Link Invariants from Finite Group Chern-Simons Theory

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1. Introduction

In [Wit89], Witten discussed the functional integral formulation of 3d Chern-Simons theory. It was made rigorous by Reshetikhin and Turaev in [RT91] via surgery presentations of 3-manifolds and certain Hopf algebras.

In [DW90], Dijkgraaf and Witten discussed the functional integral formulation of finite gauge group Chern-Simons theory by defining a particular topological quantum field theory (TQFT). Their discussion, already rigorous because the integral is over a finite set, involved triangulations of 3-manifolds and surfaces. Altschuler and Coste [AC92] developed Dijkgraaf and Witten's theory from the "Reshetikhin-Turaev" perspective of link diagrams and certain quasi-Hopf algebras. They conjectured the equivalence of their construction with that of Dijkgraaf and Witten. The equivalence was proven for closed manifolds in [KSW05] Theorem 5.2.

It is the purpose of the present paper to construct Altschuler and Coste's link and 3-manifold invariants via the simplicial perspective of [DW90]. To be more precise, let $\mathcal{L}$ be a framed directed link in S^3 whose N components are colored by simple modules $\mathbf{i} = (i_1, \ldots, i_N)$ of the twisted Drinfeld double D of a finite group G (see Section 7 for a definition of D). Let $I(\mathcal{L}, \mathbf{i})$ be the link invariant (recalled in Definition 2.4) derived from the ribbon quasi-Hopf algebra D. Let $\chi_{i_1}, \ldots, \chi_{i_N}$ be the characters of the simple modules $i_1, \ldots, i_N$. For a 3-manifold $\mathcal{M}$, let $Z(\mathcal{M})$ denote the Dijkgraaf-Witten invariant of $\mathcal{M}$. If $\mathcal{M} = S^3 \setminus \nu\mathcal{L}$, the complement of a neighborhood of $\mathcal{L}$ in S^3, it turns out that $Z(S^3 \setminus \nu\mathcal{L})$ can be thought to lie in $D^{\otimes N}$. Then (see Theorem 9.6 for a more precise statement)

Theorem. *The invariants $I(\mathcal{L}, \mathbf{i})$ and $Z(S^3 \setminus \nu\mathcal{L})$ are related by*

$$I(\mathcal{L}, \mathbf{i}) = \frac{1}{|G|^{N-1}} \langle Z(S^3 \setminus \nu\mathcal{L}), \chi_{i_1} \otimes \cdots \otimes \chi_{i_N} \rangle.$$

The main contribution of this paper is a new proof of the Theorem. In particular, in Section 4, various triangulations of 3-manifolds are constructed. Each triangulation corresponds via the Dijkgraaf-Witten TQFT to a particular algebraic object associated to D. The Theorem follows from gluing together these triangulated pieces to form the link complement $S^3 \setminus \nu\mathcal{L}$. For closed $\mathcal{M}$, the description of $Z(\mathcal{M})$ in terms of link invariants of a surgery diagram for $\mathcal{M}$ also follows immediately.

Section 5 and 6 may be of independent interest as an exposition of the finite group TQFT in an arbitrary dimension n. The simplicial perspective presented here is different from the perspective of [FQ93], [Fre94] and involves more algebraic structures than the original treatment in [DW90].

Section 2 reviews the link invariants extractable from a ribbon quasi-Hopf algebra. Section 3 covers some necessary background and sets notations. Section 4 gives important examples of triangulations used later. Section 5 recalls the finite group TQFT of Dijkgraaf and Witten, for all dimensions n. Section 6 extracts an algebra from the TQFT. Section 7 specializes to $n = 3$ and extracts a quasi-Hopf algebra structure from the TQFT. Section 8 discusses the vector space associated to a surface. Sections 9 shows how the link invariants of Section 2 follow from the TQFT of Sections 5 through 8 by using the triangulations of Section 4.

Fix throughout a finite group G and an algebraically closed field k in which $|G|$ is invertible. The letter e stands for the identity element in G.

2. Quasi-Hopf Algebras

For more information about quasi-Hopf algebras, see [EGNO15] section 5.13.

Definition 2.1. A quasi-bialgebra $(B, m, i, \Delta, \epsilon)$ is an associative unital k-algebra B with multplication m and unit i, plus an algebra morphism $\Delta : B \to B \otimes B$ (called the coproduct), an algebra morphism $\epsilon : B \to \mathsf{k}$ (called the counit), and a distinguished invertible element $\Phi \in B \otimes B \otimes B$, satisfying

$$(\mathrm{id} \otimes \Delta)(\Delta(b)) = \Phi(\Delta \otimes \mathrm{id})(\Delta(b))\Phi^{-1}$$

$$(\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\Phi)(\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\Phi) = (\Phi \otimes 1)(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\Phi)(1 \otimes \Phi)$$

$$(\epsilon \otimes \mathrm{id})(\Delta(b)) = b = (\mathrm{id} \otimes \epsilon)(\Delta(b))$$

$$(\mathrm{id} \otimes \epsilon \otimes \mathrm{id})(\Phi) = 1 \otimes 1.$$

Definition 2.2. A quasi-Hopf algebra H is a quasi-bialgebra $(H, m, i, \Delta, \epsilon)$ plus an algebra antimorphism $S : H \to H$ and distinguished elements $\alpha, \beta \in H$ satisfying (write $\Delta(a) = \sum_i a_i^1 \otimes a_i^2$)

$$\sum_i S(a_i^1)\alpha a_i^2 = \epsilon(a)\alpha \tag{1}$$

$$\sum_i a_i^1 \beta S(a_i^2) = \epsilon(a)\beta. \tag{2}$$

Let V, W, U be finite-dimensional right H-modules. As with any bialgebra, Δ turns $V \otimes W$ into an H-module and ϵ turns k into an H-module. S turns V^* into a right H-module as follows

$$\langle v, \xi a\rangle := \langle vS(a), \xi\rangle,\ v \in V,\ \xi \in V^*,\ a \in H.$$

The natural pairing

$$V \otimes V^* \to \mathsf{k}$$
$$v \otimes \xi \mapsto \langle v, \xi\rangle$$

is not a map of H-modules but

$$P_V : V \otimes V^* \to \mathsf{k}$$
$$v \otimes \xi \mapsto \langle v\beta, \xi\rangle$$

is. Write $\{e_i\}$ for a basis of V and $\{e^i\}$ for the dual basis. The natural map

$$\mathsf{k} \to V^* \otimes V$$
$$1 \mapsto \sum_i e^i \otimes e_i$$

is not a map of H-modules but

$$C_V : \mathsf{k} \to V^* \otimes V$$
$$1 \mapsto \sum_i e^i \otimes e_i\alpha$$

is. The canonical map

$$V \otimes (W \otimes U) \to (V \otimes W) \otimes U$$
$$v \otimes w \otimes u \mapsto v \otimes w \otimes u$$

is not necessarily a map of H-modules. The map

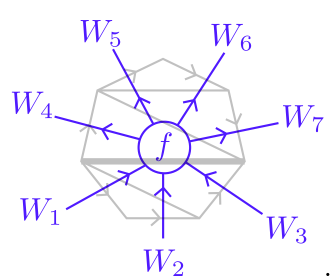
$$v \otimes w \otimes u \mapsto (v \otimes w \otimes u)\Phi$$

is, however.

It will be helpful to develop a graphical notation for H-module morphisms. Diagrams are to be read from the bottom to the top. For $\{W_i\}$ some collection of right H-modules, write a morphism of H-modules

$$f : (W_1 \otimes W_2) \otimes W_3 \to W_4 \otimes ((W_5 \otimes W_6) \otimes W_7)$$

as

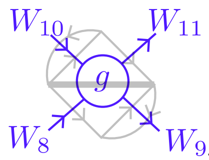


.

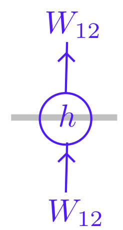
Note that the triangulation of the polygon indicates the placement of the parentheses. Allow for duals by reversing the blue arrows and inserting bigons into the triangulation: a map

$$g : W_8 \otimes W_9^* \to W_{10}^* \otimes W_{11}$$

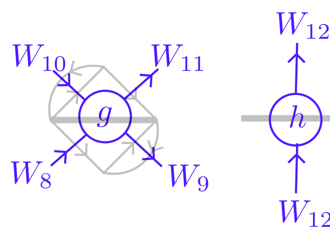
is drawn



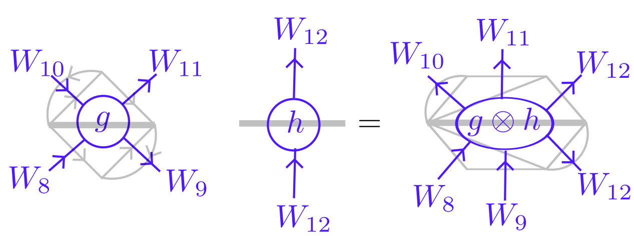
A map $h : W_{12} \to W_{12}$ is therefore drawn



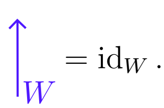
Two maps g and h placed next to each other



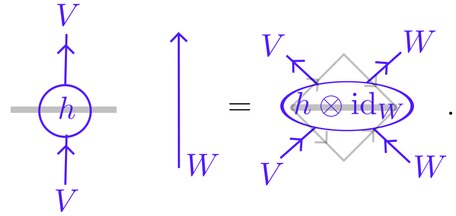
indicate the map $g \otimes h$:



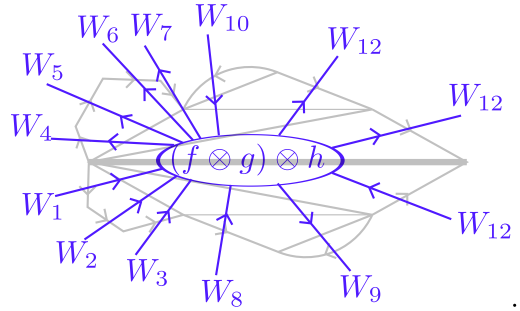
An unmarked vertical corresponds to the identity map:



Therefore, for example



The maps $(f \otimes g) \otimes h$ and $f \otimes (g \otimes h)$ are drawn on two different triangulations. Here's $(f \otimes g) \otimes h$:



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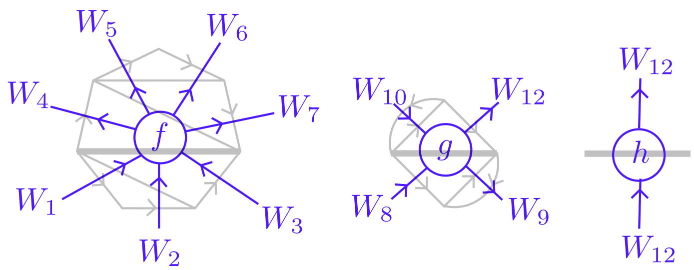
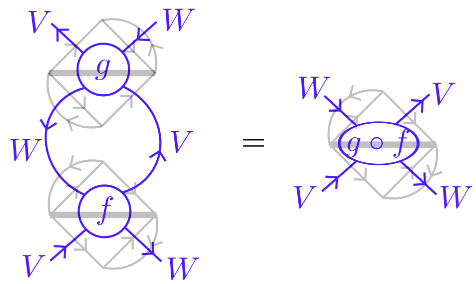
For simplicity, the maps $(f \otimes h) \otimes g$ and $f \otimes (g \otimes h)$ will still both be denoted by placing the three maps next to each other horizontally



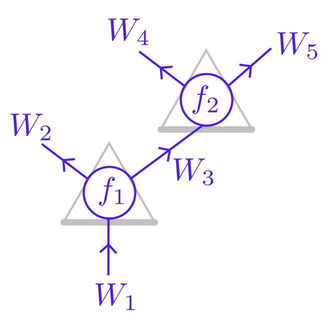
and it is left to the user to decide whether the map should be thought of as $(f \otimes g) \otimes h$ or $f \otimes (g \otimes h)$.

Gluing the ends of two same-labeled blue arrows together is defined to be composition so, for example,



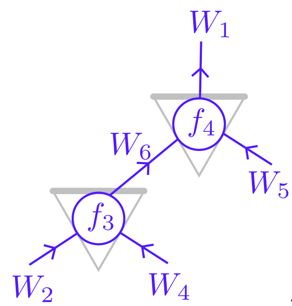
for $f : V \otimes W^* \to W^* \otimes V$ and $g : W^* \otimes V \to V \otimes W^*$.

The following denotes a map $W_1 \to W_2 \otimes (W_4 \otimes W_5)$:



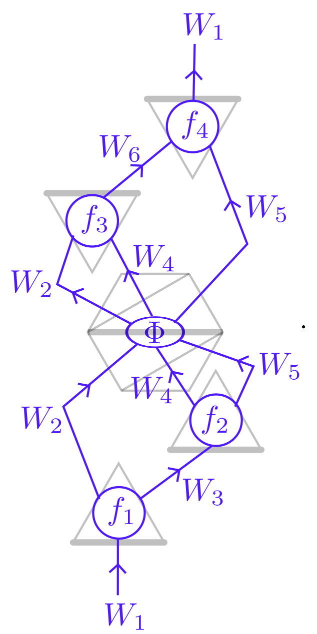
(3)

and the following denotes a map $(W_2 \otimes W_4) \otimes W_5 \to W_1$:



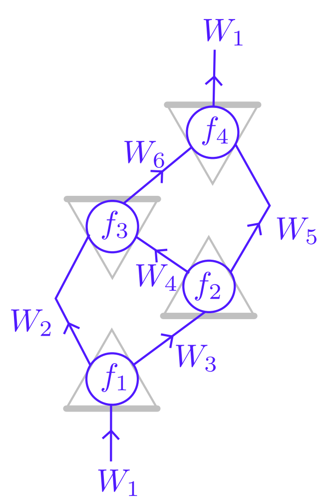
. (4)

Since the parentheses of the codomain of (3) are not the same as the parentheses of the domain of (4), one has to throw in a factor of Φ when composing



. (5)

The axioms of a quasi-bialgebra ensure that there's a unique way to combine multiple copies of Φ to provide a map between different parenthizations of $W_1 \otimes \cdots \otimes W_k$. Hence it is possible to be sloppy and write



instead of (5), leaving Φ implicit but unwritten.

Definition 2.3. (See [Som10]) A ribbon quasi-Hopf algebra is a quasi-Hopf algebra H plus a distinguished invertible central element $v \in H$ and a distinguished invertible element $R \in H \otimes H$ such that

$$R\Delta(a)R^{-1} = \Delta^{\mathrm{op}}(a),\ \forall a \in H$$

$$\Phi^{-1}((\mathrm{id} \otimes \Delta)(R))(\Phi^{-1})_{231} = R_{12}\Phi_{213}R_{13}$$
$$\Phi((\Delta \otimes \mathrm{id})(R))\Phi_{312} = R_{23}\Phi^{-1}_{132}R_{13}$$
$$R_{21}R_{12} = (v^{-1} \otimes v^{-1})\Delta(v)$$
$$S(v) = v$$

where if $R = \sum_i x_i \otimes y_i$ then, e.g., $R_{31} = \sum_i y_i \otimes 1 \otimes x_i$ and similarly for Φ.

For two right H-modules V and W, write

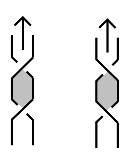
$$\sigma_{V,W} : V \otimes W \to W \otimes V$$

$$v \otimes w \mapsto w \otimes v.$$

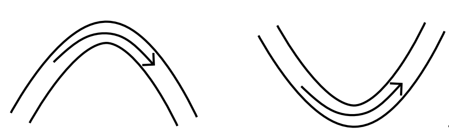
Define $\widehat{R}_{V,W} = (\rho_V \otimes \rho_W)(R) \circ \sigma_{V,W}$ (that is, switch the tensor factors then apply R). The condition $R\Delta(a)R^{-1} = \Delta^{\mathrm{op}}(a)$ implies that $\widehat{R}_{V,W}$ is a morphism of H-modules.

It is possible to construct directed framed link invariants from a quasi-Hopf algebra H. The construction works in three steps which are sketched below. For details see [RT90] or [Tur16] (for the Hopf algebra setting) and [AC92] (for the more general quasi-Hopf algebra setting).

1. Consider the directed framed link as a ribbon embedded in $\mathbb{R}^3$, one side painted white and the other black. Put it in a position so that its projection to $\mathbb{R}^2$ is generic and such that, in the projection, the white sides face up with the possible exception of some twists



and so that the only cups and caps are directed as follows:



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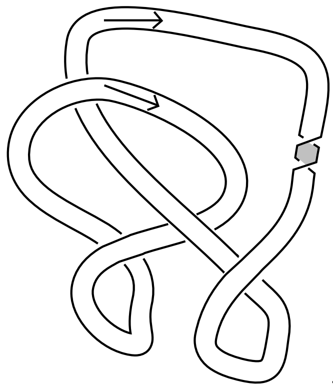
Here's an example of such a projection:



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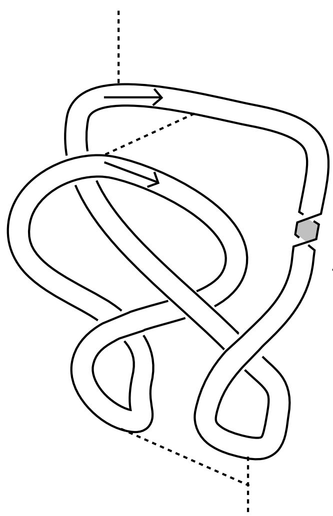
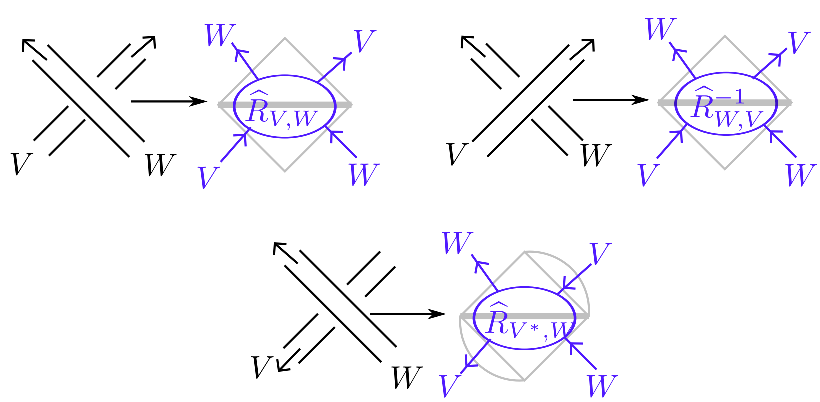
2. Add dashed lines connecting each cup or cap to some place below or above it, respectively:

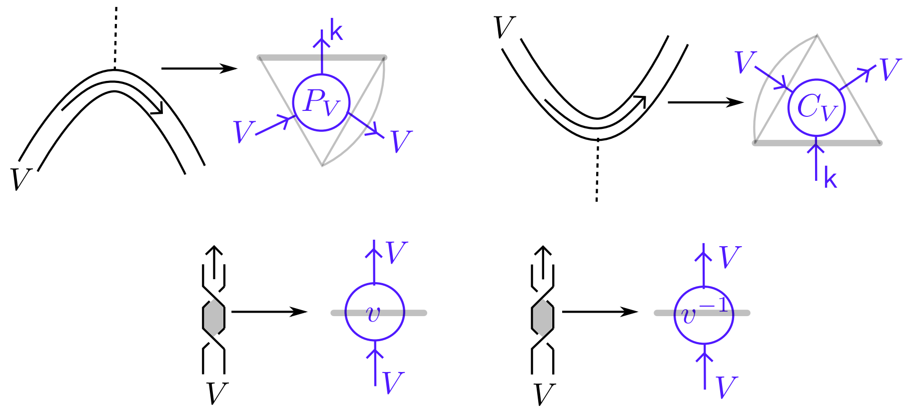


. (6)

3. Label each component of the link by a finite-dimensional module of H and label the dashed lines by the trivial module k. Peform the following replacements on the link diagram:



(and other similar ones for different directions of the crossings)



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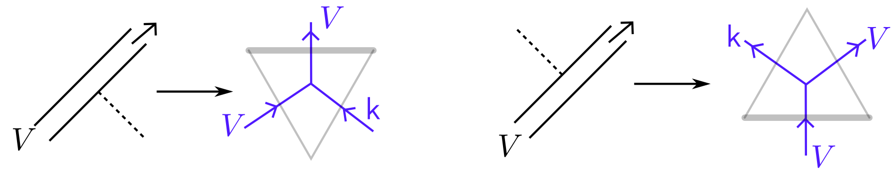
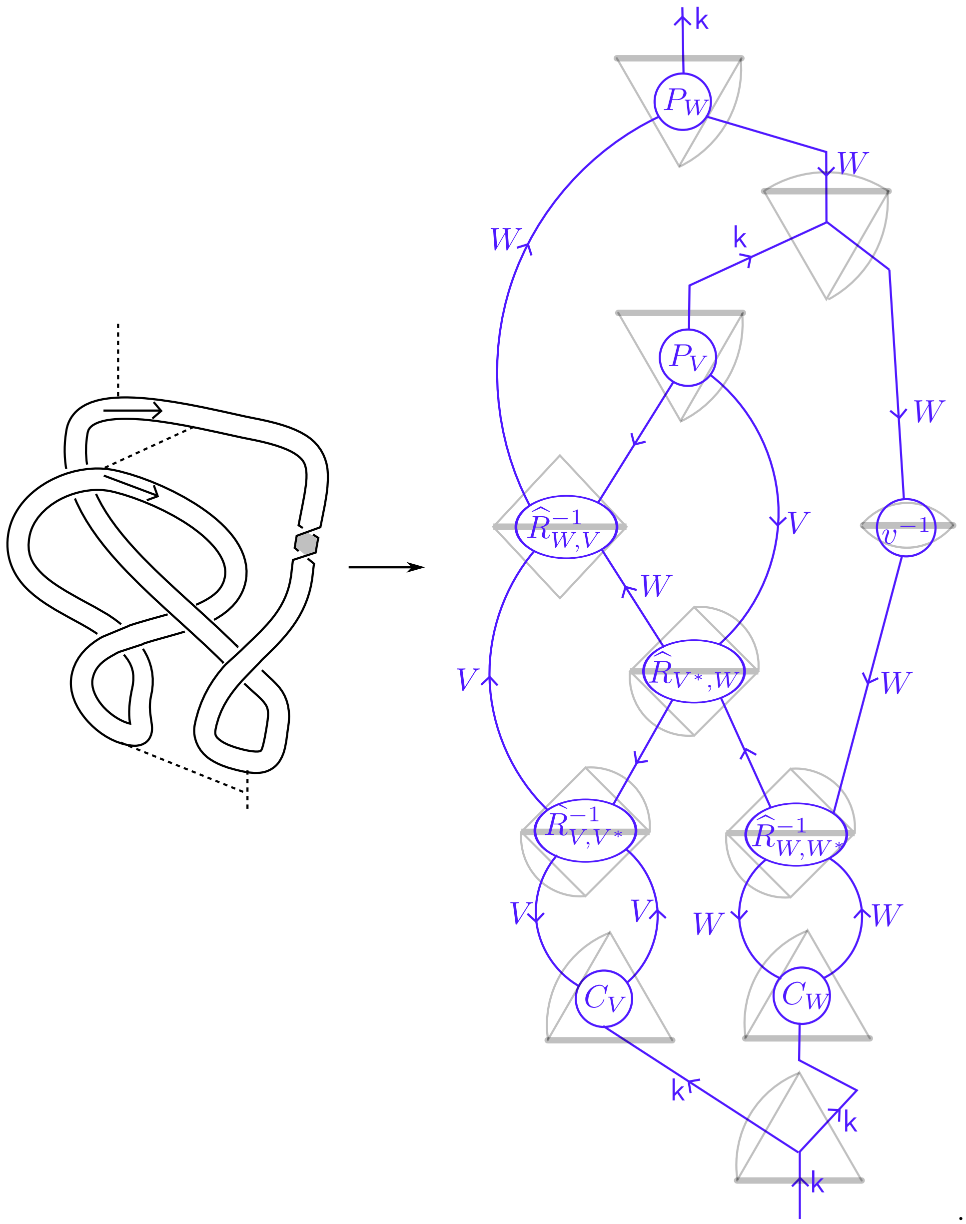


(8)

(plus other directions) where the unlabeled blue trivalent vertex corresponds to the usual canonical maps $V \otimes \mathsf{k} \to V$ or $V \to \mathsf{k} \otimes V$. The end result is a blue arrow diagram representing a map k to k. For example, the Hopf link diagram (6) gets turned into the following blue arrow diagram:



$$.\tag{9}$$

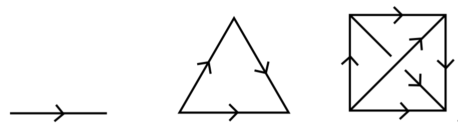
The trace of this map $\mathsf{k} \to \mathsf{k}$ is an invariant of the link. That it is an invariant involves checking invariance under Reidemeister moves. Such a check uses the axioms of a ribbon quasi-Hopf algebra.

Definition 2.4. Let $\mathcal{L}$ be a directed framed link in S^3. Let $\mathbf{i} = i_1, \ldots, i_N$ be simple modules for a ribbon quasi-Hopf algebra coloring the N components of $\mathcal{L}$. Write $I(\mathcal{L}, \mathbf{i})$ for the associated link invariant constructed by the previous three steps.

