = \sum_g (\delta_g \otimes e) \otimes (1 \otimes g) = \sum_g$$



$$= \sum_{g,h}$$



This R-matrix was constructed by following purely algebraic dictums. A priori it should have nothing to do with knot theory. It is a little surprising, then, to note that this looks a lot like the Wirtinger presentation for a knot group!



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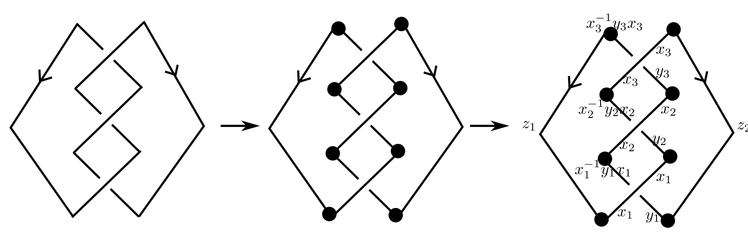
6. The Wirtinger Presentation

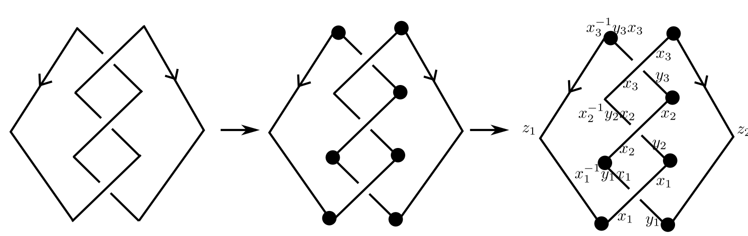
The Wirtinger presentation of a knot group can be described as follows. Divide the knot diagram into strands as in section 4 and make the following labelings:



Associate to this object the (free) group with presentation

$$\langle \{x_i, y_i, z_i\} \rangle.$$

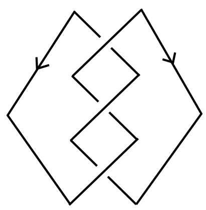
Diagramatically glue two of the strands together:



and associate to this new object the group with presentation

$$\langle \{x_i, y_i, z_i\} | x_2^{-1} y_2 x_2 = x_3 \rangle.$$

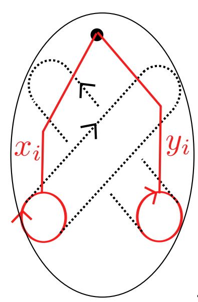
Continue in this manner gluing strands together and adding relations to the group presentation until you associate a group presentation to a diagram with a single strand:



$$\mapsto \langle \{x_i, y_i, z_i\} | x_2^{-1} y_2 x_2 = x_3,\ x_3 = z_2,\ \cdots \rangle.$$

Claim 6.1. *The group so presented is isomorphic to* $\pi_1(S^3 \setminus K)$.

Proof. Think of each crossing as a tangle complement (a ball with two holes drilled out):

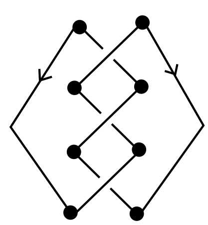


.

Put a basepoint on the top of each tangle. The fundamental group is free on two generators x_i and y_i. Glue the tangles together according the knot diagram, and use Seifert-van Kampen. □

7. Finite Group Reshetikhin-Turaev

Apply section 4 to the algebra A. That is, associate to a diagram an element of $A^{\otimes 8}$:

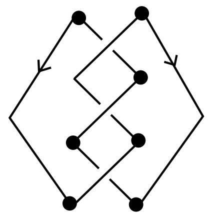


$$\mapsto R \otimes R \otimes R \otimes \mathrm{id} \otimes \mathrm{id} \in A^{\otimes 8}.$$

The element $R \otimes R \otimes R \otimes \mathrm{id} \otimes \mathrm{id}$ is a sum of many primitive elements. Because of the similarity (1) the terms in the sum are in bijection with homomorphisms:

$$\langle \{x_i, y_i, z_i\} \rangle \to G.$$

Next remove one of the dots and multiply the corresponding factors of A; so associate the new diagram and element in $A^{\otimes 7}$:

 $\mapsto$ something in $A^{\otimes 7}$.

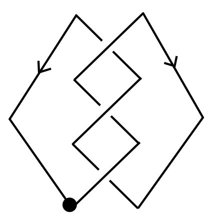
This element in $A^{\otimes 7}$ is a sum of many terms. Because of the similarity (1)—and the fact that elements in A multiply to 0 if their boundaries don't match—the terms in the sum are in bijection with homomorphisms

$$\langle \{x_i, y_i, z_i\} | x_2^{-1} y_2 x_2 = x_3 \rangle \to G.$$

Each process of removing one of the dots involves the multiplication map

$$A \otimes A \to A$$

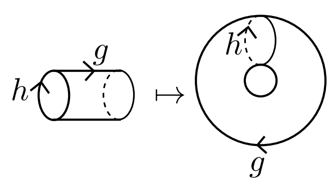
so this process should terminate with the assignment of

 $\mapsto$ something in A.