3. G-labelings of Δ-complexes

Let $(x_0, \ldots, x_n)$ be an ordered $(n+1)$-tuple of points in $\mathbb{R}^n$ whose span is not a proper affine subspace. An n-simplex is the convex hull of $(x_0, \ldots, x_n)$ plus the data of the ordering of the vertices. Ordered subsets of the vertices span subsimplices of an n-simplex. There is a unique affine map taking any n-simplex to any other, preserving the order of the vertices. Identifying all such simplices using these maps, one obtains what is called "the" n-simplex. The n-simplex is denoted Δ^n. The 1-simplex Δ^1 will also be identified with the closed interval I.

The n-simplex is drawn in such a way that arrows on the edges indicate the ordering of the vertices:



Definition 3.1. ([Hat02] Section 2.1) A Δ-complex is a topological space X together with a collection of maps $\{\sigma_i\}$, $\sigma_i : \Delta^n \to X$ (n depends on i) such that

- The restriction $\sigma_i|_{\mathrm{int}(\Delta^n)}$ is injective, and each point in X lies in one such restriction.
- The restriction of σ_i to a face of Δ^n is one of the maps $\sigma_j : \Delta^{n-1} \to X$.
- $A \subset X$ is an open set iff $\sigma_i^{-1}(A)$ is open for all i.

Informally speaking, a Δ-complex is a space obtained by gluing together n-simplices in such a way that the edge arrows match. A Δ-complex structure on a space will often be called a "triangulation."

Definition 3.2. An isomorphism $f : K \to K'$ of Δ-complexes is a homeomorphism $|K| \to |K'|$ that restricts to an affine order-preserving homeomorphism on each simplex.

Definition 3.3. Given a Δ-complex K, let $|K|$ denote the underlying topological space and let K^0 denote the set of 0-simplices of K.

Definition 3.4. Given a Δ-complex K, let $C_n(K)$ denote the integral Δ-chains with coefficients in K; that is, $C_n(K)$ is the free abelian group on the set of n-simplices of K. Let $\partial : C_n(K) \to C_{n-1}(K)$ denote the usual boundary map.

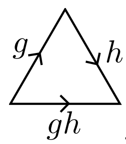
In particular, $C_n(\Delta^n)$ is canonically isomorphic to $\mathbb{Z}$, and so has a canonical generator 1. This generator will also be denoted $+$ and its additive inverse $-$.

Definition 3.5. If L is a subcomplex, let $C_n(K, L)$ denote relative Δ-chains. Let $H_n(K)$ and $H_n(K, L)$ denote the integral Δ-homology groups. Let $C^n(K; \mathsf{k}^\times)$ denote the Δ-cochains with coefficients in $\mathsf{k}^\times$. Elements of $C^n(K; \mathsf{k}^\times)$ will be written multiplicatively.

Example 3.6. Recall that G is a finite group. Define a Δ-complex BG by starting with a single vertex, gluing on an edge labeled by each element $g \in G$

$$\xrightarrow{\quad g \quad},$$

gluing on a 2-simplex to the edges g, h, and gh as follows



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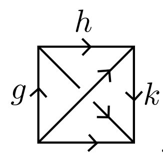
gluing on the following 3-simplex for triple (g, h, k)



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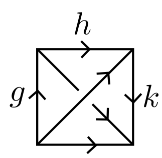
and similarly gluing on higher n-simplices. Since any n-simplex in BG is determined by an ordered n-tuple of elements of G, e.g.,



$\leftrightarrow (g, h, k)$

the group $C^n(BG; \mathsf{k}^\times)$ is naturally identified with functions $G^n \to \mathsf{k}^\times$. After this identification, the coboundary map is

$$(\delta c)(g_1, \ldots, g_{n+1}) = c(g_2, \ldots, g_{n+1})c(g_1 g_2, g_3, \ldots, g_{n-1})^{-1}c(g_1, g_2 g_3, g_4, \ldots, g_{n+1})$$
$$\cdots c(g_1, g_2, \ldots, g_n g_{n+1})^{(-1)^n} c(g_1, g_2, \ldots, g_n)^{(-1)^{n+1}}.$$

Definition 3.7. If K is a Δ-complex, say that K is manifold if $|K|$ is a manifold and $\partial|K|$ is a subcomplex, denoted ∂K.

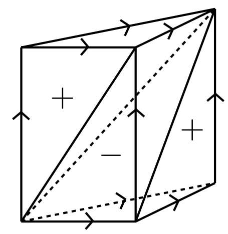
Definition 3.8. If K is an n-manifold, then an orientation class x_K is an element of $C_n(K)$ that represents a generator of $H_n(K, \partial K)$. The pair (K, x_K) will be called an oriented Δ-complex. In practice K will also be allowed to be a pseudo-manifold (a collection of simplices of the same dimension so that a codimension 1 face belongs to at most two simplices).

Remark 3.9. The canonical generator of $C_n(\Delta^n)$ gives a canonical orientation on Δ^n.

Remark 3.10. If (K, x_K) is an oriented manifold, then $(\partial K, \partial x_K)$ is also an oriented manifold.

Remark 3.11. If $K' \subset K$ is a codimension 0 submanifold, then an orientation on K restricts to an orientation on K'.

Example 3.12. Here is an oriented Δ-complex structure on $I \times \Delta^2$:



(10)

The directions on the undirected edges are implied by the indicated directions.

Definition 3.13. Let X be a topological space and $S \subset X$ a subset. Let $\pi_1(X; S)$ denote the groupoid of homotopy classes (rel endpoints) of paths in X starting and ending in S.

A group can naturally be considered as a category with a single object. For example, $\pi_1(X; \{x\})$ is naturally identified with the group $\pi_1(X, x)$. Thinking of groups as categories implies that functors $G \to H$ are the same as homomorphisms $G \to H$.

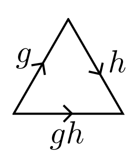
Definition 3.14. Write $\mathrm{Hom}(\pi_1(X; S), G)$ to denote functors from $\pi_1(X; S)$ to G.

Definition 3.15. Define $\mathrm{Hom}(\emptyset, G)$ to be a single element set.

The following is well-known:

Proposition 3.16. *There are natural bijections between the following three sets*

- *The set of labelings of the edges of a Δ-complex K by elements of G such that any 2-simplex is labeled by elements of the form*

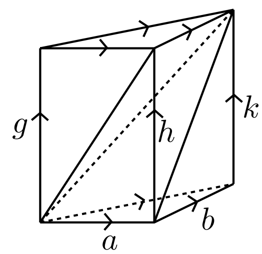


- *Relative homotopy classes of maps $(|K|, K^0) \to (BG, *)$, where $*$ is the vertex of BG*

- $\mathrm{Hom}(\pi_1(|K|; K^0), G)$.

Definition 3.17. Write $\mathrm{Hom}(K, G)$ to denote any of the three equivalent sets of Proposition 3.16. Elements of $\mathrm{Hom}(K, G)$ will be called G-labelings.

Example 3.18. Here are the G-labelings $I \times \Delta^2$:



$$g, h, k, a, b \in G \qquad (11)$$

The labelings on all the edges are determined by these five labelings.

Remark 3.19. Given $\phi \in \mathrm{Hom}(K, G)$, ϕ induces a map

$$\phi_* : C_n(K) \to C_n(BG)$$

by taking a particular labeled simplex of K to the associated labeled simplex of BG. In a similar manner, ϕ also determines a map $|\phi| : |K| \to |BG|$ by mapping a particularly labeled simplex of K to the associated labeled simplex of BG in an affine manner.

Remark 3.20. Given $f : K \to K'$ an isomorphism of complexes and $\phi \in \mathrm{Hom}(K, G)$, let $f(\phi)$ denote the labeling of K' obtained by transporting the labeling ϕ to K' via the map f.

Definition 3.21. If K is a manifold and $\phi \in \mathrm{Hom}(K, G)$, then $\partial\phi \in \mathrm{Hom}(\partial K, G)$ denotes ϕ restricted to ∂K.

Definition 3.22. Let S be a finite set. The group G^S acts on $\mathrm{Hom}(\pi_1(X; S), G)$ (on the right) as follows:

$$(\phi \cdot f)(\gamma) = f(\gamma(0))^{-1}\phi(\gamma)f(\gamma(1)),\ \phi \in \mathrm{Hom}(\pi_1(X; S, G)),\ f \in G^S,\ \gamma \in \pi_1(X; S).$$

Call this the "gauge" action.

Remark 3.23. The gauge action gives an action of G^{K^0} on $\mathrm{Hom}(K, G)$.

Remark 3.24. If $S = \{x\}$, then the gauge action is the conjugation action on $\mathrm{Hom}(\pi_1(X, x), G)$.

The following is easy:

Claim 3.25. *If $s_0, s_1 \in S$ are distinct points in the same connected component of X, then $G^{\{s_1\}}$ acts freely on* $\mathrm{Hom}(\pi_1(X; S), G)$.

Corollary 3.26. *Suppose $S \subset X$ is a finite set that intersects every path component of X. Then*

$$\frac{|\operatorname{Hom}(\pi_1(X;S),G)|}{|G|^{|S|}}$$

is independent of the choice such an S. In particular, it equals

$$\prod_{x_i} \frac{|\operatorname{Hom}(\pi_1(X,x_i),G)|}{|G|}$$

where there is one $x_i \in X$ in each path component of X.

Let $c \in C^n(BG;\mathsf{k}^\times)$. Given an n-dimensional oriented Δ-complex (K,x_K) and a G-labeling ϕ of K, a quantity of interest will be

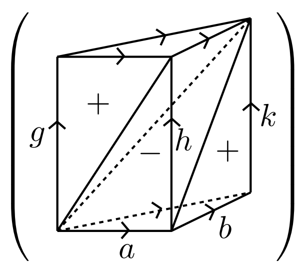
$$\langle \phi_* x_K, c\rangle \in \mathsf{k}^\times.$$

Example 3.27. If (K,x_K) is (10) and ϕ is (11) then

$$\langle \phi_* x_K, c\rangle = \frac{c(g,g^{-1}ah,h^{-1}bk)c(a,b,k)}{c(a,h,h^{-1}bk)}.$$

Often this will be notated by applying c to a picture of the oriented G-labeling:

$$\langle \phi_* x_K, c\rangle =: c$$



Δ^n can be thought of as the geometric realization of the simplicial complex given by the power set of the ordered set $[n] := \{0,\dots,n\}$. Put a poset structure on $[n]\times[m]$ by the condition

$$(a,b) \le (c,d) \Leftrightarrow a \le c \text{ and } b \le d.$$

$\Delta^n \times \Delta^m$ is the geometric realization of the simplicial complex given by the subset of strictly increasing chains in $[n]\times[m]$. Let σ be a maximal such chain (so its length is $n+m+1$). σ can be realized as a walk on the lattice $[n]\times[m]$ in the first quadrant of $\mathbb{R}^2$. Let $(-1)^\sigma = (-1)^{\#(\text{lattice points below walk})}$. Then

$$\sum_{\sigma \text{ maximal in } \Delta^n\times\Delta^m} (-1)^\sigma \sigma \in C_{n+m}(\Delta^n \times \Delta^m)$$

is an orientation on $\Delta^n \times \Delta^m$.

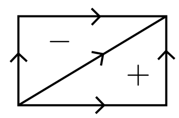
Let (K,x_K) and (L,x_L) be two oriented Δ-complexes. Working simplex by simplex, $K \times L$ inherits an oriented Δ-complex structure. Call this $(K \times L, x_K x_L)$. If the orientation on K is implicit, then write $-K$ for the same complex with the opposite orientation. Examples can be found in the next section.

4. Triangulating a Link Complement

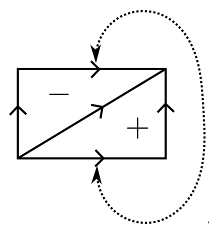
This section introduces triangulations and notation for these triangulations. Both the triangulations and the notation will be useful in later sections.

Example 4.1. Write S^1 to mean I with the endpoints identified. It inherits an oriented Δ-complex structure from I.

Example 4.2. $I \times I$ with the product orientation is

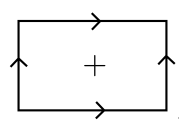


and similarly $I \times S^1$ inherits an oriented Δ-complex structure obtained by gluing the top and bottom together:



.

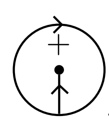
It will be convenient to indicate $I \times I$ and $I \times S^1$ by



. (12)

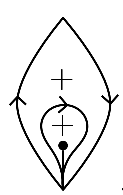
Whether or not the top and bottom edges will be identified should be clear from context. The horizontal edge of $I \times S^1$ in (12) will be called the "longitude" and the two loops on the ends will be called "meridians."

Example 4.3. Let D^2 be the oriented Δ-complex obtained by gluing together two edges of Δ^2:



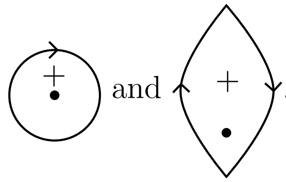
.

Glue on a copy of Δ^2 to produce an oriented Δ-complex structure on the bigon B:



.

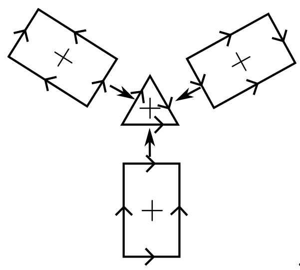
It will be convenient to abbreviate D^2 and B as



Drawing the interior vertex closer to one of the boundary vertices of the bigon helps indicate the implied Δ-complex structure.

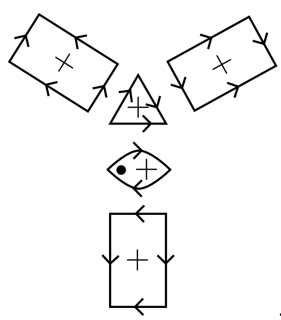
Example 4.4. This example can be profitably compared to the triangulations in Section 2.

The following oriented Δ-complexes can be glued (arrows indicating sides to be glued) to form a new one:



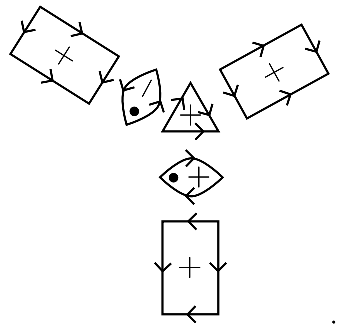
(13)

When the the long edges of each rectangle are glued together, the result is an oriented Δ-complex structure on the pair of pants. The boundary of Δ^2 has two positive intervals and one negative interval. The positive intervals should be thought to correspond to "outputs" and the negative interval to an "input." In this Δ-complex structure on the pair of pants, there is one "incoming" input and two "outgoing" outputs. To change that to one "outgoing" input and two "outgoing" outputs, add a bigon:



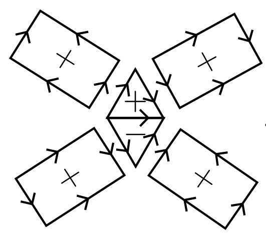
(14)

To change to one "outgoing" input, one "incoming" output, and one "outgoing" output, add another bigon:



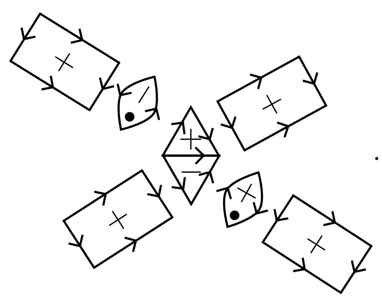
(15)

Here's a Δ-complex structure on the four-holed sphere with two "incoming" inputs and two "outgoing" outputs:



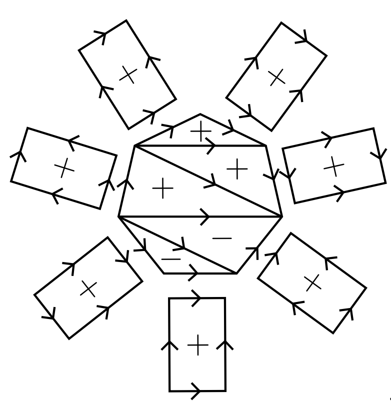
(16)

Here's a Δ-complex structure on the four-holed sphere with one "incoming" and one "outgoing" input, and one "incoming" and one "outgoing" output:



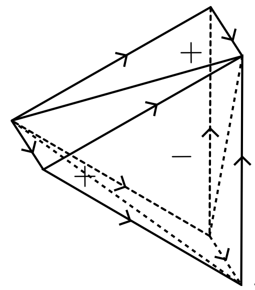
(17)

This can be generalized to triangulations of many-holed spheres with a partition of the boundary components into input and output components. For example, here's a seven-holed sphere:

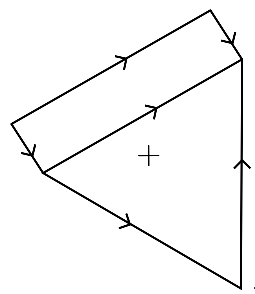


(18)

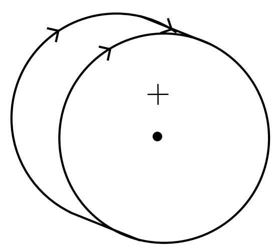
Example 4.5. The oriented Δ-complex structure on $\Delta^2 \times I$ is:



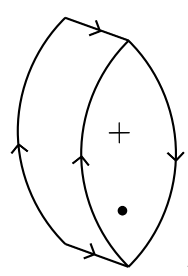
This will be abbreviated by



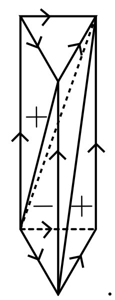
Gluing two of the faces of $\Delta^2 \times I$ together produces an oriented Δ-complex structure on $D^2 \times I$, abbreviated by



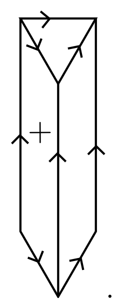
and adding on another copy of $\Delta^2 \times I$ produces a Δ-complex structure on $B \times I$, denoted by



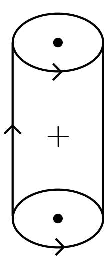
The oriented Δ-complex structure on $I \times \Delta^2$ is:



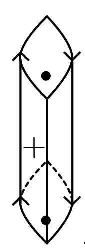
This will be abbreviated by



Gluing two of the faces together produces a triangulation of $I \times D^2$, abbreviated

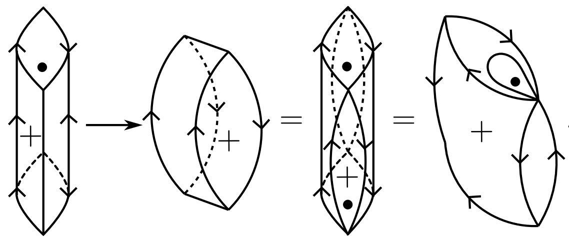


and gluing on another copy of $I \times \Delta^2$ produces $B \times I$, abbreviated



Note that $I \times \Delta^2$ is distinguished from $\Delta^2 \times I$ in these pictures in that the first cartesian factor is drawn larger than the second: the I direction in $I \times \Delta^2$ is drawn longer than the I direction in $\Delta^2 \times I$.

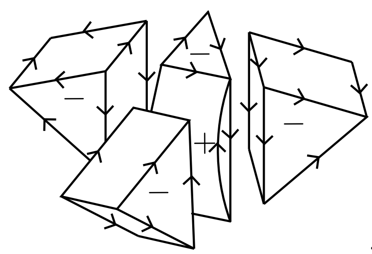
Example 4.6. It will be convenient to abbreviate the triangulation obtained by gluing together $B \times I$ and $I \times B$:



(19)

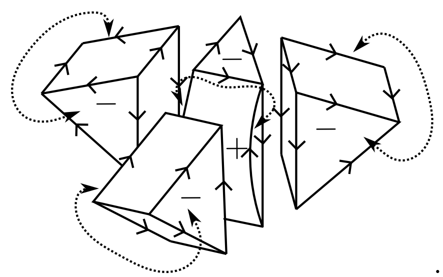
The orientation on this will be called $+$.

Example 4.7. Let P be the pair of pants. $\partial(I \times P)$ is divided into five parts: two copies of P in $\partial I \times P$ and three cylinders in $I \times \partial P$. Collapsing the three cylinders to circles produces a homeomorphic manifold. The following gives an oriented Δ-complex structure on this manifold:



(20)

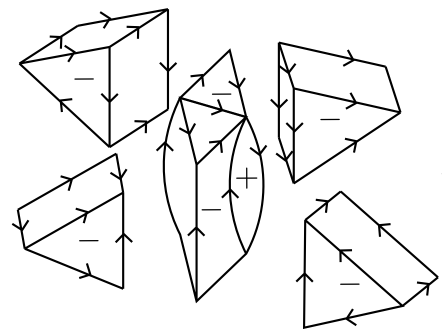
Some explanation is warranted: there are three copies of $-\Delta^2 \times I$ here and one copy of $-I \times \Delta^2$. There is one copy of $I \times B$. Opposite triangles are glued together, as are the two opposite bigons:



Such identifications will be suppressed from the notation for the rest of the section.

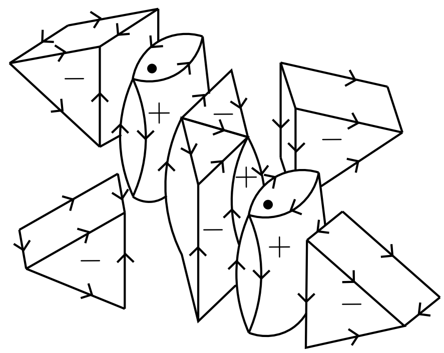
The boundary of this oriented Δ-complex is a copy of (13) on top and a copy of its opposite on the bottom.

Example 4.8. Let S be the four-holed sphere. $\partial(I \times S)$ has six parts to its boundary: two copies of S in $\partial I \times S$, and four cylinders in $I \times \partial S$. Collapse the cylinders to circles to create a homeomorphic manifold. The result can be given the following oriented Δ-complex structure:



. (21)

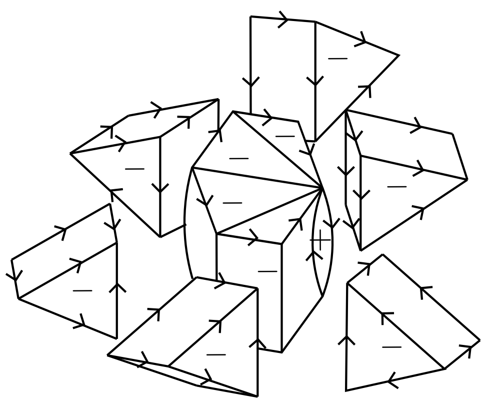
Note that the top is a copy of (16). The following is also a Δ-complex structure on the same manifold:



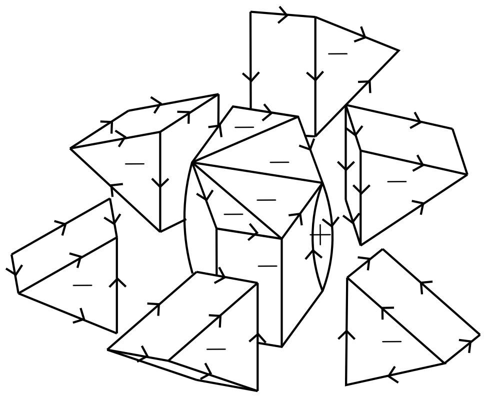
(22)

whose top is a copy of (17).

Example 4.9. Similar to the previous example, here are two analogs for the six-holed sphere:



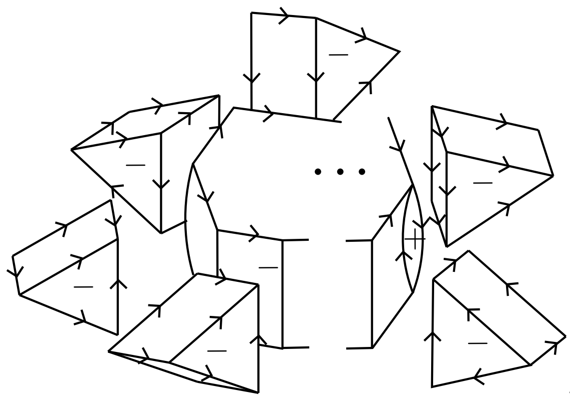
(23)



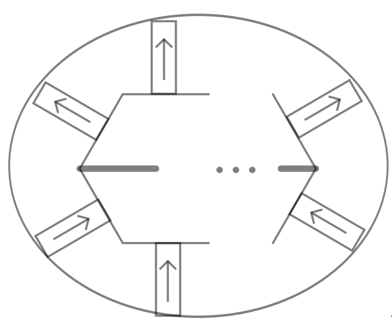
(24)

Note the different triangulations on the hexagonal prism in the middle.

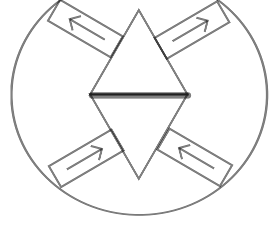
Example 4.10. It should not be hard to determine how to triangulate analogs of the previous three examples for spheres with more holes and with varying input and output configurations, as well as varying triangulations of the central polygon:



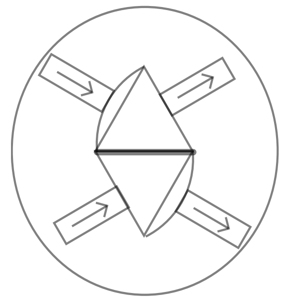
Example 4.11. Denote the last example as



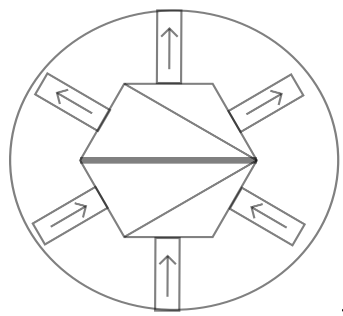
This is best clarified with explicit examples. The following denotes (21):



the following denotes (22):

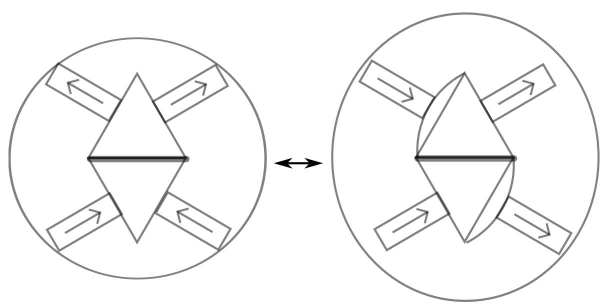


and the following denotes (23):



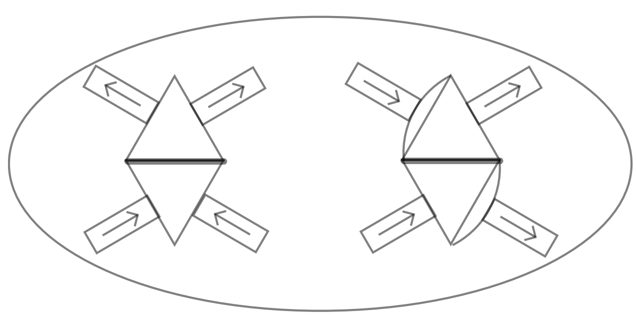
.

Example 4.12. One can boundary sum two of these things together:



(25)

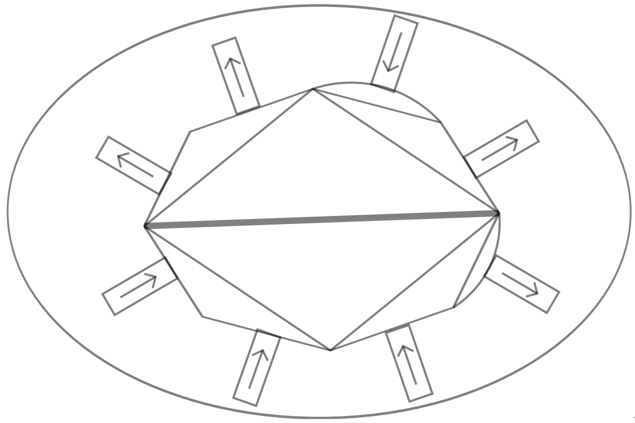
to obtain



(26)

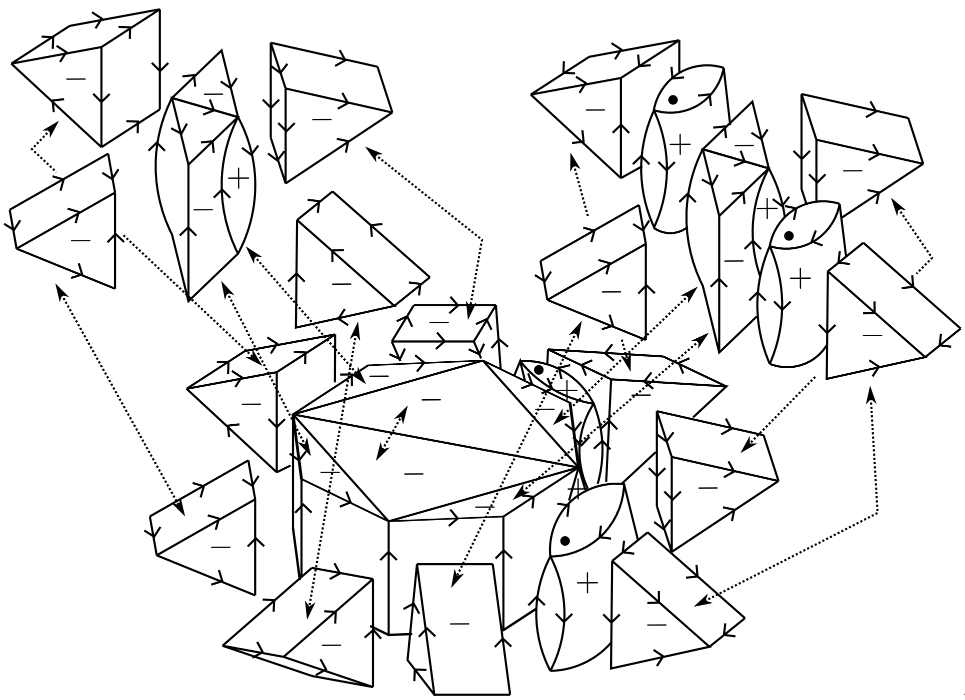
the triangulation of which is, by definition, obtained by taking each part of

(25) and pressing it onto



.

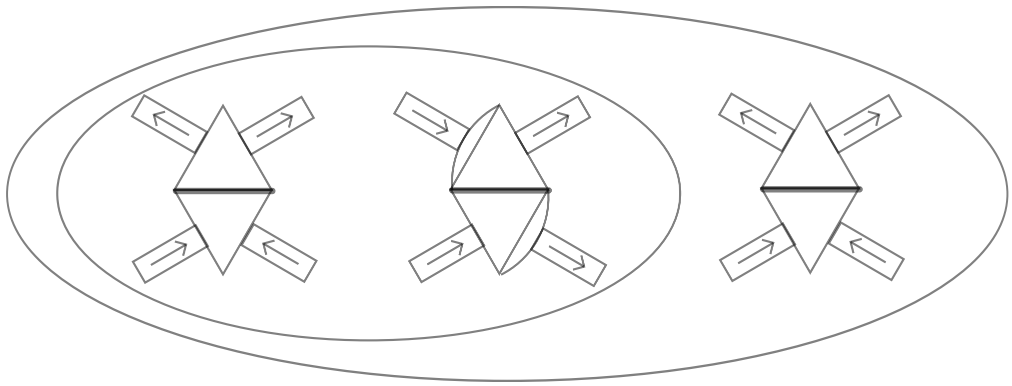
Explicitly:

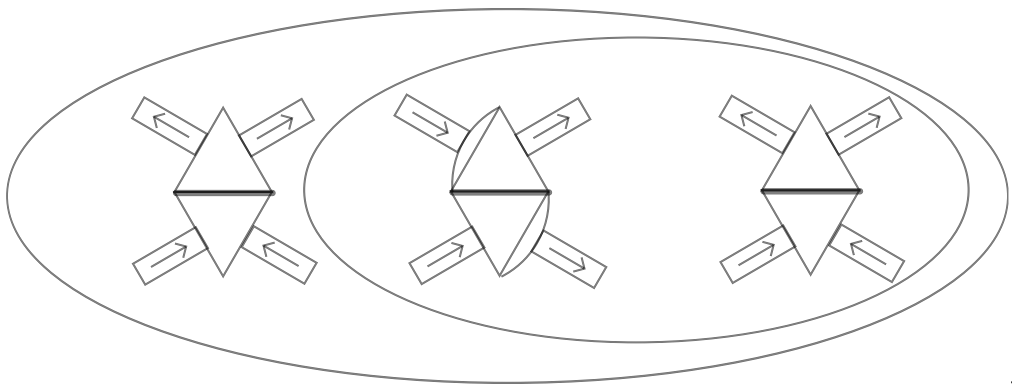


$$. \quad (27)$$

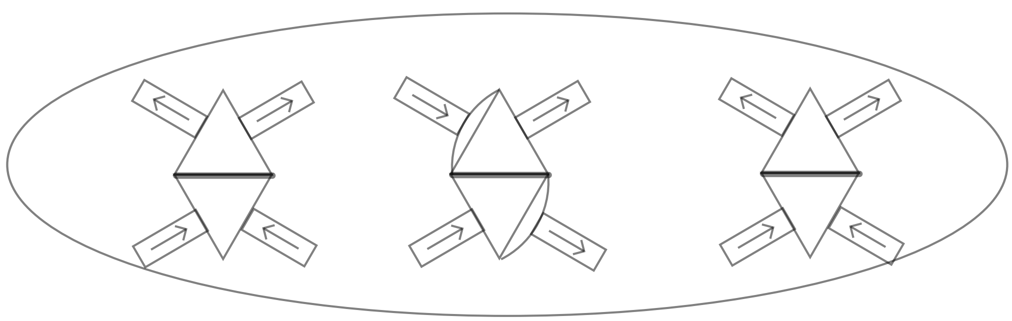
Complexes like (26) will be called “generalized tangles.” If there are n inputs and m outputs, it will be called a generalized (n, m)-tangle. To parallel the language of [RT90], the strips with arrows are called ribbons and the larger polygons with the dark line through the middle are called “coupons.” The coupons here need to keep track of the input and output triangulations, unlike those of Reshetikhin-Turaev.

Example 4.13. There are two different ways of triangulating the following generalized $(6, 6)$-tangle:



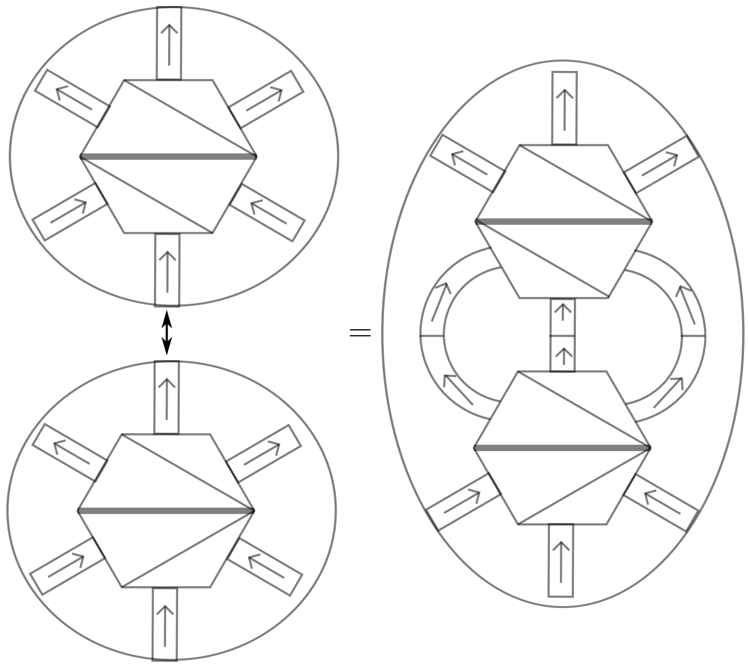


They will both be denoted by



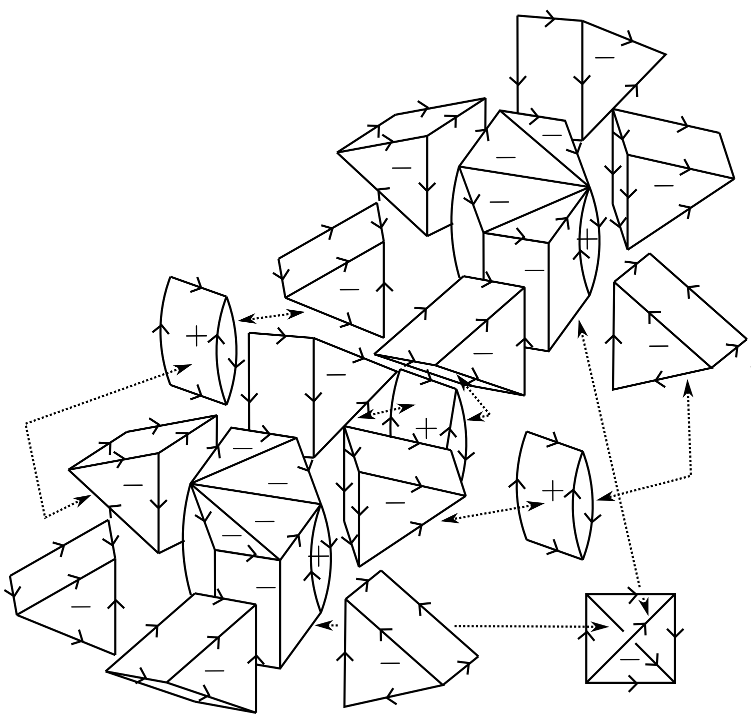
and it will be left to the user to decide which of the two triangulations to use.

Example 4.14. A generalized (n, m)-tangle can be stacked with a generalized (m, k)-tangle to form a generalized (n, k)-tangle. Here is an example of two generalized $(3,3)$-tangles stacked together:



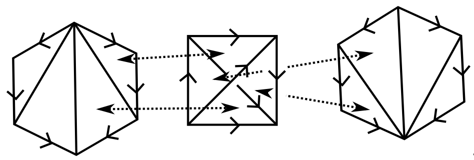
(28)

where the triangulation is

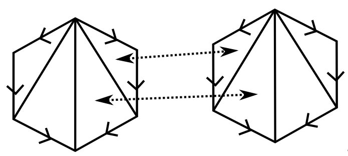


(29)

Note that three copies of $B \times S^1$ need to be glued in. Here is how the bottom hexagons are glued to the copy of Δ^3:

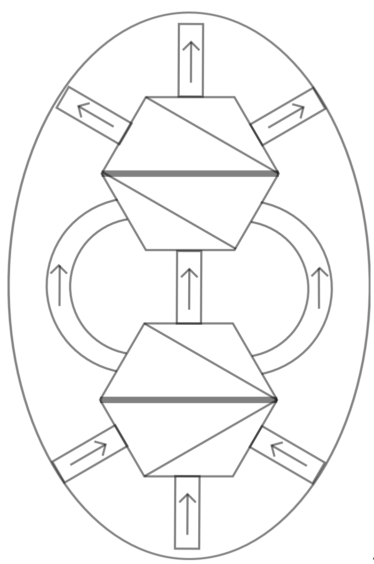


The copy of Δ^3 is needed because the output triangulation of the first generalized $(3,3)$-tangle does not match the input triangulation of the second generalized $(3,3)$-tangle. If the triangulations matched, there would be no copy of Δ^3 and the gluing on the bottom would be simpler:



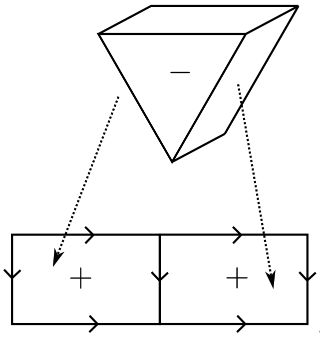
In general one must glue on suitably oriented copies of Δ^3 to change from the output triangulation of the first generalized tangle to the input triangulation from the second generalized tangle, then glue together the two matching triangulations.

After performing the gluing (29), cylinders passing from one the bottom coupon to the top are triangulated into two cylinders each. This is notated in the horizontal lines dividing the corresponding ribbons. One might want to fuse these pairs of cylinders each into a single cylinder, to be notated like:



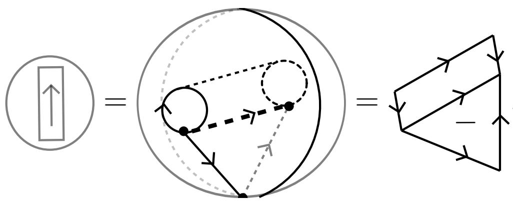
. (30)

To accomplish this, glue on copies of $-\Delta^2 \times S^1$ (in this case, three of them):



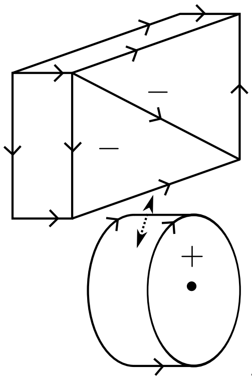
(31)

Example 4.15. The following $(1,1)$-tangle will denote the singly-gored ball:

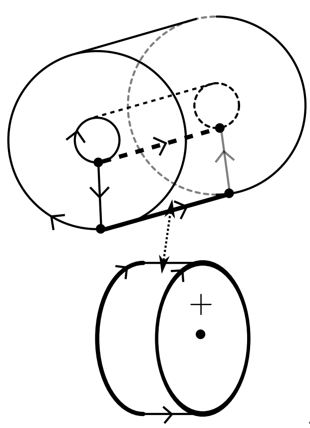


(32)

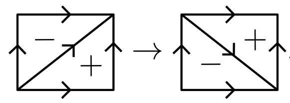
The following is a different triangulation of the single-gored ball, but with the same boundary triangulation:



Drawn another way:

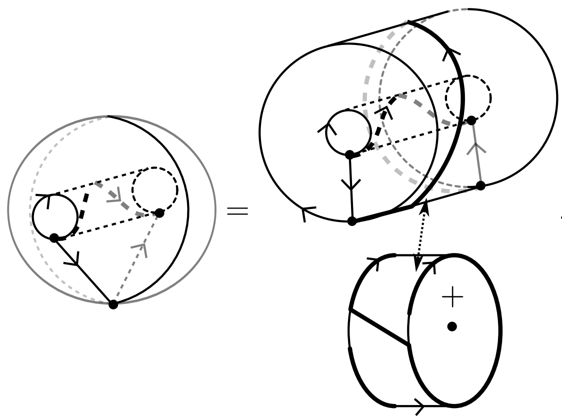


The Dehn twist on the cylinder that goes against the meridian direction changes the triangulation of the cylinder as



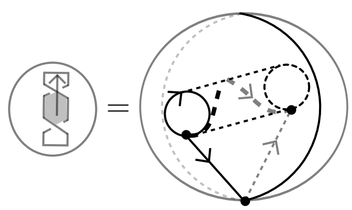
(33)

Therefore

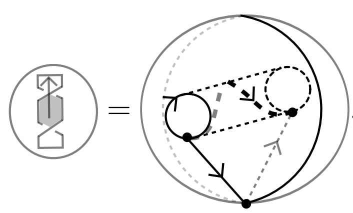


(34)

The bottom solid torus has a different boundary than the usual one: it is changed by (33). Denote this complex by

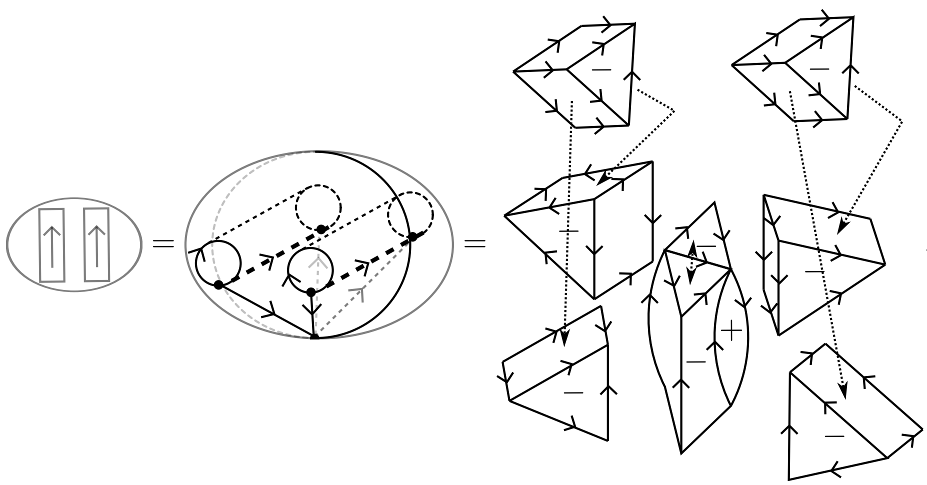


and similarly the other twist direction by



(35)

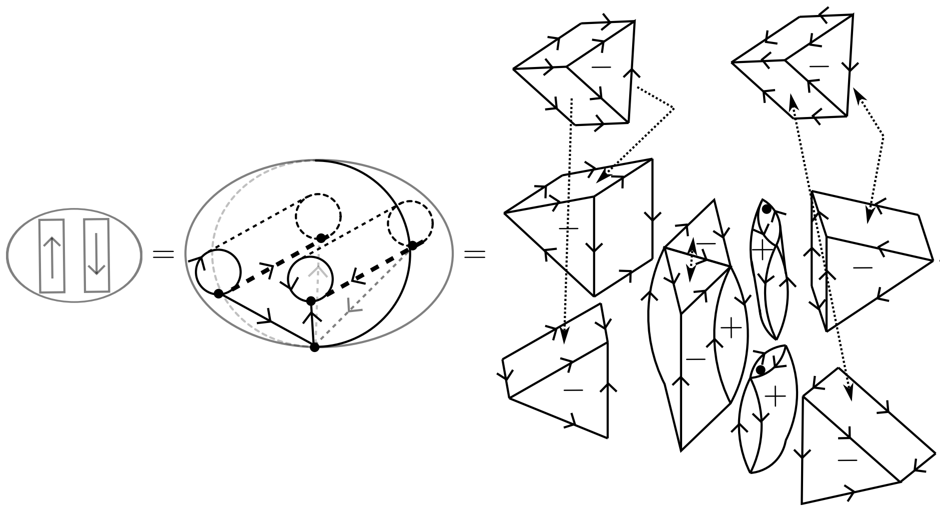
Example 4.16. The twice-gored ball is



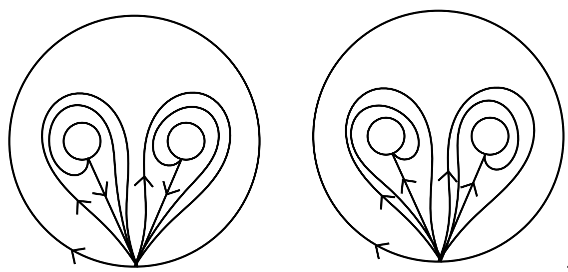
(36)

Note that the two triangles in the center are glued together.

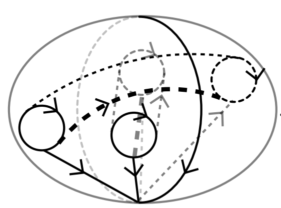
Example 4.17. To reverse one of the directions in the twice-gored ball glue on suitably oriented copies of (19):



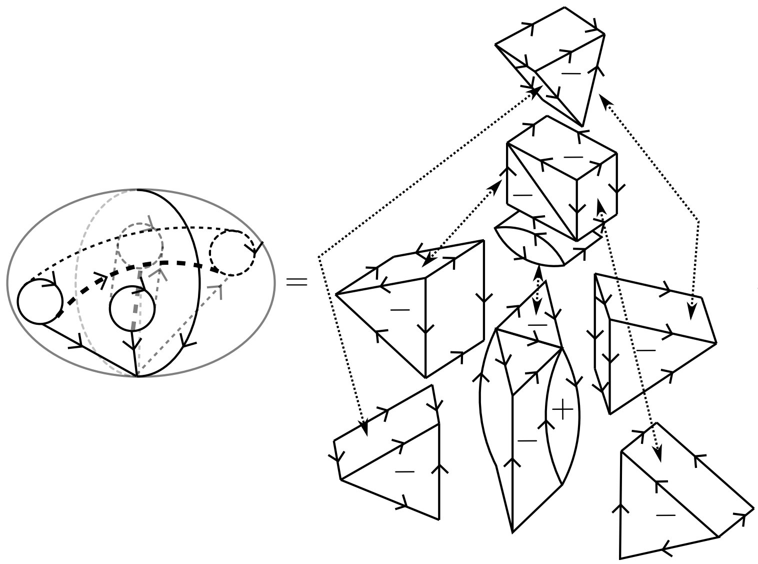
Example 4.18. The twice-gored ball (36) has a front and back that are triangulated like



A different complex with the same front and back boundary triangulations can be obtained by creating a half twist in the two interior cylinders:

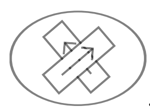


This complex should be thought of as the complement of a tubular neighborhood of a crossing from a link diagram. An explicit triangulation is:



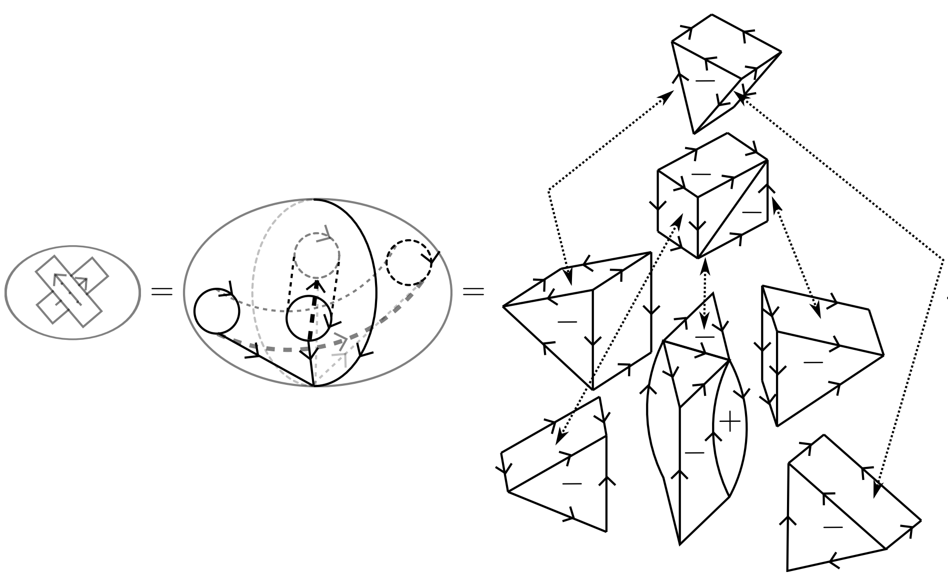
. (37)

The following will be shorthand for (37):



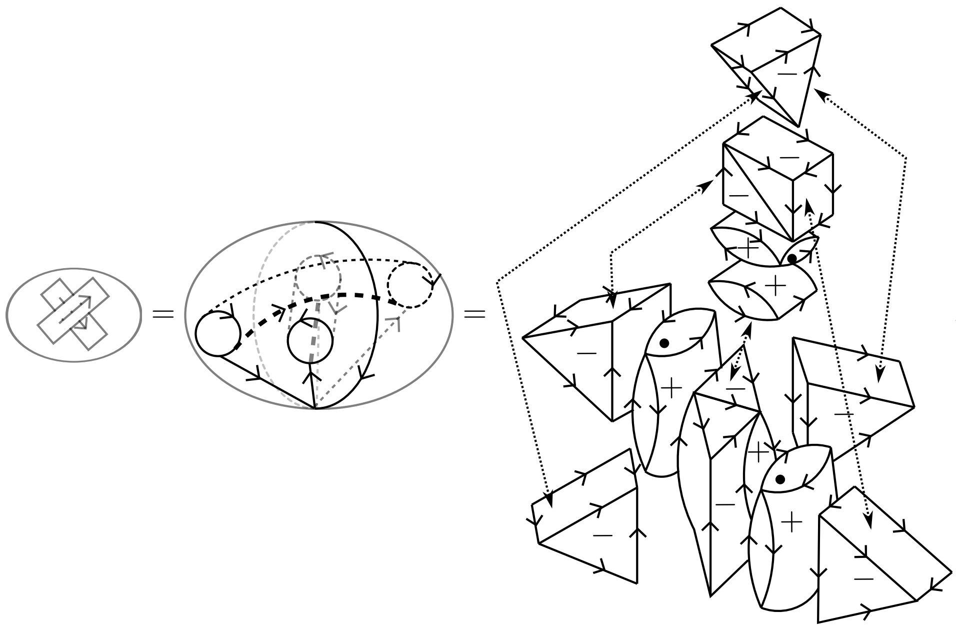
.