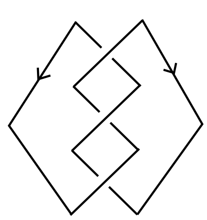
However there's a "self-multiplication" map for A. Let X be the free complex vector space on the set $\mathrm{Hom}(\pi_1 T^2, G)$. Then there's a map

$$A \to X$$

given by gluing the ends of a cylinder together:



(and 0 if the ends don't match up). In this way you make an association

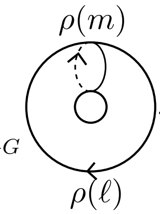
 $\mapsto$ something in X.

The element in X is a sum of many terms. The terms in the sum are in bijection with homomorphisms

$$\langle \{x_i, y_i, z_i\} | x_2^{-1} y_2 x_2 = x_3,\ x_3 = z_2,\ \cdots \rangle \cong \pi_1(S^3 \setminus K) \to G.$$

In fact, if you follow through the construction, the element in X assigned to the knot diagram is

$$\sum_{\rho:\pi_1(S^3\setminus K)\to G}$$



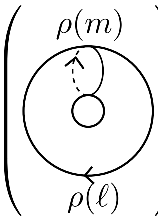
.

Here m is the meridian of the knot and ℓ is the blackboard longitude. For V an A-module the trace tr_V descends to a map

$$\mathrm{tr}_V : X \to \mathbb{C}.$$

In this manner, the finite group Reshetikhin-Turaev invariant $I(K, V)$ is

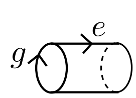
$$\sum_{\rho:\pi_1(S^3\setminus K)\to G} \mathrm{tr}_V \left(\quad \right). \qquad (2)$$



This invariant, unlike the one for U_h, has an immediate 3-dimensional interpretation.

In light of the similarity between ω and the regular representation, it is worth examining the invariant $I(K, A)$. Here A acts on itself on the right—this is the "regular representation" for A. It's a permutation representation so $\mathrm{tr}_A(x)$ is the number of algebra generators fixed by x. A moment's thought shows that x only fixes something if it's of the form

$$x =$$



in which case it fixes $|G|$ elements. Hence

$$I(K, A) = |G| \# \{\rho : \pi_1(S^3 \setminus K) \to G | \rho(\ell) = e\} = |G| \# \{\rho : \pi_1(S^3(K)) \to G\} \qquad (3)$$

which is manifestly an invariant of the surgered manifold $S^3(K)$!

8. Witten's Integral

The Reshetikhin-Turaev 3-manifold invariant was first conceived by Witten as the following integral [Wit89]:

$$\int_{\mathcal{A}} e^{i\langle \mathrm{CS}(A), [M] \rangle} DA.$$

Here $\mathcal{A}$ is the space of SU(2) connections on a closed 3-manifold M, $\mathrm{CS}(A)$ is some 3-form on M associated to the connection A, and DA is some (hypothetical) measure on $\mathcal{A}$. Because the measure doesn't exist this construction is not a rigorous 3-manifold invariant. Despite this, Witten was able to predict some of its properties.

Reshetikhin and Turaev were partly inspired by this integral when they constructed their invariants. Their invariant satisfies all of the properties predicted for Witten's invariant and so can be considered the rigorous realization of Witten's invariant.

Witten's integral works for any compact Lie group G. Similarly, the Reshetikhin-Turaev construction works for any of the quantum groups $U_h(\mathfrak{g})$. In [DW90] Dijkgraaf and Witten set out to construct Witten's integral for G a finite group. They fix a cocycle $c \in H^3(BG; \mathbb{R}/\mathbb{Z})$ and consider the quantity

$$\int_{\phi\in[M,BG]} e^{i\langle\phi^* c,[M]\rangle} d\phi.$$

For finite G all connections on a G-bundle are flat. Such G bundles with flat connection are in bijection with $[M, BG]$, which is why this appears in place of $\mathcal{A}$. $[M, BG] \cong \mathrm{Hom}(\pi_1 M, G)/G$ is a finite set and $d\phi$ is a natural measure that descends from the Haar measure on G—so the "integral" written here is actually a finite sum. Up to a constant factor it's just

$$\sum_{\phi\in[M,BG]} e^{i\langle\phi^* c,[M]\rangle}. \tag{4}$$

The algebra A can be "twisted" by the cocycle c (see [DPR90]) to form an algebra A^c. When A^c is plugged into the Reshetikhin-Turaev machine ([AC92]), out pops the sum (4). If I had more patience I would have constructed this note around the algebra A^c and not A; but A is simpler to deal with. Of course $A = A^0$ and so the invariant (4) is, for A, just $\#\{\phi : M \to BG\} = \#\{\rho : \pi_1 M \to G\}$. Up to a constant, this was recovered in (3).

The point is, the algebras U_h and A share a common lineage back to Witten's integral. The invariant (2) therefore has more than a formal relationship to the Jones polynomial.

References

[AC92] Daniel Altschüler and Antoine Coste, *Invariants of 3-manifolds from finite groups*, Proceedings of the XXth International Conference on Differential Geometric Methods in Theoretical Physics, Vol. 1, 2 (New York, 1991), 1992, pp. 219–233.

[DPR90] R. Dijkgraaf, V. Pasquier, and P. Roche, *Quasi Hopf algebras, group cohomology and orbifold models*, Nuclear Phys. B Proc. Suppl. **18B** (1990). Recent advances in field theory (Annecy-le-Vieux, 1990).

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[Wit89] Edward Witten, *Quantum field theory and the Jones polynomial*, Comm. Math. Phys. **121** (1989), no. 3, 351–399.