The opposite crossing can be triangulated as



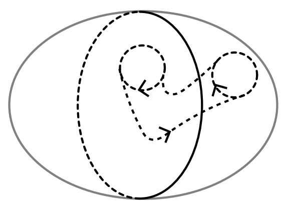
. (38)

Switching one of the directions on the longitudes results in a slightly different triangulation, including a copy of (19):

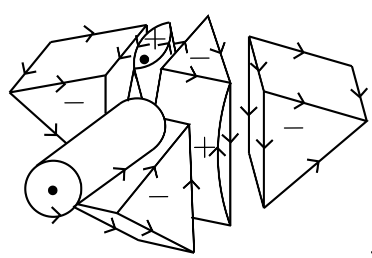


$$. \quad (39)$$

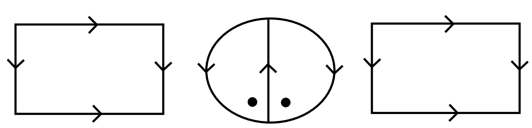
Example 4.19. Here is a complement of a "cup" in a link



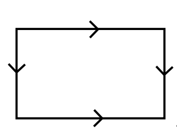
which may be triangulated by



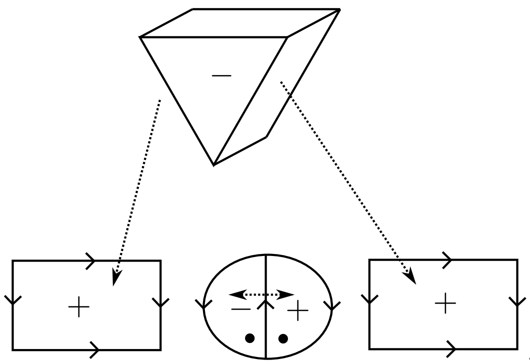
This is not exactly the triangulation you want since the cylinder part of the boundary (which sits on the top of the Δ-complex pictured) is triangulated by



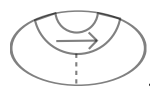
when it should be triangulated like a usual cylinder



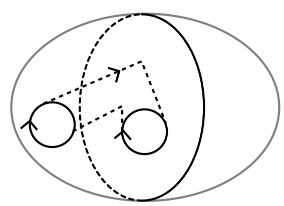
As such, fold the two bigons together and glue on a copy of $-\Delta^2 \times S^1$:



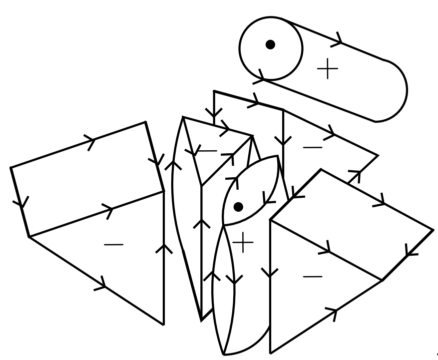
This example should be compared to the cup in (7). Denote this Δ-complex by



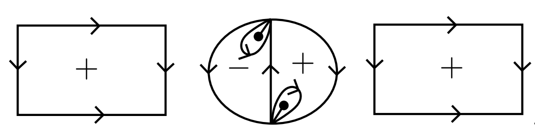
Example 4.20. Here is a complement of a "cap" in a link



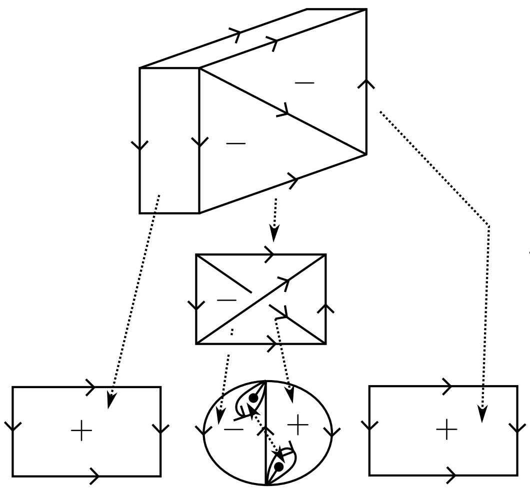
which may be triangulated by



This is not exactly the triangulation you want since the cylinder part of the boundary (which sits on the top of the Δ-complex pictured) is triangulated by

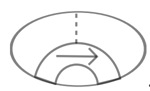


To get the correct triangulation, glue on a copy of $-\Delta^3$, two copies of $-\Delta^2 \times S^1$, and glue the two disks together:

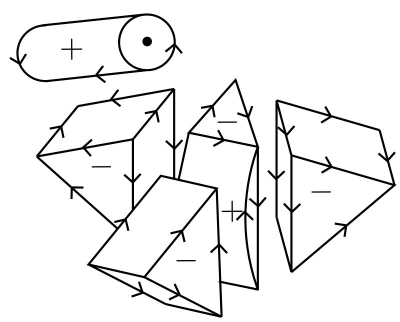


(40)

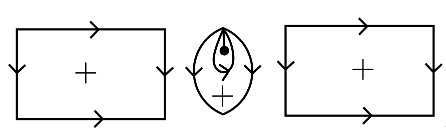
This example should be compared to the cap in (7). Denote this Δ-complex by



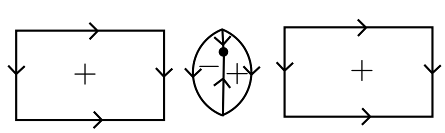
Example 4.21. Here's another triangulation of the singly-gored ball:



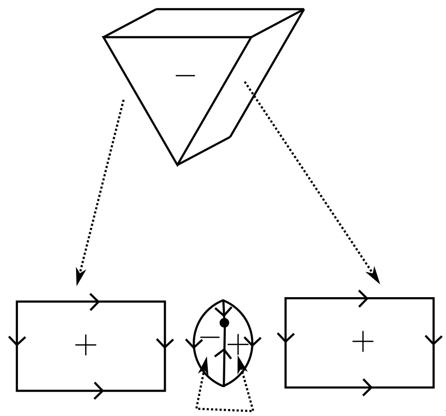
except that the "interior" cylinder of the singly-gored ball, which sits on the top here, does not have the correct triangulation. It an extra disk and triangle:



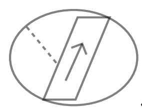
Glue on a copy of Δ^3 to obtain



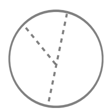
and then fold up the two faces in the middle and glue in a copy of $-\Delta^2 \times S^1$:



This example should be compared with the right side of (8). Denote this Δ-complex by

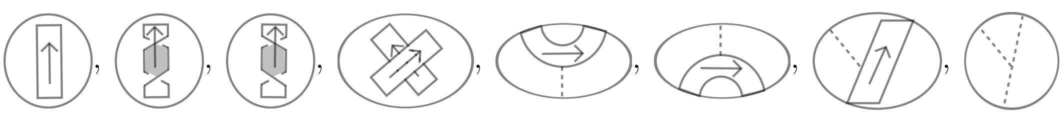

.

Let

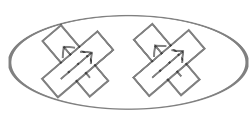


denote the result of gluing in a solid cylinder.

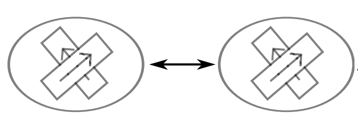
Example 4.22. The tangles



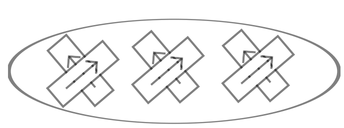
can be glued together either horizontally (as in Example 4.12) or vertically (as in Example 4.14). Write, for example,



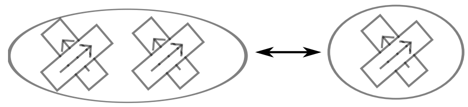
to denote two crossing complements glued together horizontally:


.

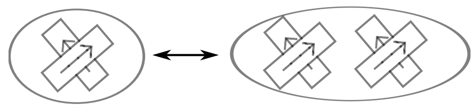
As in Example 4.13, write



to denote both

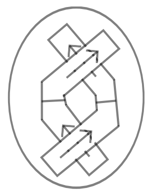


and



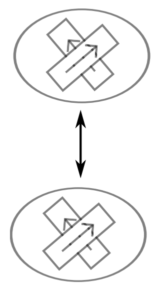
even though these give slightly different triangulations.

Similarly write

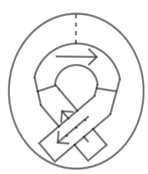


(41)

to denote the vertical stacking

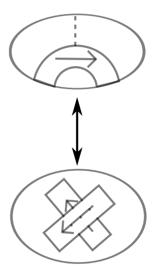


or write



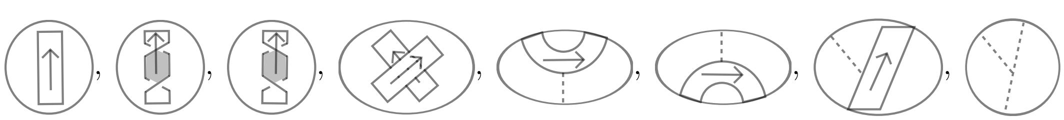
(42)

to denote the vertical stacking

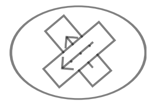


etc.

Example 4.23. It is now possible to triangulate a link complement. First put the link complement in the form (6) so that cups and caps are directed only to the right and that there are dotted lines coming out of cups and caps. Divide the link projection into pieces like

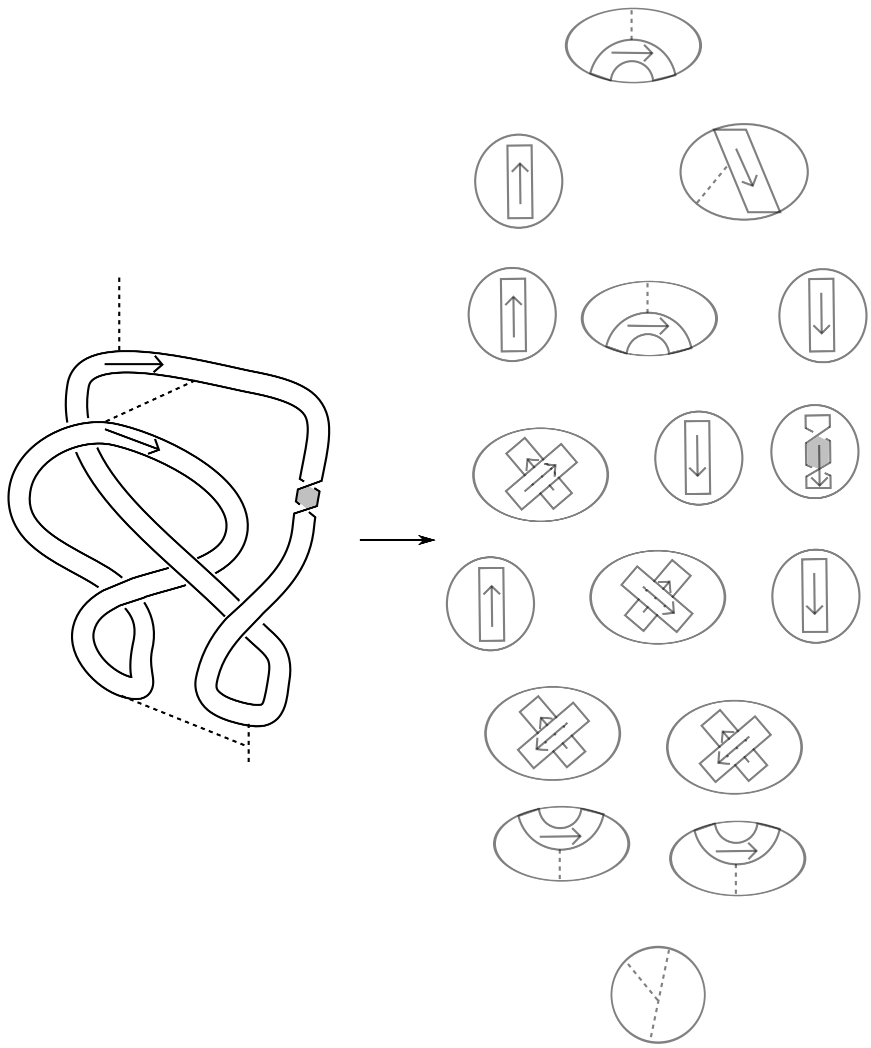


and glue them together as prescribed in Example 4.22. There are some details that were not presented here, for example when various arrows are turned the other way:

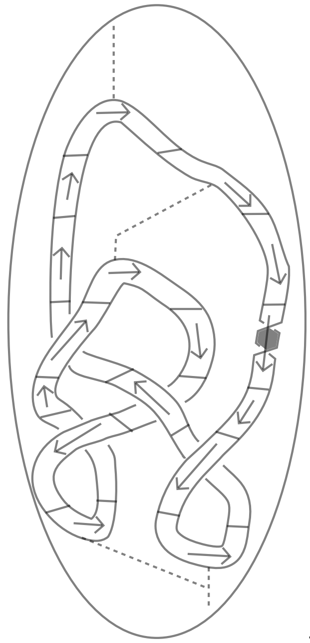


or when certain of these generalized tangles are turned upside down. These details are left to the reader.

Here is a complete example (compare with (9)):



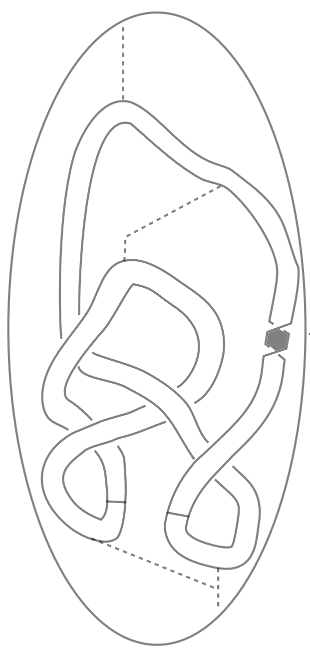
which also might be written, as in Example 4.22,



.

This is a generalized tangle in the ball. Glue one side of the ball to the other to get a triangulation of the link complement in S^3.

The boundary of each tube around each link component here is divided into many cylinders. It can be reduced to a single cylinder by gluing on many copies of $-\Delta^2 \times S^1$ (as in (31)), the result of which will be denoted by



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5. Dijkgraaf-Witten Theory

(The TQFT defined below in Definition 5.1 is not quite the same as the one defined by Dijkgraaf and Witten (and technically isn't even a TQFT in the sense of Atiyah). You can, however, construct the usual Dijkgraaf-Witten TQFT from it. See Remark 5.13 below.)

Fix a cocycle $c \in C^n(BG; \mathsf{k}^\times)$ so that $c(g_1, \ldots, g_n) = 1$ if some $g_i = e$.

Definition 5.1. Let (L, x_L) denote an oriented $(n-1)$-manifold and (K, x_K) an oriented n-manifold. The n-dimensional Dijkgraaf-Witten TQFT is a collection of vector spaces $Z_n(L, x_L)$, operations on those vector spaces, and vectors $Z_n(K, x_K) \in Z_n(\partial K, \partial x_K)$. These data are described as follows:

- For each (L, x_L), define a vector space $Z_n(L, x_L) = \mathsf{k}\,\mathrm{Hom}(L, G)$, the free k-module on the set $\mathrm{Hom}(L, G)$. (This vector space does not depend on the whole Δ-complex structure on L, but the following operations do.)

- Suppose ∂L decomposes as a union of three subcomplexes M_1, M_2, and M_3 such that the M_i do not pairwise intersect in a top-dimensional face. Suppose $f : M_1 \to M_2$ is an orientation reversing isomorphism of Δ-complexes. Write L_f to denote L glued along f. Then orientation x_L descends to an orientation on L_f, call it x_{L_f}. A G-labeling ϕ of L descends to a G-labeling of L_f if $f(\phi_{M_1}) = \phi_{M_2}$. Define a gluing map

$$\mathrm{glue} : Z_n(L, x_L) \to Z_n(L_f, x_{L_f})$$

$$\phi \mapsto \begin{cases} \phi & \text{if } f(\phi_{M_1}) = \phi_{M_2} \\ 0 & \text{otherwise} \end{cases}.$$

- The same notation as in the previous bullet. Define an ungluing map

$$\mathrm{unglue} : Z_n(L_f, x_{L_f}) \to Z_n(L, x_L)$$

$$\phi \mapsto \phi.$$

  Note that the image of the ungluing map is in those labelings ϕ where $f(\phi|_{M_1}) = \phi|_{M_2}$.

- Define a pairing map

$$\mathrm{pair} : Z_n(L, x_L) \otimes Z_n(L, -x_L) \to \mathsf{k} \tag{43}$$

  by

$$\phi \otimes \psi \mapsto \begin{cases} 1 & \text{if } \phi = \psi \\ 0 & \text{otherwise} \end{cases}.$$

- For each (K, x_K) define

$$Z_n(K, x_K) = \frac{1}{|G|^{|K^0 \setminus (\partial K)^0|}} \sum_{\phi \in \mathrm{Hom}(K,G)} \langle \phi_* x_K, c \rangle \partial \phi$$

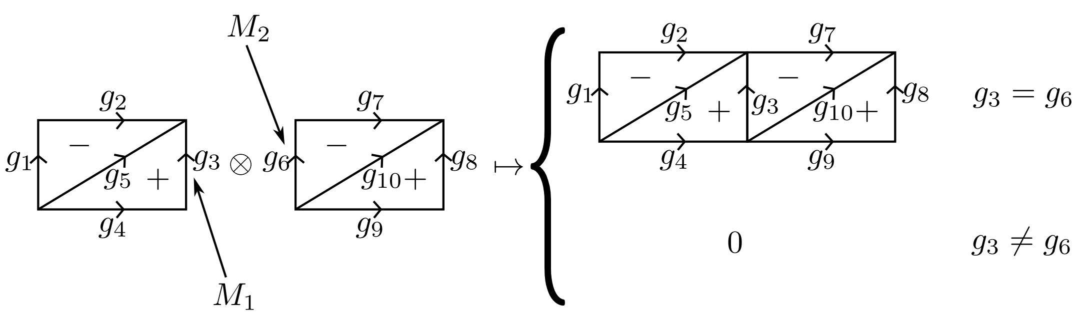
an element of $Z_n(\partial K, \partial x_K)$.

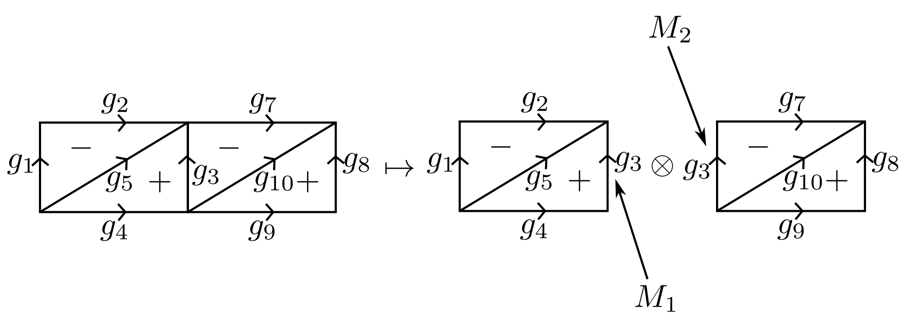
Remark 5.2. If the orientation K or L is implied from context, often the x_K or x_L will be omitted: $Z_n(K)$, $Z_n(L)$, etc.

Remark 5.3. If $f : L_1 \to L_2$ is an orientation-preserving isomorphism of Δ-complexes, recall that f induces a map $\mathrm{Hom}(L_1, G) \to \mathrm{Hom}(L_2, G)$ and so induces a map (also called f)

$$f : Z_n(L_1, x_{L_1}) \to Z_n(L_2, x_{L_2}).$$

Example 5.4. Here are examples of the gluing and ungluing maps when $n = 2$:





(44)

Remark 5.5. There's a natural isomorphism between $Z_n(L_1 \sqcup L_2, x_{L_1} + x_{L_2})$ and $Z_n(L_1, x_{L_1}) \otimes Z_n(L_2, x_{L_2})$. The tensor product in (44) is an example a convenience that will be used freely in what follows: identify $Z_n(L, x_L)$ with $\bigotimes_i Z_n(L_i, x_{L_i})$ where L_i are the connected components of L.

Claim 5.6. *The quantity $Z_n(K, x_K)$ depends only the oriented Δ-complex structure on ∂K, the set K^0, and the singular homology class of the Δ-chain x_K.*

Proof. For Φ, Φ' maps from $(|K|, K^0)$ to $(BG, *)$ say $\Phi \sim \Phi'$ if Φ is homotopic to Φ' through homotopies that are the identity on ∂K. Let $\overline{c} \in C^n_{\mathrm{sing}}(|BG|, \mathsf{k}^\times)$ be a singular cocycle whose restriction of the complex BG is cohomologous to

c in $C^n(BG; \mathsf{k}^\times)$. Using the fact that $\bar{c}$ is a cocycle, it is not hard to check that

$$Z_n(K, x_K) = \frac{1}{|G|^{|K^0 \setminus (\partial K)^0}} \sum_{\psi \in \mathrm{Hom}(\partial K, G)} \sum_{\substack{\Phi \in \mathrm{Maps}((|K|, K^0), (|BG|, *)) \\ \Phi|_{\partial K} = |\psi|}} \langle \Phi_* x_K, \bar{c} \rangle \psi$$

where Φ_* is the induced map on singular chains.

This description of $Z_n(K, x_K)$ only makes a reference to the oriented Δ-complex structure on ∂K, the set K^0, and the singular homology class of the Δ-chain x_K. $\square$

Corollary 5.7. *If K is a closed manifold, $Z_n(K, x_K)$ is an invariant of underlying oriented manifold and does not depend on the chosen triangulation.*

Claim 5.8 ("Gluing Law")**.** *Let (K, x_K) be an oriented n-manifold. Write $\partial K = L_1 \cup L_2 \cup L_3$ where the subcomplexes L_i do not pairwise share any $(n-1)$-simplices. Write x_{L_i} for the restriction of ∂x_K to L_i. Suppose $f : L_1 \to L_2$ is an orientation reversing isomorphism of Δ-complexes. Let K_f denote K glued along f. The orientation x_K descends to an orientation x_{K_f} on K_f. Write (abusively)* unglue$(Z_n(K, x_K))$ *to denote the image of $Z(K, x_K)$ under the map*

$$Z_n(\partial K, \partial x_K) \overset{\text{unglue}}{\to} Z_n(L_1 \sqcup L_2 \sqcup L_3, x_{L_1} + x_{L_2} + x_{L_3})$$

$$\to Z_n(L_1, x_{L_1}) \otimes Z_n(L_2, x_{L_2}) \otimes Z(L_3, x_{L_3})$$

where the second arrow is the natural isomorphism from Remark 5.5. Then

$$Z_n(K_f, x_{K_f}) = \frac{1}{|G|^{\#\text{newly interior vertices}}} \mathrm{pair}_{12}((f \otimes \mathrm{id} \otimes \mathrm{id})(\mathrm{unglue}(Z_n(K, x_K))).$$

Here pair$_{12}$ *means apply the pairing map (43) to the first two tensor factors. A "newly interior vertex" is a vertex that was on the boundary of K but becomes interior in K_f.*

Proof. Exercise. $\square$

Remark 5.9. The pairing map is nondegenerate and will be used to identify $Z(L, -x_L)$ with $Z(L, x_L)^*$.

G^{L^0} acts (on the right, say) on $\mathrm{Hom}(L, G)$ via the gauge action (see Definition 3.22) and so it acts on $Z(L, x_L) = \mathsf{k}\,\mathrm{Hom}(L, G)$ by permuting the canonical basis. For $q \in G^{L^0}$ and $\psi \in \mathrm{Hom}(L, G)$, let $\psi \cdot q$ denote q acting on ψ.

It will be useful to modify the gauge action to incorporate the cocycle c. Observe that a G-labeling of the Δ-complex for $I \times L$ is determined by a labeling of $\{0\} \times L$ and $I \times L^0$. The latter labeling is nothing but an element $q \in G^{L^0}$. Therefore $\mathrm{Hom}(I \times L, G)$ is in bijection with pairs (ψ, q) where

$\psi \in \mathrm{Hom}(L, G)$ and $q : L^0 \to G$. Let $\phi_{\psi,q}$ be the corresponding G-labeling of $I \times L$.

The condition that $c(g_1, \ldots, g_n) = 1$ if some $g_i = e$ ensures that $\langle (\phi_{\psi,q})_* x_I x_L, c\rangle$ vanishes if q is the identity element in the group G^{L^0}. Since c is a cocycle, considering the obvious G-labeling on $\Delta^2 \times L$ shows that

$$\langle (\phi_{\psi,q_1q_2})_* x_I x_L, c\rangle = \langle (\phi_{\psi,q_1})_* x_I x_L, c\rangle \langle (\phi_{\psi\cdot q_1,q_2})_* x_I x_L, c\rangle.$$

It follows that one may make the following definition

Definition 5.10. Define a linear action of G^{L^0} on $Z_n(L, x_L)$ by

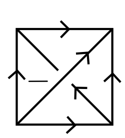
$$\psi \bullet q := \langle (\phi_{\psi,q})_* x_I x_L, c\rangle \psi \cdot q.$$

Example 5.11. $Z_n(I \times L, x_I x_L)$ is an element of $Z_n(L, x_L)^* \otimes Z_n(L, x_L)$ (see Remarks 5.5 and 5.9) and thus can be thought of an endomorphism of $Z_n(L, x_L)$. It is not hard to see that $Z_n(I \times L, x_I x_L)$ is $|G|^{|L^0|}$ times the projection to the trivial part of the $\bullet$ action. It is also not hard to use the gluing law (Claim 5.8) in conjunction with gluing a collar neighborhood of ∂K onto K to see that $Z_n(K, x_K)$ is always in the G^{K^0}-trivial part of $Z_n(\partial K, \partial x_K)$.

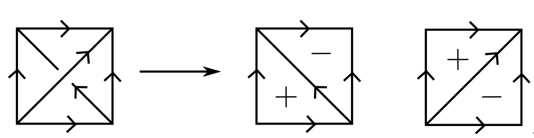
Definition 5.12. Let $Z_n(L, x_L)^{G^{L^0}}$ denote the trivial part of the $\bullet$ action on $Z_n(L, x_L)$.

Remark 5.13. To recover Dijkgraaf and Witten's original TQFT, define, for an oriented $(n-1)$-manifold Σ, $\widetilde{Z}_n(\Sigma) := Z_n(L, x_L)^{G^{L^0}}$ for some oriented Δ-complex structure L on Σ. One can show that $\widetilde{Z}_n(\Sigma)$ is, up to canonical isomorphism, independent of the Δ-complex structure L. Use $\frac{1}{|G|^{|L^0|}}$ times the pairing (43) to identify $\widetilde{Z}_n(-\Sigma)$ with $\widetilde{Z}_n(\Sigma)^*$. $Z_n(K, x_k)$ is always in the $G^{(\partial K)^0}$-trivial part and so for an oriented n-manifold $\mathcal{M}$ one can define $\widetilde{Z}_n(\mathcal{M}) := Z_n(K, x_K)$ for some Δ-complex structure K on $\mathcal{M}$. Then $\widetilde{Z}_n$ becomes an actual functor from the category of n-cobordisms to vector spaces. For example, because of the restriction to G^{L^0}-trivial part and the extra factor of $\frac{1}{|G|^{|L^0|}}$ in the identification of $Z_n(L, -x_L)$ with $Z_n(L, x_L)^*$, the linear map $\widetilde{Z}_n(I \times \Sigma)$ is the identity map on $\widetilde{Z}_n(\Sigma)$.

Example 5.14. Given a copy of $-\Delta^3$, drawn like

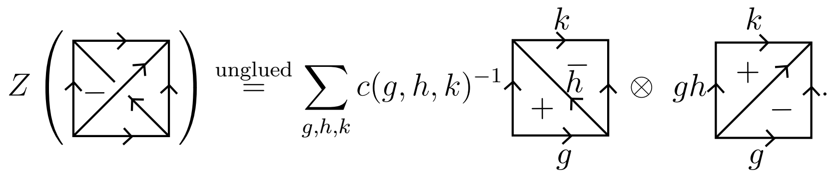


one can unglue the boundary in the following way:



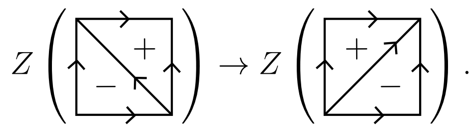
.

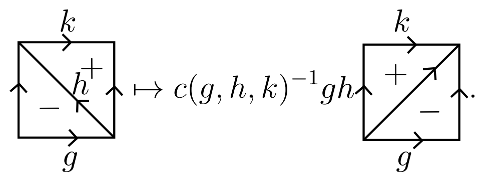
And[1]



(45)

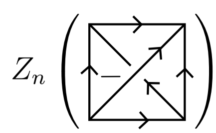
and so this is a map





(46)

Of course there are many other ways that



can be viewed as a map of vector spaces: simply decompose the $\partial\Delta^3$ in another way.

Example 5.15. If $c = 1$, then

$$Z_n(K, x_K) = \frac{|\operatorname{Hom}(K, G)|}{|G|^{|K^0|}}$$

(see Corollary 3.26).

Example 5.16. Let K be an oriented manifold with $\partial K = L_1 \sqcup L_2$. Suppose there exists a Δ-complex (L, x_L) and Δ-complex isomorphisms $f_1 : (L_1, x_{L_1}) \to (L, -x_L)$ and $f_2 : (L_2, x_{L_2}) \to (L, x_L)$. Then $Z(K, x_K)$ can be thought to lie in $Z_n(L, x_L)^* \otimes Z_n(L, x_L) \cong \operatorname{End}(Z_n(L, x_L))$. Let K' denote the result of gluing L_1 to L_2, with induced orientation $x_{K'}$. Then the gluing law implies that

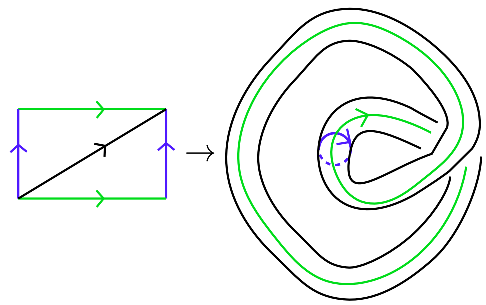
$$Z_n(K', x_{K'}) = \frac{1}{|G|^{|L^0|}} \operatorname{tr}_{Z_n(L,x_L)}(Z_n(K, x_K)).$$

In particular

$$Z_n(S^1 \times L, x_{S^1} x_L) = \dim Z_n(L, x_L)^{G^{L^0}}.$$

[1]The notation $\overset{\text{unglue}}{=}$ in equation (45) means the appropriate ungluing map has been applied to the left hand side.

Example 5.17. Let $\mathcal{L} = \mathcal{K}_1 \sqcup \cdots \sqcup \mathcal{K}_N \subset S^3$ be a link made of knots $\mathcal{K}_i$. Let $\nu\mathcal{L}$ denote a tubular neighborhood of $\mathcal{L}$ in S^3. For each $\mathcal{K}_i$ choose a directed meridian and a directed longitude on $\partial(\nu\mathcal{K}_i)$. In particular this turns $\mathcal{L}$ into a directed framed link. The meridians and longitudes define a Δ-complex structure on $\partial(S^3 \setminus \nu\mathcal{L})$:



Extend this Δ-complex structure to the rest of $S^3 \setminus \nu\mathcal{L}$. Pick your favorite orientation on S^3. After possibly gluing on some copies of $S^1 \times B$ to the boundary (which will reverse some meridians), the induced orientations on the boundary agree with $+S^1 \times S^1$ (the first cartesian factor of S^1 is the longitude). Write (K, x_K) for the oriented Δ-complex structure on $S^3 \setminus \nu\mathcal{L}$ thus constructed. Then

$$Z_3(K, x_K) \in Z_3(S^1 \times S^1)^{\otimes N}$$

is an invariant of the directed framed link $\mathcal{L}$. (In fact, $Z_3(K, x_K)$ sits inside $(Z_3(S^1 \times S^1)^G)^{\otimes N}$ (see Example (5.11)).

When $c = 1$ and $\mathcal{L} = \mathcal{K}$ is a knot, then $Z_3(K, x_K)$ admits a simple description: it is the sum of the boundary restrictions of all elements $\mathrm{Hom}(\pi_1(S^3 \setminus \nu\mathcal{K}), G)$.

Example 5.18. Any closed oriented 3-manifold $\mathcal{M}$ can be written as surgery on a framed link $\mathcal{L}$. An oriented Δ-complex structure on $\mathcal{M}$ can be obtained taking K in the last example and gluing on a copy of $-D^2 \times S^1$ on each torus boundary component.

By the gluing law,

$$Z_3(\mathcal{M}) = \frac{1}{|G|^N} \langle Z_3(K, x_K), \underbrace{Z_3(-D^2 \times S^1) \otimes \cdots \otimes Z_3(-D^2 \times S^1)}_{N \text{ copies}} \rangle$$

where $\langle , \rangle$ denotes the pairing map (43). Thus Dijkgraaf and Witten's 3-manifold invariant can be expressed in terms of link invariants of a surgery presentation.

6. An Algebra from the TQFT

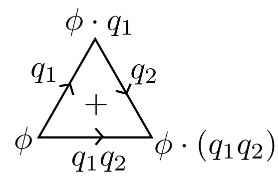
In this section fix a positive integer n, a cocycle $c \in C^n(BG; \mathsf{k}^\times)$ such that $c(g_1, \ldots, g_n) = 1$ if any $g_i = e$, and fix (M, x_M) an oriented closed $(n-2)$-manifold. Often the orientation on M will be supressed and $-M$ will be written for $(M, -x_M)$. The orientation on (e.g.) $I \times M$ will be $x_I x_M$, as usual.

A G-labeling on $I \times M$ is determined by a G-labeling ϕ of $\{0\} \times M$ and a G-labeling q of $I \times M^0$. Such labelings form bases of $Z_n(I \times M)$ and $Z_n(-I \times M) \cong Z(I \times M)^*$ drawn (respectively):

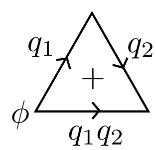
$$\phi \xrightarrow[+]{q}, \quad \phi \xrightarrow[-]{q}.$$

Tracing through the identification of $Z(-I \times M)$ with $Z(I \times M)^*$, one sees that these labelings give dual bases. Note that q can and will be thought of as an element of the group G^{M^0}. The letter e will (abusively) also be used to denote the identity in G^{M^0} as well as in G.

A typical G-labeling of $\Delta^2 \times M$ can be denoted

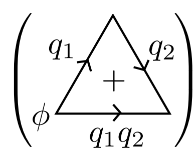


and this will be simplified to



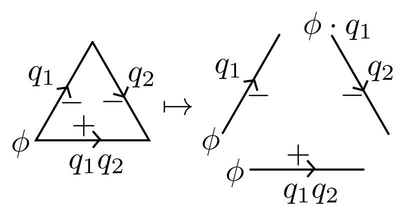
Define

$$c_\phi(q_1, q_2) := c$$



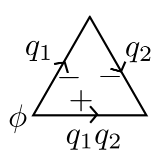
Since $\partial\Delta^2$ is three intervals glued together, there is an ungluing map:

$$Z_n(-(\partial\Delta^2) \times M) \to Z_n(-I \times M) \otimes Z_n(-I \times M) \otimes Z_n(I \times M)$$



(47)

the right side is drawn in $Z_n((-I \times M) \sqcup (-I \times M) \sqcup (I \times M))$ but this canonically isomorphic to $Z_n(-I \times M) \otimes Z_n(-I \times M) \otimes Z(I \times M)$. It equally well could be drawn



$$\mapsto \phi \xrightarrow[-]{q_1} \otimes \ \phi \cdot q_1 \xrightarrow[-]{q_2} \otimes \ \phi \xrightarrow[+]{q_1 q_2} \tag{48}$$

The notation (48) does not keep track of the data of which tensor factor corresponds to which part of the original boundary, whereas the notation (47) does.

After ungluing,

$$Z_n(-\Delta^2 \times M) \overset{\text{unglued}}{=} \sum_{\substack{\phi \in \mathrm{Hom}(M,G) \\ q \in G^{M^0}}} c_\phi(q_1, q_2)^{-1} \phi \xrightarrow[-]{q_1} \otimes \phi \cdot q_1 \xrightarrow[-]{q_2} \otimes \phi \xrightarrow[+]{q_1 q_2}. \tag{49}$$

Equation (49) describes an algebra structure on M with unit

$$\sum_\phi \phi \xrightarrow[+]{e}.$$

Note that the unit is $Z_n(D^2 \times M)$ unglued into $Z_n(I \times M)$. Write A to denote $Z_n(I \times M)$ with this unital algebra structure. Denote the multiplication in A by $\star$ so that

$$\phi_1 \xrightarrow[+]{q_1} \star \phi_2 \xrightarrow[+]{q_2} = \begin{cases} \phi_1 \xrightarrow[+]{q_1 q_2} & \text{if } \phi_1 \cdot q_1 = \phi_2 \\ 0 & \text{otherwise} \end{cases}.$$

Remark 6.1. $Z_n(\Delta^2 \times M)$ similarly defines a map $A \to A \otimes A$ and $Z(-D^2 \times M)$ defines a map $A \to \mathsf{k}$. One can check that this extra structure turns A into a Frobenius algbera. This point will not be pursued here.

Definition 6.2. Given $\phi \in \mathrm{Hom}(M, G)$, let $A_\phi \subset A$ denote the subalgebra spanned by those elements

$$\{\phi \xrightarrow[+]{q}\}_{q \in \mathrm{Stab}(\phi)}$$

i.e, those elements for which $\{0\} \times M$ and $\{1\} \times M$ are both labeled by ϕ.

Claim 6.3. *A_ϕ is isomorphic to a central extension of the group algebra $\mathsf{k}\,\mathrm{Stab}(\phi)$ by the cocycle $c_\phi^{-1} \in C^2(B\,\mathrm{Stab}(\phi); \mathsf{k}^\times)$.*

Proof. Easy. $\square$

Corollary 6.4. *A_ϕ is semisimple.*

Write $\{\pi_j^{A_\phi}\}_j$ to denote the simple projectors in A_ϕ. Let $\{\alpha\}$ denote the collection of G^{M^0}-orbits of $\mathrm{Hom}(M, G)$. Fix representatives $\phi_\alpha \in \alpha$ for each. Let $\{W_{\phi_\alpha,j}\}_{j \in J_\alpha}$ denote a complete collection of simple modules for A_{ϕ_α}. Define

$$V_{\phi_\alpha,j} := W_{\phi_\alpha,j} \underset{A_{\phi_\alpha}}{\otimes} A.$$

The $V_{\phi_\alpha,j}$ form a complete set of simple right A-modules. It is possible to form a projector onto $V_{\phi_\alpha,j}$ by smearing the projectors for A_{ϕ_α} around A:

$$\pi_{\phi_\alpha,j} = \frac{1}{|\mathrm{Stab}(\phi_\alpha)|} \sum_{q \in G^{M^0}} \phi_\alpha \cdot q^{-1} \xrightarrow[+]{q} \star \pi_j^{A_{\phi_\alpha}} \star \phi_\alpha \xrightarrow[+]{q}.$$

Corollary 6.5. *A is semisimple.*

Corollary 6.6. $A \cong \bigoplus_\alpha \bigoplus_{j \in J_\alpha} \mathrm{End}(V_{\phi_\alpha,j})$.

From now on, index a complete set of simples of A by i, so that

$$A \cong \bigoplus_i \mathrm{End}(V_i) \cong V_i^* \otimes V_i. \tag{50}$$

Remark 6.7. If V is an A-module, then V admits a grading by $\mathrm{Hom}(M, G)$:

$$V \cong \bigoplus_{\phi \in \mathrm{Hom}(M,G)} V_\phi$$

where $V_\phi := VA_\phi$. Note that the grading provides a map

$$V \to V \otimes \mathsf{k}\,\mathrm{Hom}(M, G)$$

definined on pure elements in V_ϕ by $v \mapsto v \otimes \phi$. Since A can be viewed as a bimodule over itself, it is graded by $\mathrm{Hom}(M, G) \times \mathrm{Hom}(M, G)$. The grading records the labelings on $\{0\} \times M$ and $\{1\} \times M$. There's an associated grading map

$$\mathrm{gr} : A \mapsto \mathsf{k}\,\mathrm{Hom}(M, G) \otimes A \otimes \mathsf{k}\,\mathrm{Hom}(M, G)$$
$$\phi \underset{+}{\overset{q}{\longrightarrow}} \mapsto \phi \ \otimes \ \phi \underset{+}{\overset{q}{\longrightarrow}} \ \otimes \ \phi \cdot q.$$

The isomorphism

$$A \cong \bigoplus_i V_i^* \otimes V_i$$

is an isomorphism of bigraded A-modules.

Remark 6.8. Some thought with the definition for $\dim V_{\phi_\alpha,j}$ shows that

$$\dim V_{\phi_\alpha,j} = |\phi_\alpha \cdot G^{M^0}| \dim W_{\phi_\alpha,j}.$$

Example 6.9. A natural example of A-modules comes from cylinders of $(n-1)$-manifolds. Let (L, x_L) be an oriented $(n-1)$-manifold, possibly with boundary. Write $x_{\partial L} = \partial x_L$. $\partial(I \times L)$ is divided into three parts: $\{0\} \times L$, $I \times \partial L$, and $\{1\} \times L$. In $\partial(x_I x_L)$, the first part gets the orientation $-x_L$, the second part gets the orientation $-x_I x_{\partial L}$, and the third part gets the orientation x_L. Thus after ungluing these three parts apart, $Z(I \times L, x_I x_L)$ is an element in

$$Z(L, -x_L) \otimes Z(I \times \partial L, -x_I x_{\partial L}) \otimes Z(L, x_L)$$
$$\cong Z(L, x_L)^* \otimes Z(I \times \partial L, x_I x_{\partial L})^* \otimes Z(L, x_L)$$

and this almost provides an action of $Z(I \times \partial L, x_I x_{\partial L})$ on $Z(L, x_L)$. One reason that it is not an action is that the identity element in $Z(I \times \partial L, x_I \partial x_L)$ acts as $|G|^{|L^0 \setminus (\partial L)^0|}$ times the projection to $Z_n(I \times \partial L, x_I x_L)^{G^{L^0 \setminus (\partial L)^0}}$. However, when multiplied by $|G|^{-|L^0 \setminus (\partial L)^0|}$ and restricted to $Z_n(I \times \partial L, x_I x_L)^{G^{L^0 \setminus (\partial L)^0}}$, the element $Z(I \times L, x_I x_L)$ does give an action.

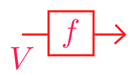
Since gluing top to bottom is a trace times $|G|^{-|L^0 \setminus (\partial L)^0|}$, (see Example 5.16), it follows that

$$Z(S^1 \times L, x_I x_L) \in Z(S^1 \times \partial L, -x_{S^1} x_{\partial L}) \overset{\text{unglue}}{\to} Z(I \times \partial L, -x_I x_{\partial L})$$

is the character of the module $Z(I \times L, x_I x_L)^{L^0 \setminus (\partial L)^0}$.

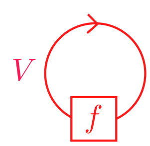
It will be helpful to develop some diagrammatic notation for k-linear maps. This notation will look similar to the blue arrow notation of Section 2 but is different in several important respects, including the facts that the arrows are not forced to lie in a plane and do not have to be read in just a single direction.

Write



to denote a linear map $f : V \to V$; alternatively, an element of $V^* \otimes V$. Reading an arrow in reverse is the adjoint map, gluing arrows end to end is composition (equivalently, contraction of $V \otimes V^* \to \mathsf{k}$), and therefore

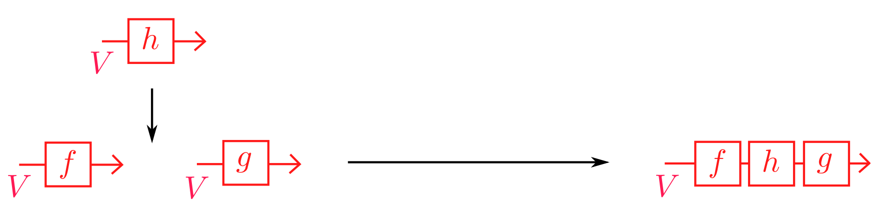
is the identity map $V \to V$ and



is the trace of the map $f : V \to V$. Disjoint union stands for tensor product, so

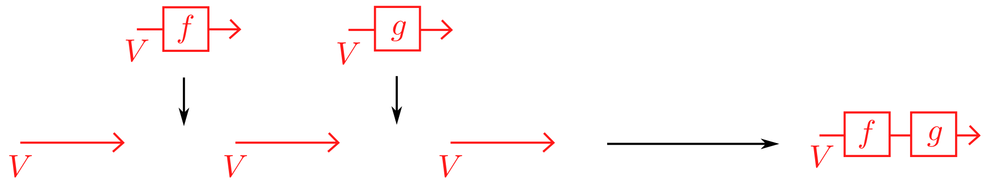
(51)

stands for $f \otimes g : V \otimes V \to V \otimes V$. One can also, however, think of (51) as a map $\mathrm{End}(V) \to \mathrm{End}(V)$ as follows:



namely, precomposition by f and postcomposition by g. Similarly,

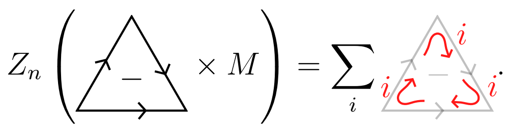
gives composition of endomorphisms



i.e., multiplication in the algebra $\mathrm{End}(V)$. Let $\{V_i\}$ be a collection of vector spaces. Then the multiplication in the algebra $\bigoplus_i \mathrm{End}(V_i)$ is

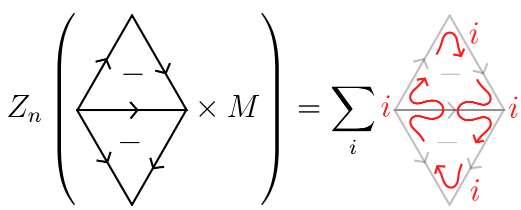
If $\{V_i\}$ is a complete collection of simple modules for A, this describes the multiplication for A under the isomorphism $A \cong \bigoplus_i \mathrm{End}(V_i)$.

The red arrows permit the following very elegant notation:

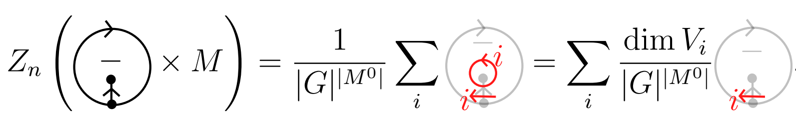


The simplex is drawn on the right side only to indicate context (i.e., where are the two inputs and where is the output). It should be regarded like comment text in a program.

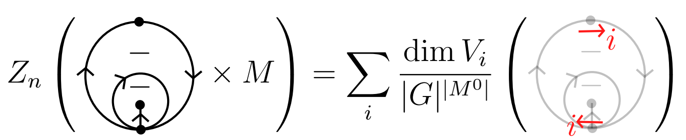
According to the gluing theorem, gluing together simplices along oppositely oriented faces corresponds to contracting A with A^*. Under the isomorphism $A \cong \bigoplus_i \mathrm{End}(V_i)$, this translates to gluing the ends of arrows together if the labeling i matches on the two arrows. For example



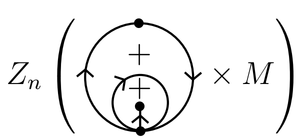
and



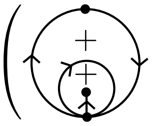
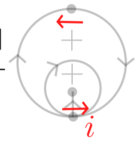
so therefore

 (52)

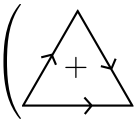
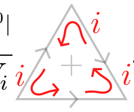
Think of (52) as a map $A \to A^*$. Then its inverse is given by



so that

$$Z_n\Big($$  $$\times M\Big) = \sum_i \frac{|G|^{|M^0|}}{\dim V_i}$$  $.$ (53)

Gluing two of these onto $-\Delta^2$ and also gluing on a copy of (52), then

$$Z_n\Big($$  $$\times M\Big) = \sum_i \frac{|G|^{|M^0|}}{\dim V_i}$$  $.$

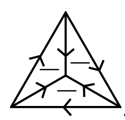
This notation allows a simple proof of the following theorem:

Theorem 6.10. *Let Σ_g be the closed surface of genus g. Then*

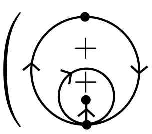
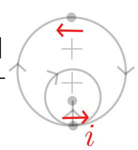
$$Z_n(\Sigma_g \times M) = \sum_i \left(\frac{|G|^{M^0}}{\dim V_i}\right)^{2g-2}$$

where the sum on the right hand side ranges over the simple $Z(I \times M)$-modules (in particular the result is independent of orientation on Σ_g).

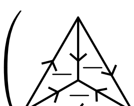
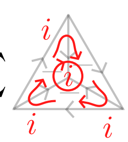
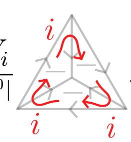
Proof. Let τ be some triangulation of Σ_g. Let $V(\tau)$, $E(\tau)$, $F(\tau)$ denote the vertices, edges, and faces of τ. Splice in a bigon B at each edge and replace each face by

 .

This gives Σ_g the structure oriented Δ-complex. Note that

$$Z_n\Big($$  $$\times M\Big) = \sum_i \frac{|G|^{|M^0|}}{\dim V_i}$$  $.$ (54)

and

$$Z_n\Big($$  $$\times M\Big) = \frac{1}{|G|^{|M^0|}} \sum_i$$  $$= \sum_i \frac{\dim V_i}{|G|^{|M^0|}}$$  $.$ (55)

After placing each of these red arrows diagrams in the surface and contracting, one is left with a small circle around each vertex of τ. Each circle contributes a factor of $\dim V_i$. There's an overall factor of $|G|^{|V(\tau)|}$ in the definition of Z_n, so each vertex of τ contributes in total a factor $\frac{\dim V_i}{|G|^{|M^0|}}$. Equation (54) shows that each edge of τ contributes a factor of $\frac{|G|^{|M^0|}}{\dim V_i}$. Equation (55) shows that each face of τ contributes a factor of $\frac{\dim V_i}{|G|^{|M^0|}}$. Thus

$$Z_n(\Sigma_g \times M) = \sum_i \left(\frac{\dim V_i}{|G|^{|M^0|}}\right)^{|V(\tau)|} \left(\frac{|G|^{|M^0|}}{\dim V_i}\right)^{|E(\tau)|} \left(\frac{\dim V_i}{|G|^{|M^0|}}\right)^{|F(\tau)|}$$

$$= \sum_i \left(\frac{|G|^{|M^0|}}{\dim V_i} \right)^{-\chi(\Sigma_g)}.$$

$\square$

Remark 6.11. It may seem like the quantity $\frac{|G|^{|M^0|}}{\dim V_i}$ depends on the vertices of M, but:

$$\frac{|G|^{|M^0|}}{\dim V_{\phi_\alpha,j}} = \frac{|G|^{|M^0|}}{|\phi_\alpha \cdot G^{M^0}| \dim W_{\phi_\alpha,j}} = \frac{|\operatorname{Stab}(\phi_\alpha)|}{W_{\phi_\alpha,j}}$$

and the isomorphism type of $\operatorname{Stab}(\phi_\alpha)$ is independent of the Δ-complex structure on M, essentially because of Claim 3.25.

7. A Quasi-Hopf Algebra for $n = 3$

In this section set $n = 3$ and fix a cocycle $c \in C^3(BG; \mathsf{k}^\times)$ such that $c(g_1, \ldots, g_n) = 1$ if $g_i = e$ for some i. Write $Z = Z_3$.

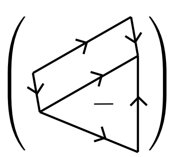
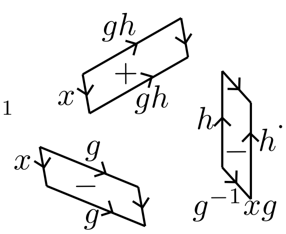
Definition 7.1. Write D to denote the algebra $Z(I \times S^1)$ with multiplication as defined in the previous section. Note that D depends on G and c.

The choice of letter D refers to the fact that D is called the "twisted Drinfeld double" of G [Wil08]. The goal of this section is to use the 3d TQFT to imbue D with the structure of a quasi-Hopf algebra.

Since $M = S^1$, $\operatorname{Hom}(M, G)$ is in bijection with G and

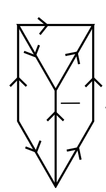
$$c_x(g, h) = \frac{c(x, g, h)c(g, h, (gh)^{-1}x(gh))}{c(g, g^{-1}xg, h)}.$$

Drawn in the notation of Section 4, equation (49) becomes

Z  $\overset{\text{unglued}}{=} \sum_{x,g,h} c_x(g,h)^{-1}$ 

As used in Section 4, each rectangle represents a cylinder so the G-labelings on the two long edges are the same. For simplicity, only one labeling will be notated from now on. Also recall that $Z(D^2 \times S^1)$ provides the unit for D.

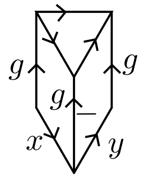
In Section 4, $-I \times \Delta^2$ was drawn like

 (56)

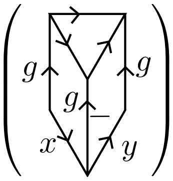
In almost all instances in Section 4 where $-I \times \Delta^2$ appeared, its three vertical edges were identified. This is the convention in this section:

Convention 7.2. The picture (56) refers to $-I \times \Delta^2$ with its three vertical edges identified.

In particular, its G-labelings are



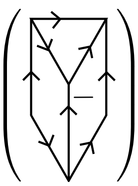
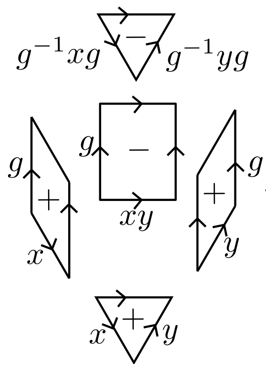
and note that

$$c\left(\right.$$  $$\left. \right) = c^g(x,y)^{-1}$$

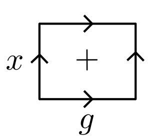
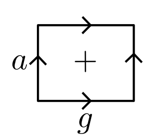
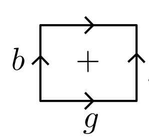
where

$$c^g(x,y) := \frac{c(g, g^{-1}xg, g^{-1}yg)c(x,y,g)}{c(x,g,g^{-1}yg)}.$$

Hence

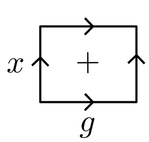
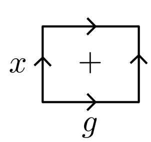
$$Z\left(\right.$$  $$\left. \right) \overset{\text{unglued}}{=} \sum_{g,x,y} c^g(x,y)^{-1}$$  $$. \tag{57}$$

The right side of (57) is almost a map $D \to D \otimes D$: if you were to forget the top and bottom triangles, it'd be

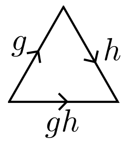
 $$\mapsto \sum_{ab=x} c^g(a,b)^{-1}$$  $$\otimes$$  $$. \tag{58}$$

In fact, define $\Delta : D \to D \otimes D$ to be the map (58). The top and bottom triangles can be "remembered" using the grading map:

$$\mathrm{gr} : D \mapsto \mathsf{k}G \otimes D \otimes \mathsf{k}G$$

 $$\mapsto x \otimes$$  $$\otimes g^{-1}xg.$$

More precisely, identify $Z(\pm\Delta^2)$ with $\mathsf{k}G \otimes \mathsf{k}G$ via

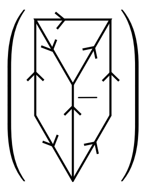
 $\mapsto g \otimes h$

and then (57) is given by the composition

$$D \xrightarrow{\Delta} D \otimes D \xrightarrow{\mathrm{gr} \otimes \mathrm{gr}} \mathsf{k}G \otimes D \otimes \mathsf{k}G \otimes \mathsf{k}G \otimes D \otimes \mathsf{k}G \tag{59}$$

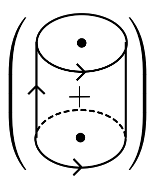
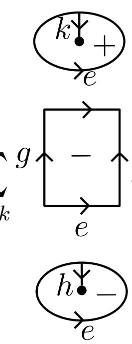
$$\to \mathsf{k}G \otimes \mathsf{k}G \otimes D \otimes D \otimes \mathsf{k}G \otimes \mathsf{k}G$$

where the last map is a reordering of the tensor factors. In what follows, the top and bottom labelings will not be detailed in the notation. For example, write

Z  $\overset{\text{unluged}}{=} \Delta \otimes$(top and bottom triangles). (60)

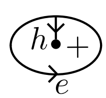
The actual top and bottom labelings can be recovered using (59).

As described in Example 6.9, $Z(I \times D^2)$ is a module for $A = Z(I \times S^1)$. Namely,

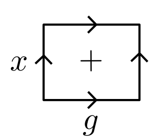
Z  $\overset{\text{unglued}}{=} \sum_{g,h,k}$  . (61)

(This calculation uses the fact that $c(g_1, g_2, g_3) = 1$ if some $g_i = e$.) Identify $Z(D^2)$ with $\mathsf{k}G$ by

$$Z(D^2) \to \mathsf{k}G$$

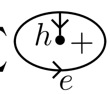
 $\mapsto h.$

Then the action of D on $Z(D^2)$ can be read off (61): the element



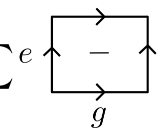
acts by $|G|$ times a projection onto the trivial part of $\mathsf{k}G \cong Z(D^2)$ if $x = e$, and 0 otherwise.

Let $V_0 = Z(D^2)^G$. Example (6.9) indicates that D acts on V_0. V_0 is the 1-dimensional space:

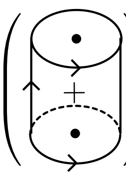
$V_0 = \mathrm{span}\left\{ \sum_h \right.$  $\left. \right\}$.

$V_0^* \otimes V_0$ is canonically identified with k by sending the identity transformation in $V_0^* \otimes V_0 \cong \mathrm{End}(V_0)$ to $1 \in \mathsf{k}$. Define

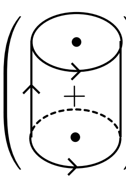
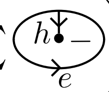
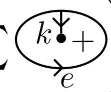
$$\epsilon : D \to \mathsf{k}$$

$\epsilon = \sum_g$ 

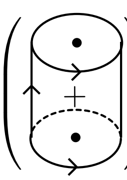
is the representation of D into $\mathrm{End}(V_0) \cong \mathsf{k} \cong \mathrm{End}(\mathsf{k})$. V_0 is called the trivial representation. Then

$Z\Big($  $\Big) \overset{\text{unglued}}{=} \epsilon \otimes (\text{top and bottom disks})$ (62)

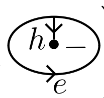
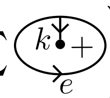
which is shorthand for

$Z\Big($  $\Big) \overset{\text{unglued}}{=} \epsilon \otimes \Big(\sum_h$  $\Big) \otimes \Big(\sum_k$  $\Big)$.

As before, the details of the top and bottom labelings will often be omitted from the notation:

$Z\Big($  $\Big) \overset{\text{unglued}}{=} \epsilon \otimes (\text{top and bottom disks})$.

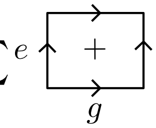
Remark 7.3. In later bookkeeping of factors of $|G|$, the reader may find it helpful to note that the pairing between

$\Big(\sum_h$  $\Big)$ and $\Big(\sum_k$  $\Big)$

is $|G|$. When the gluing law is applied, this factor of $|G|$ will often cancel with a factor of $|G|^{-1}$ from the vertex in D^2.

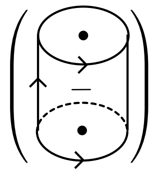
Remark 7.4. Observe that the projector in D for V_0, call it π_0, is

$$\pi_0 = \frac{1}{|G|}\sum_g$$



and therefore

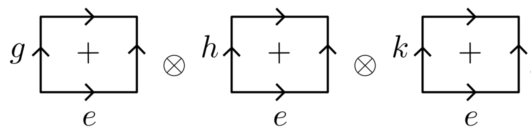
$$Z$$



$$\overset{\text{unglued}}{=} |G|\pi_0 \otimes (\text{top and bottom disks}). \qquad (63)$$

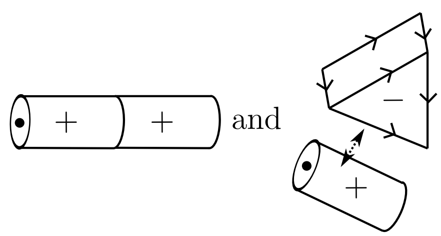
Define $\Phi \in D \otimes D \otimes D$ by

$$\Phi = \sum_{g,h,k} c(g,h,k)$$



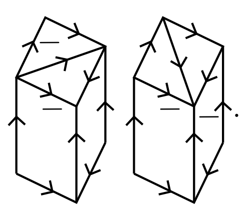
Proposition 7.5. *Δ, ϵ, and Φ give D the structure of a quasi-bialgebra.*

Sketch of Proof. ϵ is an algebra morphism because



have the same boundary Δ-complex structure, and so Z applied to each is the same. A similar proof shows that Δ is an algebra morphism.

$(\Delta \otimes \mathrm{id}) \circ \Delta$ and $(\mathrm{id} \otimes \Delta) \circ \Delta$ are obtained by applying Z to the following complexes (vertical edges identified, as usual):



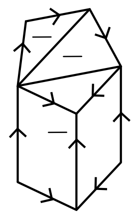
To the left complex, glue a copy of $-\Delta^3$ to the top and a copy of $+\Delta^3$ to the bottom to produce a complex with the same boundary Δ-complex structure as the right complex. This proves that

$$(\mathrm{id} \otimes \Delta)(\Delta(a)) = \Phi(\Delta \otimes \mathrm{id})(\Delta(a))\Phi^{-1},\ \forall a \in A.$$

The identity

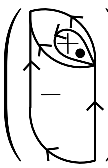
$$(\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\Phi)(\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\Phi) = (\Phi \otimes 1)(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\Phi)(1 \otimes \Phi)$$

has a similar proof, by considering

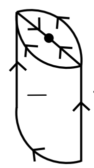


and using that c is a cocycle.

Combining (60) and (62) gives

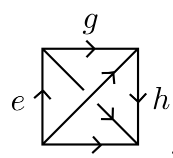
$$Z\Bigg($$  $$\Bigg) = (\epsilon \otimes \mathrm{id}) \circ \Delta \otimes (\text{top and bottom bigons}). \tag{64}$$

Change the triangulation on the top and bottom by gluing on copies of $\pm\Delta^3$ to the top and bottom:



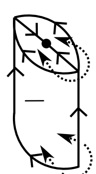
. (65)

The labelings on these copies of Δ^3 will be of the form



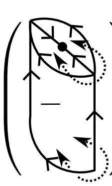
.

Since $c(e, g, h) = 1$ (by assumption) gluing on the copies of Δ^3 does not change (64). Take (65) and pinch the top and bottom together:

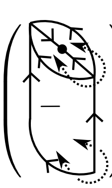
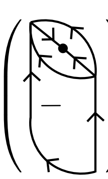
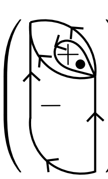


.

On the one hand,

$$Z\Bigg($$  $$\Bigg) \overset{\text{unglued}}{=} \mathrm{id}\,.$$

On the other hand, the gluing law says that

$$Z\Bigg($$  $$\Bigg) = \frac{|G|^2}{|G|^2} Z\Bigg($$  $$\Bigg) = Z\Bigg($$  $$\Bigg) \tag{66}$$

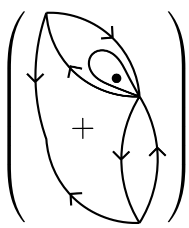
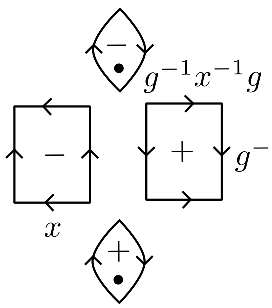
the denominator factor of $|G|^2$ coming from two newly interior vertices and one numerator factor of $|G|$ coming from each pairing of the a pair of triangles. It follows that $(\epsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id}$. A similar triangulation shows that $(\mathrm{id} \otimes \epsilon) \circ \Delta = \mathrm{id}$.

The identity

$$(\mathrm{id} \otimes \epsilon \otimes \mathrm{id})(\Phi) = 1 \otimes 1$$

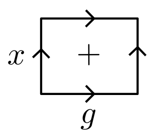
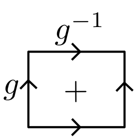
follows immediately from the assumption $c(g, e, h) = 1$. □

Next consider applying Z to (19):

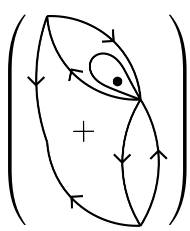
Z  $\overset{\text{unglued}}{=} \sum_{x,g} c_{x^{-1}}(g, g^{-1}) c^g(x, x^{-1}) g$  g^{-1}.

This would be a map $D \to D$, if it weren't for the top and bottom bigons. Define

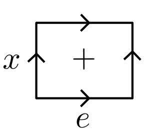
$$S : D \to D$$

 $\mapsto c_{x^{-1}}(g, g^{-1}) c^g(x, x^{-1})_{g^{-1}x^{-1}g}$ 

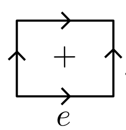
and write

Z  $\overset{\text{unglued}}{=} S \otimes (\text{top and bottom bigons}).$ (67)

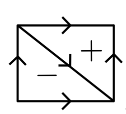
Define

$\alpha := \sum_x x$ 

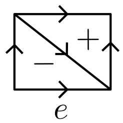
(the unit in D), and

$\beta := \sum_x c(x, x^{-1}, x)^{-1} x$ .

The element β has the following interpretation. There is a different natural triangulation of the cylinder, namely:

. (68)

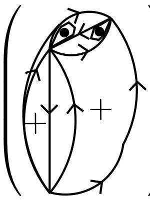
β is the image of

$$\sum_x x$$ 

under the map (46).

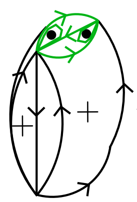
Claim 7.6. *For all $a \in D$, $S^2(a) = \beta a \beta^{-1}$.*

Proof. Note that

$Z\Bigg($ 

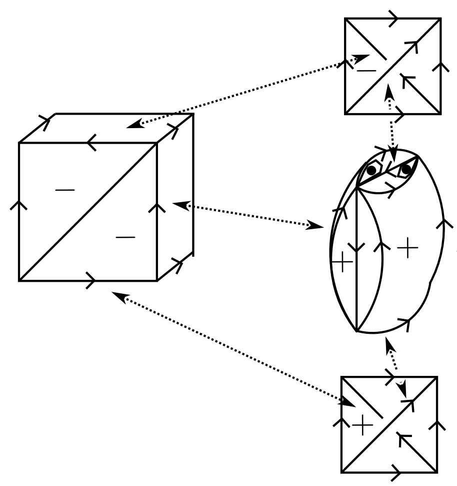
$\Bigg) = S^2 \otimes(\text{top and bottom}).$

The top (and, with opposite orientation, the bottom) has a copy of the triangulation (68):



.

Glue on a copy of $-\Delta^3$ on top and Δ^3 on the bottom to turn change this triangulation. Then glue on two multiplication maps:

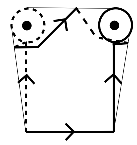


. (69)

Applying Z to this complex and ungluing the two boundary cylinders gives a map

$$a \mapsto \beta^{-1} S^2(a) \beta.$$

Each factor of β corresponds to one of the two copies of Δ^3 glued to the top and bottom. It is not hard to see that the triangulation resulting from (69) looks like



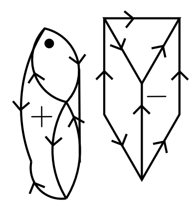
with the left and right edges identified. Z applied to this manifestly gives the identity map, whence

$$a = \beta^{-1} S^2(a)\beta.$$

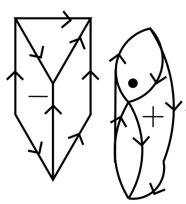
□

Proposition 7.7. *S, α, β give D the structure of a quasi-Hopf algebra.*

Proof. One can check each of the axioms of a quasi-Hopf algebra directly using the fact that c is a cocycle. The reader might find it informative, though, to relate (1) and (2) to the respective triangulations



and

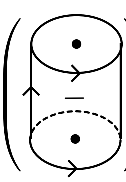


.

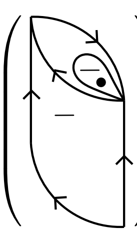
β will enter the second one in a manner similar to the proof of Claim 7.6. □

8. The Vector Space Associated to a Surface

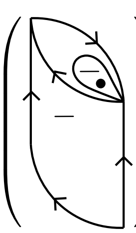
Recall from (63) that

Z () $\overset{\text{unglued}}{=} |G|\pi_0.$

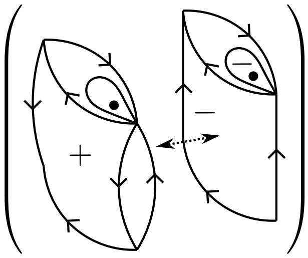
Therefore

Z () $\overset{\text{unglued}}{=} |G|\Delta(\pi_0).$

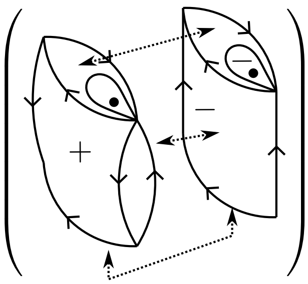
As usual the tensor factors corresponding to the top and bottom parts of the complex are dropped from the notation. Write

Z () $\overset{\text{unglued}}{=} |G|\Delta(\pi_0).$

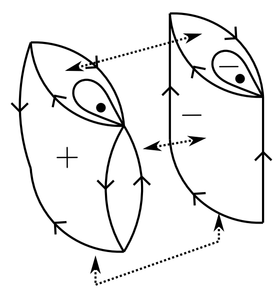
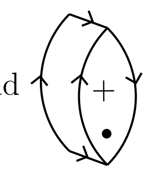
Then

Z  $\overset{\text{unglued}}{=} |G|((S \otimes \mathrm{id}) \circ \Delta)(\pi_0).$

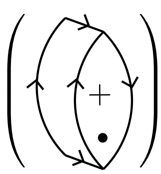
Pinch the top and bottom together: this introduces two new interior vertices but these cancel with factors of $|G|$ coming from the pairing maps on the top and the bottom, so

Z  $\overset{\text{unglued}}{=} |G|((S \otimes \mathrm{id}) \circ \Delta)(\pi_0).$

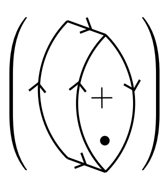
Also:

 and 

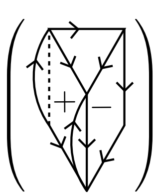
have the same boundary Δ-complex, hence

Z  $\overset{\text{unglued}}{=} |G|((S \otimes \mathrm{id}) \circ \Delta)(\pi_0).$ (70)

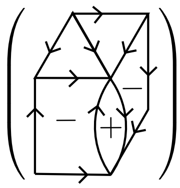
Example 8.1. For two D-modules W_1 and W_2,

Z 

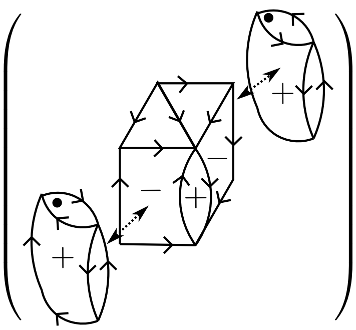
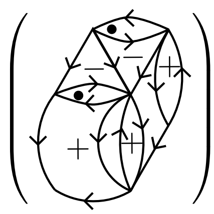
acts in $W_1^* \otimes W_2$ as $|G|$ times the projection onto the trivial part; and

Z  $\overset{\text{unglued}}{=} |G|((\mathrm{id} \otimes \Delta)(S \otimes \mathrm{id})\Delta)(\pi_0).$

acts in $W_1^* \otimes W_2 \otimes W_3$ as $|G|$ times the projection onto the trivial part; and

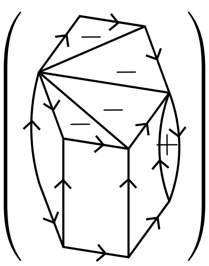
Z  $\overset{\text{unglued}}{=} |G|((\Delta \otimes \Delta)(S \otimes \mathrm{id})\Delta)(\pi_0)$

acts in $(W_1 \otimes W_2)^* \otimes W_3 \otimes W_4$ as $|G|$ times the projection onto the trivial part; and

Z  $= Z$ 

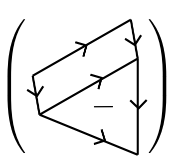
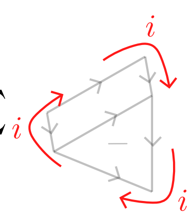
$$\overset{\text{unglued}}{=} |G|(((S \otimes \mathrm{id})(\mathrm{id} \otimes S))(\Delta \otimes \Delta)(S \otimes \mathrm{id})\Delta)(\pi_0)$$

acts in $(W_1 \otimes W_2^*)^* \otimes (W_3^* \otimes W_4)$ as $|G|$ times the projection onto the trivial part; and

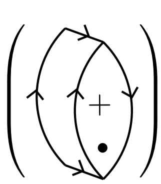
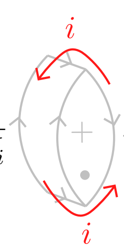
Z  $\overset{\text{unglued}}{=} |G|((((\Delta \otimes \mathrm{id})\Delta) \otimes ((\Delta \otimes \mathrm{id})\Delta))(S \otimes \mathrm{id})\Delta)(\pi_0)$

acts in $((W_1 \otimes W_2) \otimes W_3)^* \otimes ((W_4 \otimes W_5) \otimes W_6)$ acts as $|G|$ times the projection onto the trivial part. Etc.

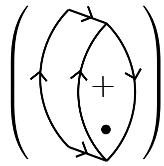
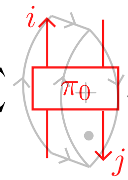
In the red arrow notation of section 6, the multiplication in D is

Z  $\overset{\text{unglued}}{=} \sum_i$  (71)

and

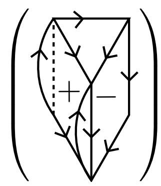
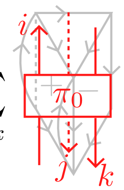
Z  $\overset{\text{unglued}}{=} \sum_i \frac{|G|}{\dim V_i}$  .

On the other hand, (70) implies that

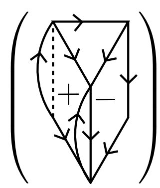
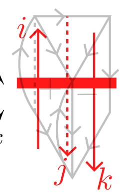
Z  $\overset{\text{unglued}}{=} |G| \sum_{i,j}$ 

.

Here π_0 is understood (read top to bottom) as acting in $V_i^* \otimes V_j$—this is indicated by the triangulation behind the red arrows. Similarly,

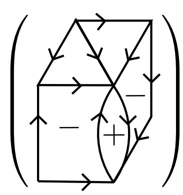
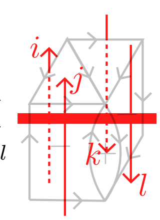
Z  $\overset{\text{unglued}}{=} |G| \sum_{i,j,k}$ 

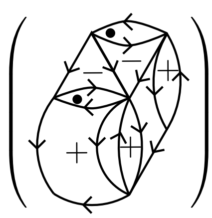
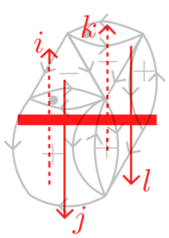
where again the gray triangulation behind the red arrows indicates that π_0, read top to bottom, acts in $V_i^* \otimes (V_j \otimes V_k)$. Equivalently, it is the projection from $\mathrm{Hom}(V_i, V_j \otimes V_k)$ to $\mathrm{Hom}_D(V_i, V_j \otimes V_k)$. It becomes a bit cumbersome writing π_0 everywhere, so instead use a solid red bar to denote π_0:

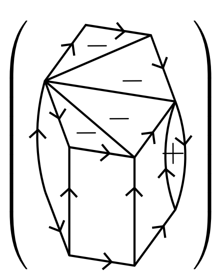
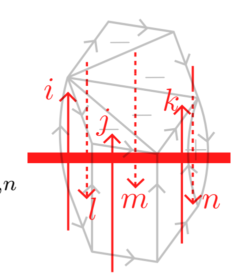
Z  $\overset{\text{unglued}}{=} |G| \sum_{i,j,k}$ 

.

Thus:

Z  $\overset{\text{unglued}}{=} |G| \sum_{i,j,k,l}$ 

Z  $\overset{\text{unglued}}{=} |G| \sum_{i,j,k,l}$ 

Z  $\overset{\text{unglued}}{=} |G| \sum_{i,j,k,l,m,n}$ 

.

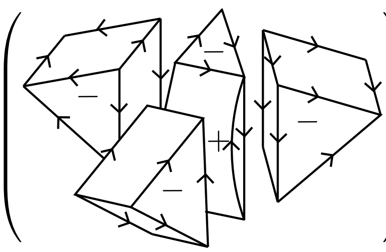
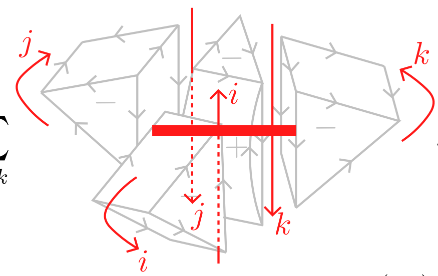
For example, in the last picture the red arrow diagram acts in

$$\mathrm{Hom}((V_i \otimes V_j) \otimes V_k, (V_l \otimes V_m) \otimes V_n)$$

as projection to

$$\mathrm{Hom}_D((V_i \otimes V_j) \otimes V_k, (V_l \otimes V_m) \otimes V_n).$$

The parenthization is indicated by the gray triangulation drawn below the red arrow. Gluing on copies of $-\Delta^2 \times S^1$ and contracting the red arrows from (71) shows that, for example

$Z\Bigg($  $\Bigg) \overset{\text{unglued}}{=} |G| \sum_{i,j,k}$  .

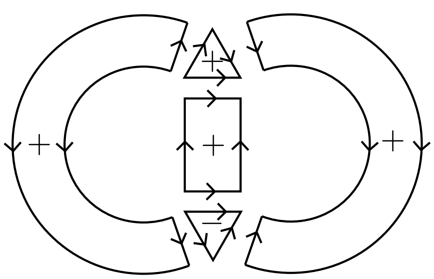
(72)

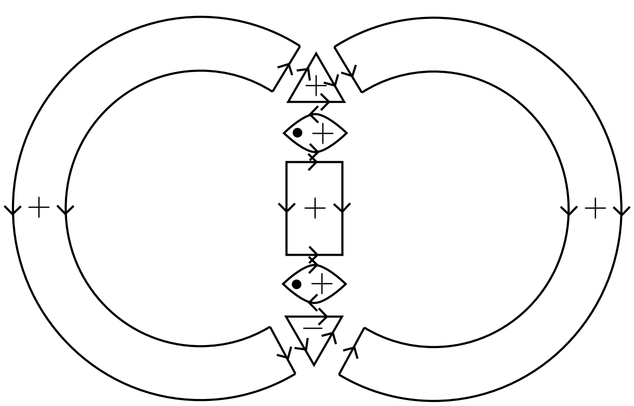
Note that the right hand side of (72) sits inside[2] $D^{\otimes 6}$. Since

$$D \cong \bigoplus_i V_i^* \otimes V_i$$

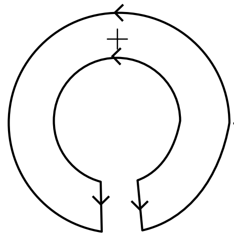
each end of a boundary cylinder in (72) corresponds to an end of a red arrow, one end corresponding to V_i^* for some i and the other to V_i for the same i.

Let (L, x_L) be an oriented closed surface built from pieces like (13)-(18). For example, here are two genus 2 surfaces and one torus:

$L_1 =$  (73)

$L_2 =$ 

[2]actually, in $D^{\otimes 6} \otimes$(top and bottom triangles)

$S^1 \times S^1 =$  .

There's a injective ungluing map

$$Z(L, x_L) \overset{\text{unglue}}{\hookrightarrow} D^{\otimes N} \otimes Z(\pm\Delta^2)^{\otimes N'}$$

where N is the number of cylinders and N' is the number of triangles in decomposition of the surface. The triangle part will be dropped from the notation:

$$Z(L, x_L) \overset{\text{unglue}}{\hookrightarrow} D^{\otimes N}.$$

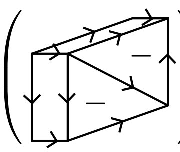
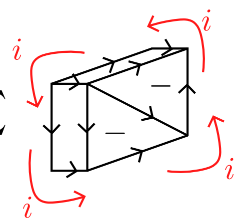
Similarly, there's also a gluing map

$$D^{\otimes N} \overset{\text{glue}}{\twoheadrightarrow} Z(L, x_L).$$

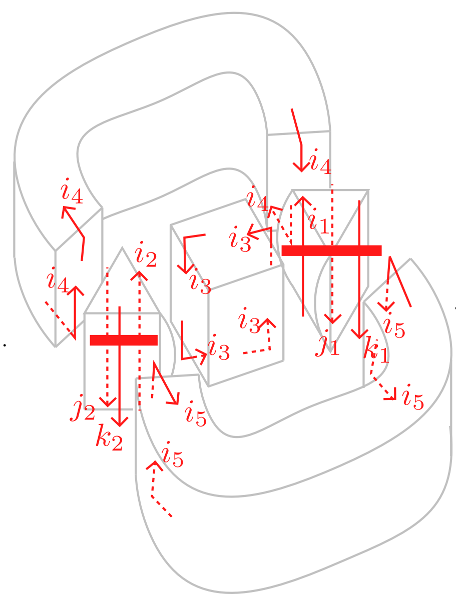
Recall that

$$Z(I \times L, x_I x_L) : Z(L, x_L) \to Z(L, x_L).$$

Precompose with the gluing map and postcompose with the ungluing map to think of $Z(I \times L, x_I x_L)$ as a map $D^{\otimes N} \to D^{\otimes N}$. Since

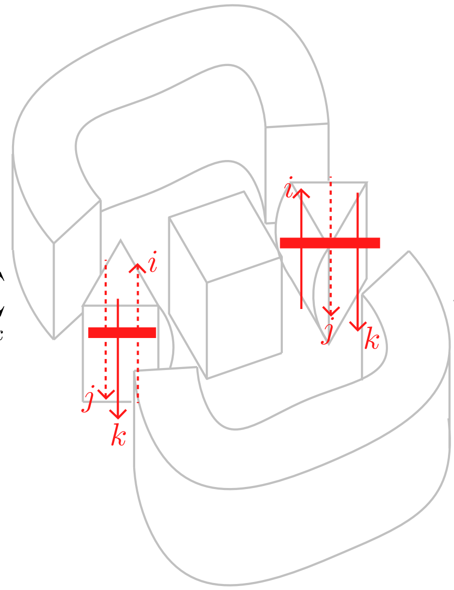
$Z\Bigg($  $\Bigg) = \sum_i$ 

then by the gluing law, $Z(I \times L_1, x_I x_{L_1})$ is the contraction of

$|G|^2 \sum_{i_1, j_1, \ldots}$  .

The contraction simply sets $k_1 = i_5 = k_2$ etc. Therefore,

$$Z(I \times L_1, x_I x_{L_1}) = |G|^2 \sum_{i,j,k} \tag{74}$$

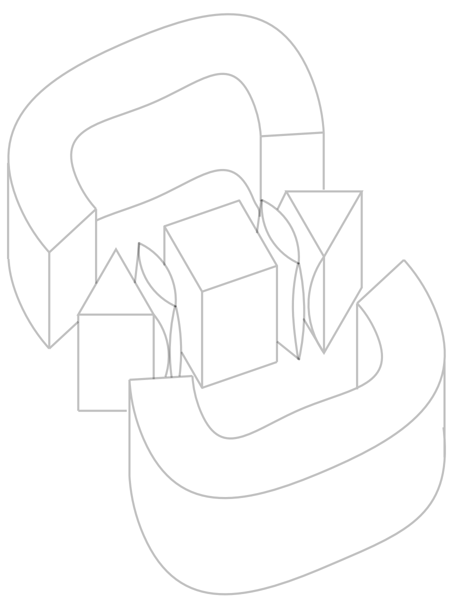


In particular,

$$Z(L_1, x_{L_1})^{G^2} = \mathrm{image}(Z(I \times L_1, x_I x_{L_1})) \tag{75}$$

$$\cong \bigoplus_{i,j,k} \mathrm{Hom}_D(V_i, V_j \otimes V_k)^* \otimes \mathrm{Hom}_D(V_j \otimes V_k, V_i)^*. \tag{76}$$

The reason for the duals is that the projections π_0 in (74) are read top to bottom while the map $Z(I \times L_1, x_I x_{L_1})$ is read bottom to top. Thus the map $Z(I \times L_1, x_I x_{L_1})$ is the adjoint of the projectors. In a similar way, the triangulation

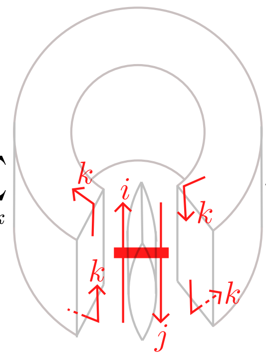


leads to

$$Z(L_2, x_{L_2})^{G^2} \cong \mathrm{Hom}_D(V_i^*, V_j \otimes V_k)^* \otimes \mathrm{Hom}_D(V_j \otimes V_k, V_i)^*.$$

Note that there is an isomorphism $Z(L_1, x_{L_1})^{G^2} \cong Z(L_2, x_{L_2})^{G^2}$, but the two different triangulations L_1 and L_2 lead to different expressions of these vector spaces in terms of the modules of D. By the gluing law, $Z(I \times S^1 \times S^1)$ is the contraction of

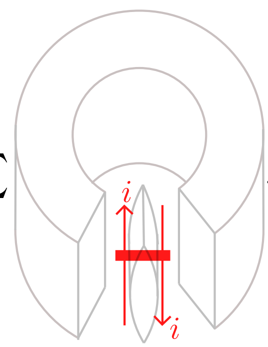
$$|G| \sum_{i,j,k}$$



.

Contracting just sets $i = j = k$, so

$$Z(I \times S^1 \times S^1) = |G| \sum_{i}$$



. (77)

In other words, $Z(I \times S^1 \times S^1)$ is $|G|$ times the projection of D to its center and $Z(S^1 \times S^1)^G$ is isomorphic to the center of D. In notation more in line with (75),

$$Z(S^1 \times S^1)^G \cong \bigoplus_i \mathrm{Hom}_D(V_i, V_i)^*. \tag{78}$$

Remark 8.2. Recall from (50) that

$$D \cong \bigoplus_i \mathrm{End}(V_i) \cong \bigoplus_i V_i^* \otimes V_i.$$

(78) might seem a little strange since, at first sight, it might seem like $\mathrm{Hom}_D(V_i, V_i)^*$ should be part of $D^* \cong \bigoplus_i \mathrm{End}(V_i)^*$, whereas the ungluing map

$$Z(S^1 \times S^1) \overset{\text{unglue}}{\hookrightarrow} D$$

identifies $Z(S^1 \times S^1)^G$ with a subset of D. But by looking at the triangulation, $\mathrm{Hom}_D(V_i, V_i)^* \cong V_i \otimes V_i^*$ is identified with the $V_i^* \otimes V_i$ part of D with the two tensor factors switched. Under these identifications, pairing an element of $\mathrm{Hom}_D(V_i, V_i)^*$ with id_{V_i} is the same as pairing the corresponding element of D by χ_i. See Example 8.4 below.

Remark 8.3. In general, there is an expression like (76) for any surface obtained by piecing together parts like (13)-(18). This level of generality will not be needed here and details are left to the reader.

Example 8.4. Suppose ∂K is (isomorphic as an oriented Δ-complex to) a disjoint union of N copies of $S^1 \times S^1$.

By using the isomorphism (78), $Z(K, x_K)$ can be thought of as an element in $\mathrm{Hom}_D(V_i, V_i)^*$ and

$$\langle Z(K, x_K), \mathrm{id}_{V_{i_1}} \otimes \mathrm{id}_{V_{i_2}} \otimes \cdots \otimes \mathrm{id}_{V_{i_N}} \rangle \tag{79}$$

is an invariant of the oriented manifold K together with a marking on its boundary.

Alternatively, by using the ungluing map

$$Z(S^1 \times S^1)^G \overset{\text{unglued}}{\hookrightarrow} D$$

one can think of $Z(K, x_K)$ in D and

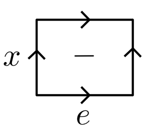
$$\langle Z(K, x_K), \chi_{i_1} \otimes \chi_{i_2} \otimes \cdots \otimes \chi_{i_N} \rangle \tag{80}$$

is an invariant of the oriented manifold K together with a marking of its boundary. By Remark 8.2, (79) and (80) are the same. The perspective (80) will the one preferred in what follows.

If $|K| = S^3 \setminus \nu\mathcal{L}$ as in Example 5.17, then (80) produces invariants of directed framed links where components are "colored" by irreps of D.

Example 8.5. In Example 5.18, a closed oriented 3-manifold $\mathcal{M}$ was constructed by surgery on a framed link $\mathcal{L}$. Since

$$Z(-D^2 \times S^1) \overset{\text{unglued}}{=} \sum_x x$$



$$= \sum_i \frac{\dim V_i}{|G|} \chi_i$$

it follows that $Z(\mathcal{M})$ is

$$\frac{1}{|G|^{2N}} \sum_{i_1, i_2, \ldots, i_N} (\dim V_{i_1})(\dim V_{i_2}) \cdots (\dim V_{i_N}) \langle Z(K, x_K), \chi_{i_1} \otimes \chi_{i_2} \otimes \cdots \otimes \chi_{i_N} \rangle.$$

In particular, the closed 3-manifold invariant for the TQFT can be written in terms of the link invariants. This should be compared to the relation between the papers [Wit89] and [RT91]: the former (conjecturally) develops a TQFT for all manifolds and the latter (rigorously) constructs the closed manifold invariants of that TQFT from its link invariants via a surgery presentations of manifolds.

Example 8.6. Suppose K is such that ∂K is (isomorphic as an oriented Δ-complex to) the triangulation (73). Then, using the isomorphism (76), $Z(K, x_K)$ sits inside

$$\bigoplus_{i,j,k} \mathrm{Hom}_D(V_i, V_j \otimes V_k)^* \otimes \mathrm{Hom}_D(V_j \otimes V_k, V_i)^*.$$

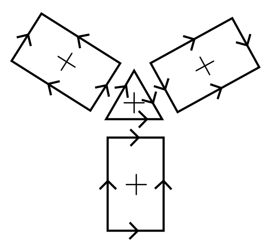
In particular, for $\alpha^i_{jk} \in \mathrm{Hom}_D(V_i, V_j \otimes V_k)$ and $\beta^{jk}_i \in \mathrm{Hom}_D(V_j \otimes V_k, V_i)$,

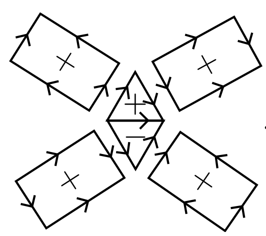
$$\langle Z(K, x_K), \alpha^i_{jk} \otimes \beta^{jk}_i \rangle$$

is an invariant of K plus its boundary triangulation. This sort of idea can be generalized to produce, for example, invariants of directed ribbon graphs in S^3 with edges colored by representations of D.

This example can be generalized to manifolds with boundary triangulated in other ways, as in Remark 8.3.

Consider the oriented surfaces

$(L_3, x_{L_3}) :=$ 

$(L_4, x_{L_4}) :=$  .

Each of these triangulations has a single interior basepoint and G acts on the set of G-labelings via this basepoint. Equation (72) shows that

$$Z(L_3, x_{L_3})^G \cong \bigoplus_{i,j,k} V_i^* \otimes V_j \otimes V_k \otimes \mathrm{Hom}_D(V_i, V_j \otimes V_k)^*. \tag{81}$$

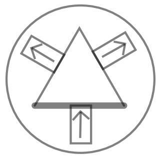
An equation analogous to (72) shows that

$$Z(L_4, x_{L_4})^G \cong \bigoplus_{i,j,k,l} V_i^* \otimes V_j^* \otimes V_k \otimes V_l \otimes \mathrm{Hom}_D(V_i \otimes V_j, V_k \otimes V_l)^*. \tag{82}$$

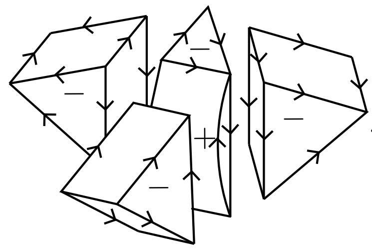
By example (6.9), $\frac{1}{|G|} Z(I \times L_3, x_I x_{L_3})$ defines an action of $D^{\otimes 3}$ on $Z(L_3, x_{L_3})^G$ and $\frac{1}{|G|} Z(I \times L_4, x_I x_{L_4})$ defines an action of $D^{\otimes 4}$ on $Z(L_4, x_{L_4})^G$. The isomorphisms (81) and (82) are isomorphisms of $D^{\otimes 3}$ and $D^{\otimes 4}$ modules, respectively. Isomorphisms like (81) and (82) for arbitrary compact surfaces with boundary can be constructed by the interested reader.

9. The Link Invariant

Recall from Example 4.11 that

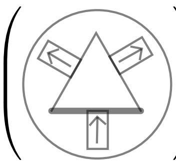


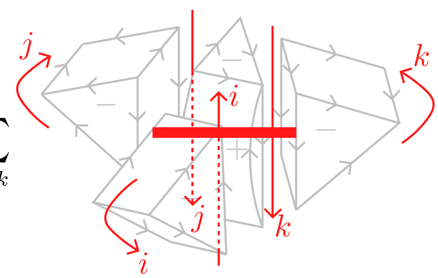
$=$



.

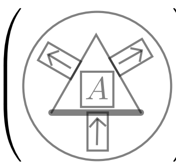
Therefore (see (72))

$Z\Bigg($  $\Bigg) \overset{\text{unglued}}{=} |G| \sum_{i,j,k}$

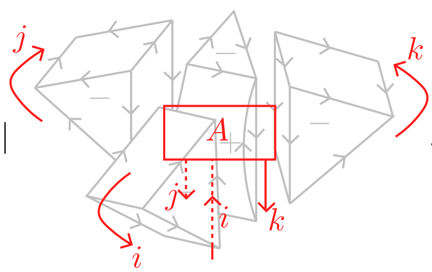


(83)

To unclutter notation, for $A \in \mathrm{Hom}_D(V_i, V_j \otimes V_k) \subset V_i^* \otimes (V_j \otimes V_k)$, let

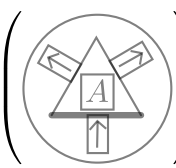
$Z\Bigg($ 

$\Bigg) = |G|$

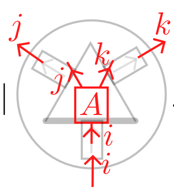


.

In other words, A has been "plugged into" (83), viewed as a map from top to bottom. To further simplify notation superimpose the red arrows on just the bottom triangulation of (20):

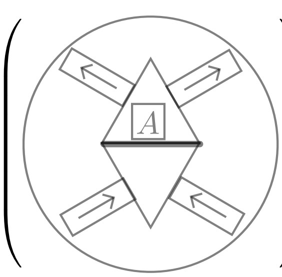
$Z\Bigg($ 

$\Bigg) = |G|$

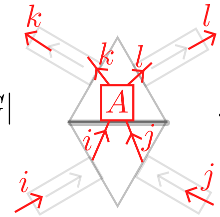


. (84)

Generalize the notation (84) in the obvious way so that, for example, for $A \in \mathrm{Hom}_D(V_i \otimes V_j, V_k \otimes V_l)$:

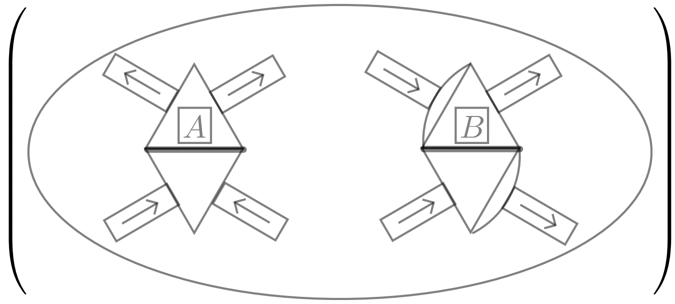
$Z\Bigg($ 

$\Bigg) = |G|$

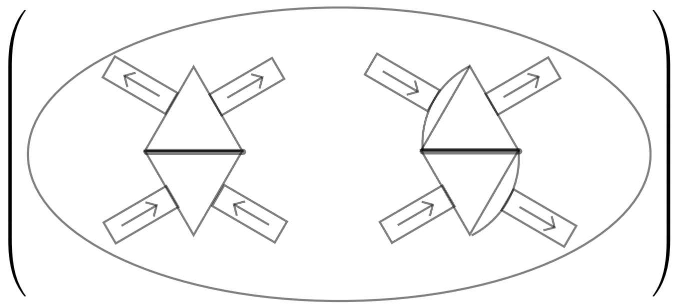


.

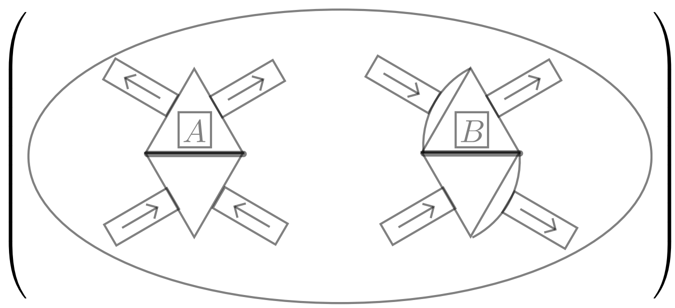
For $A \in \mathrm{Hom}_D(V_{i_1} \otimes V_{j_1}, V_{k_1} \otimes V_{l_1})$ and $B \in \mathrm{Hom}_D(V_{i_2} \otimes V_{j_2}^*, V_{k_2}^* \otimes V_{l_2})$, let

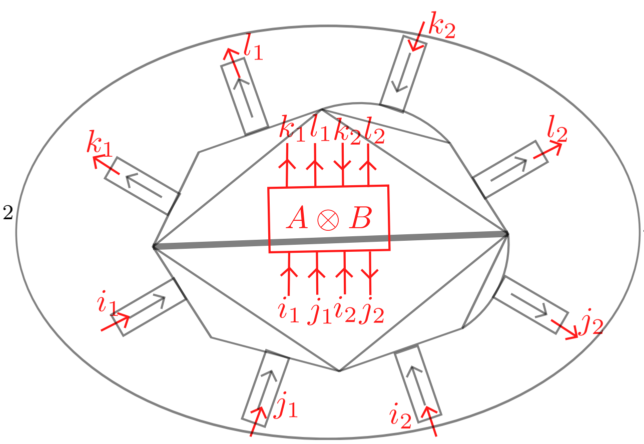
Z 

denote A and B "plugged into"

Z  .

Because of (27), it follows that

Z 

$= |G|^2$  . (85)

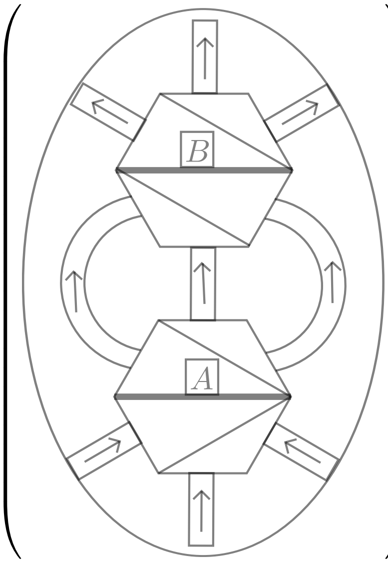
As before, the red arrows are superimposed on the "bottom" triangulation of (27).

Given

$$A \in \mathrm{Hom}_D(V_{i_1} \otimes (V_{i_2} \otimes V_{i_3}), V_{j_1} \otimes (V_{j_2} \otimes V_{j_3})$$

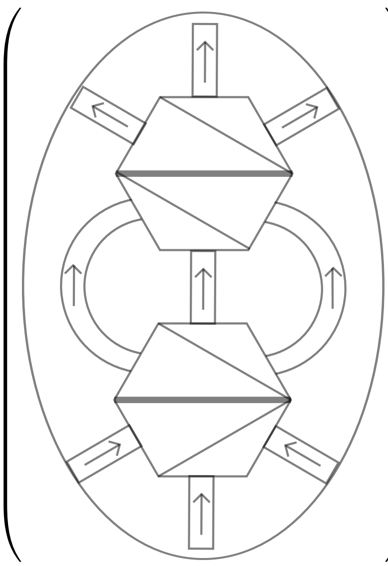
$$B \in \mathrm{Hom}_D((V_{j_1} \otimes V_{j_2}) \otimes V_{j_3}, V_{k_1} \otimes (V_{k_2} \otimes V_{k_3}))$$

let

$Z\Bigg($ 

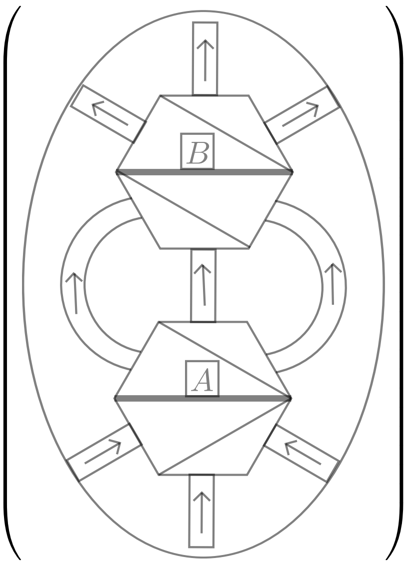
$\Bigg)$

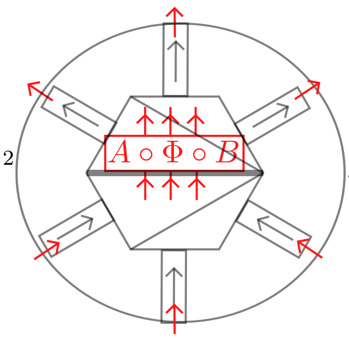
be the result of "plugging in" A and B into

$Z\Bigg($  $\Bigg)$.

Then, using (29), it follows that

$$Z\left(\vcenter{\text{diagram}} \right) = |G|^2 \vcenter{\text{diagram}} . \qquad (86)$$





Two comments are in order. First, Φ gets placed here because of the 3-simplex in (29). This 3-simplex is there because the parenthizations of the codomain of A and the domain of B do not match. Second, there is some cancellation that goes into the identity (86). Because of (53), there's a factor of

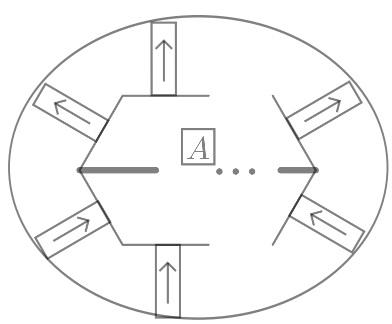
$$\frac{|G|}{\dim V_{j_1}} \frac{|G|}{\dim V_{j_2}} \frac{|G|}{\dim V_{j_3}}$$

that comes from the three copies of $B \times S^1$. To go from (29) to (30), three copies of $-\Delta^2 \times S^1$ are glued on. These introduce three new interior vertices and hence a factor of $|G|^{-3}$. These also introduce three new red circles labeled j_1, j_2, and j_3. Therefore gluing on these three copies of $-\Delta^2 \times S^1$ multiplies the expression by

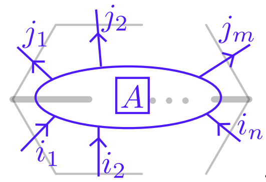
$$\frac{\dim V_{j_1}}{|G|} \frac{\dim V_{j_2}}{|G|} \frac{\dim V_{j_3}}{|G|}$$

exactly canceling the contribution from the three copies of $B \times S^1$.

The result of (85) and (86) and their easy generalizations is that the pieces

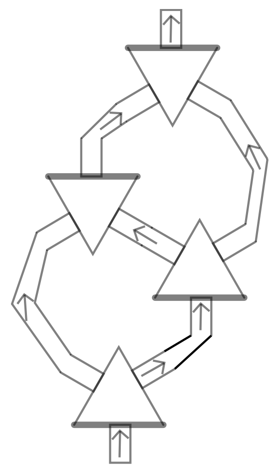


glue together horizontally and vertically in the same way as the blue arrow pieces from section 2:

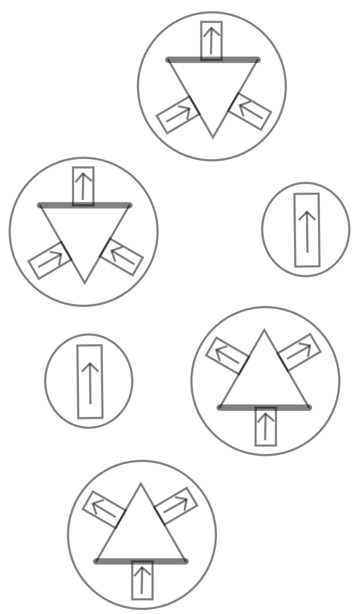


.

For example, form the generalized tangle

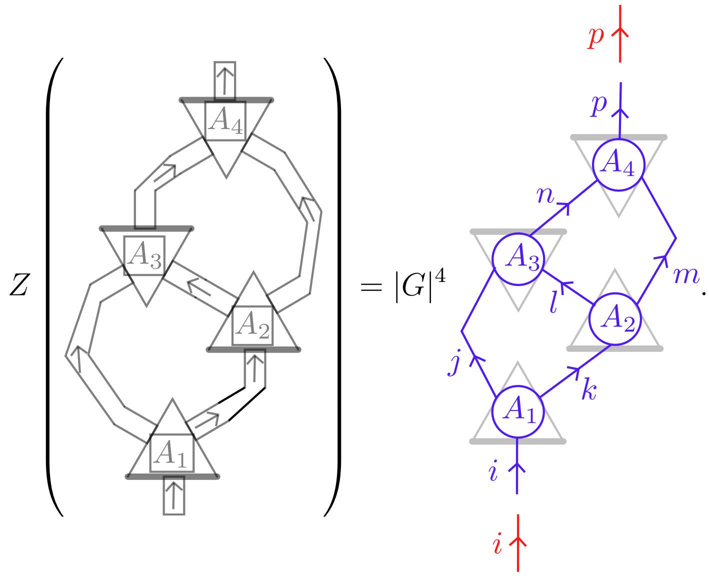


by gluing together the following generalized tangles



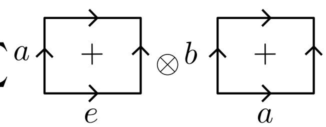
If $A_1 \in \mathrm{Hom}_D(V_i, V_j \otimes V_k)$, $A_2 \in \mathrm{Hom}_D(V_k, V_l \otimes V_m)$, $A_3 \in \mathrm{Hom}_D(V_j \otimes V_k, V_n)$

and $A_4 \in \mathrm{Hom}_D(V_n \otimes V_l, V_p)$, then

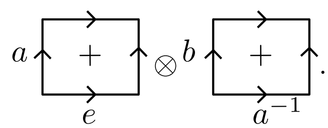


Recall the convention (see (5)) that copies of Φ are omitted from the blue arrow notation.

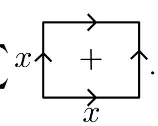
Define

$$R = \sum_{a,b}$$ 

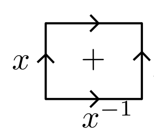
so that

$$R^{-1} = \sum_{a,b} c_b(a, a^{-1})$$ 

Also define

$$v = \sum_x$$ 

so that

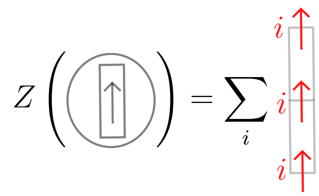
$$v^{-1} = \sum_x c_x(x, x^{-1})$$ 

Claim 9.1. *The elements R and v give D the structure of a ribbon Hopf algebra.*

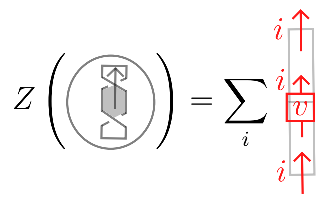
This claim is proved in [AC92] and will not be proved here. The interested reader might prove the axioms related R, v, and Φ by using the triangulations in Section 4.

In the following proposition an algebra element a applied to a red arrow labeled i means the linear map $\rho_i(a)$, where $\rho_i : D \to \mathrm{End}(V_i)$ is the ith representation.

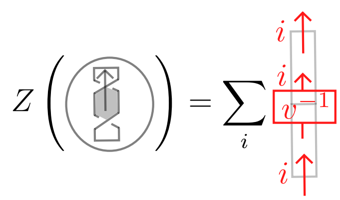
Proposition 9.2. *The identities (87)-(96) hold:*



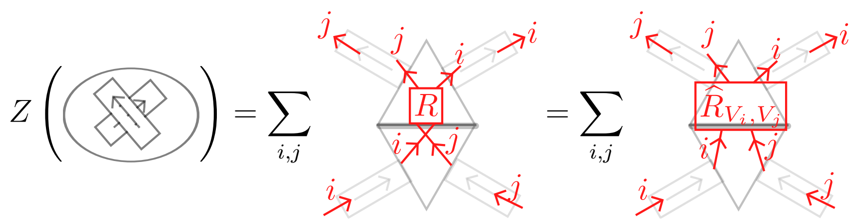
(87)



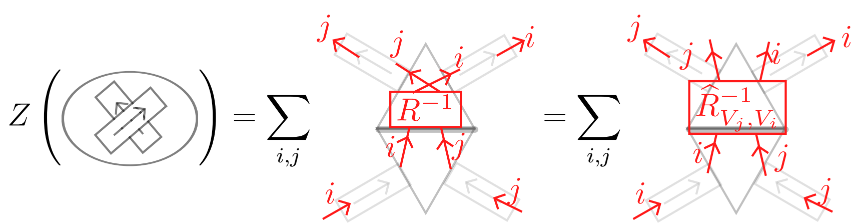
(88)



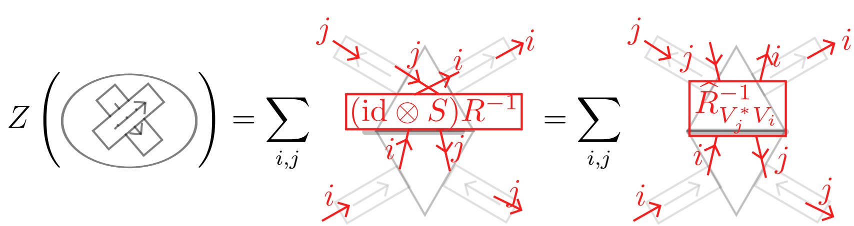
(89)



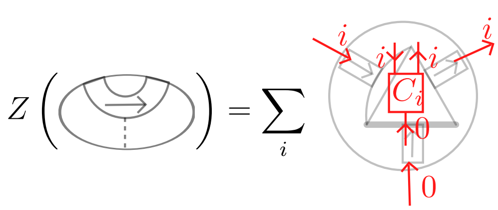
(90)



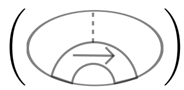
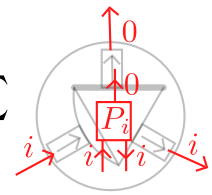
(91)

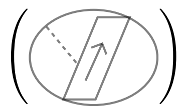
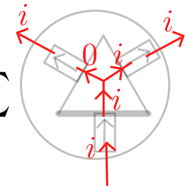


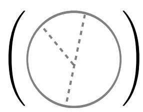
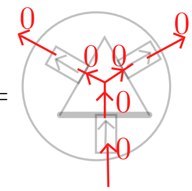
(92)



(93)

Z  $= \sum_i$  (94)

Z  $= \sum_i$  (95)

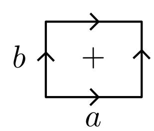
Z  $=$  (96)

There are other slight permutations involving flipping some of the generalized tangles upside down and/or reversing some of the directions of the arrows.

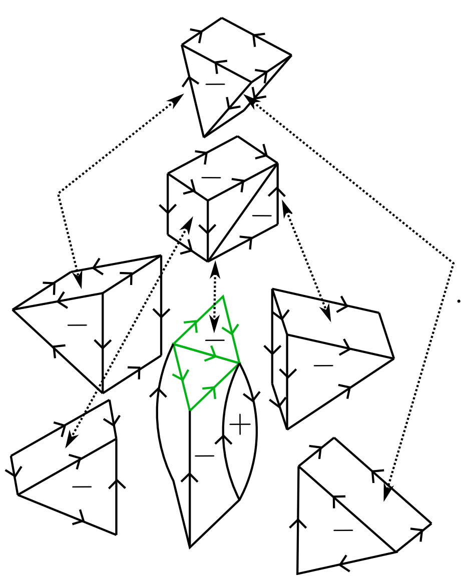
Proof. Each of these identities follows from applying Z to the relevant triangulations in Section 4.

The solid torus glued on the bottom of the right side of (34) doesn't have the usual triangulation from $D^2 \times S^1$ because of (33). One can check that Z applied to this solid torus is v. Then (88) follows immediately and (89) is similar.

The



part of R comes from the following cylinder in (38), emphasized here in green:



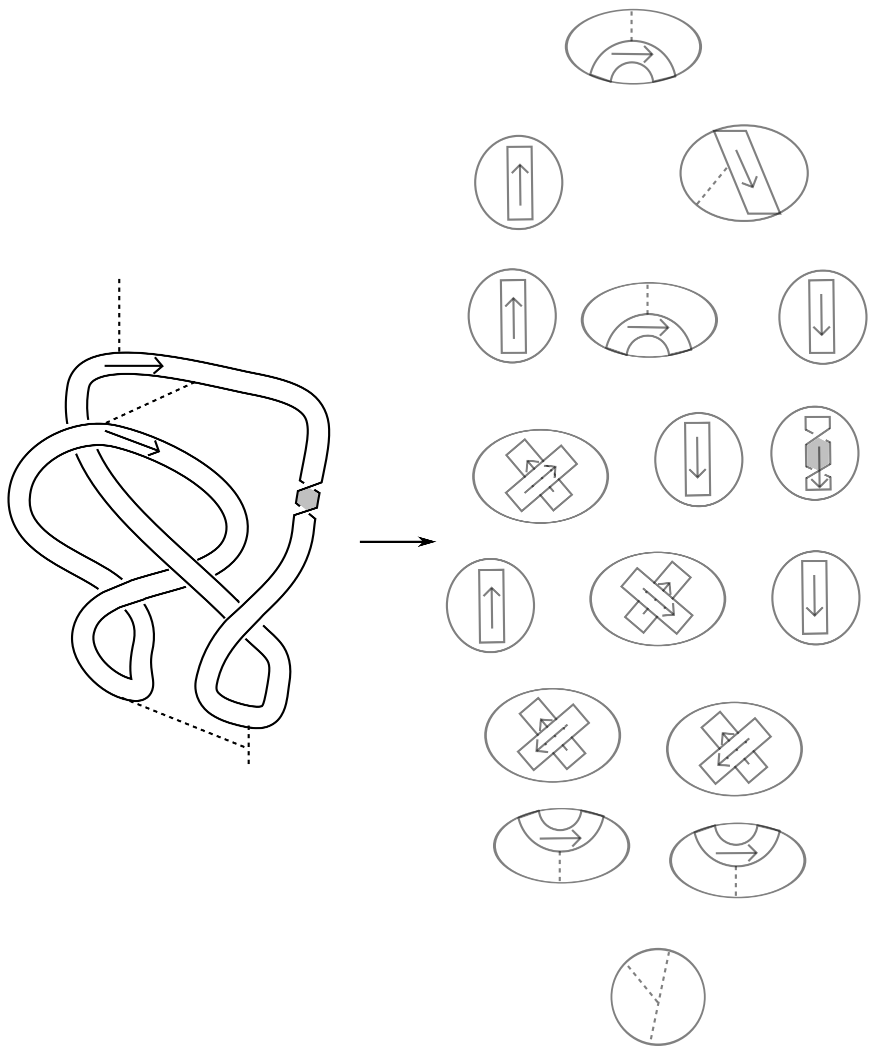
Note that the top and bottom of the green quadrilateral are identified when the top copy of $-\Delta^2 \times S^1$ is glued in, so the green quadrilateral is indeed a cylinder. Then (90) follows. (91) and (92) are similar, using triangulations (37) and (39), respectively.

(93) and (94) follow from Examples 4.19 and 4.20, respectively. The only subtle point is the factor of β in C_i comes from the extra $-\Delta^3$ in (40).

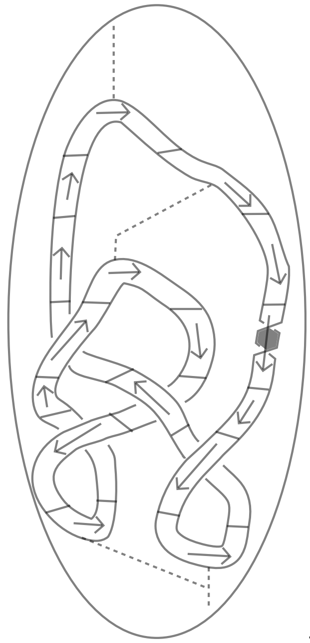
(95) and (96) follow from Example 4.21. □

Remark 9.3. Since $\mathrm{End}(V_0)$ is identified with k by sending the identity map to $1 \in \mathsf{k}$, all isolated red arrows labeled by 0 can be dropped from the notation.

Since a link diagram can be built from the elementary tangles of Proposition 9.2 (plus some small variations, like reversing directions of arrows), it follows that $Z(S^3 \setminus \nu\mathcal{L})$ can be written in terms of the link invariant detailed in Section 2. Take the same example as in Section 2 and decompose it into tangles

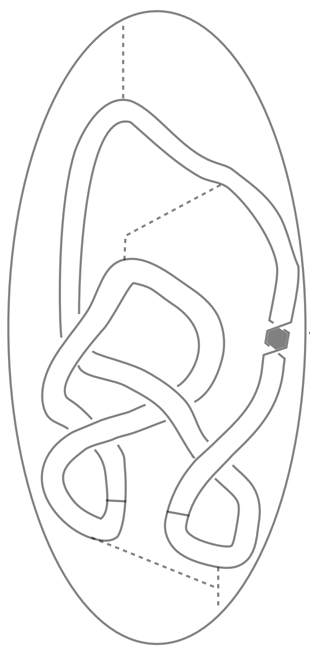


The tangles on the right glue to



.

but many copies of $-\Delta^2 \times S^1$ can be glued on to produce

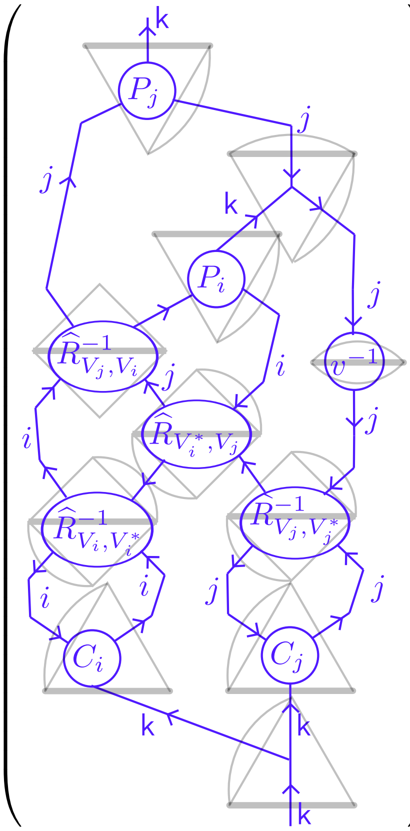


. (97)

Note that, in the process of gluing together, there is generally one copy of $-\Delta^2 \times S^1$ for every copy of $B \times S^1$, and these produce canceling factors $\frac{\dim V_i}{|G|}$ and $\frac{|G|}{\dim V_i}$. However, as the lines on each of the two link components indicate, there are two copies of $B \times S^1$ that are not matched with copies of $-\Delta^2 \times S^1$.

It follows that, for T the tangle in 97,

$$Z\left(T\right) = \sum_{i,j} \frac{|G|}{\dim V_i} \frac{|G|}{\dim V_j}$$



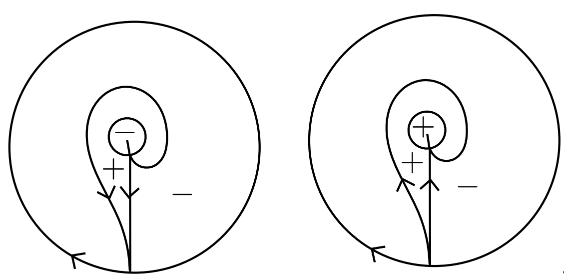
$$\pi_i \otimes \pi_j$$

Remark 9.4. The copies of π_i and π_j come from the two red arrows not cancelled by copies of $-\Delta^2 \times S^1$.

Each of the coefficients of $\pi_i \otimes \pi_j$ in $Z(T)$ are link invariants. It follows that

Theorem 9.5. *For the ribbon quasi-Hopf algebra D, the recipe in section 2 for constructing a directed framed link invariant actually produces an invariant of the directed framed link.*

More can be said, however. The link (97) sits inside a ball whose boundary is triangulated by the following two disks glued together along their boundaries:



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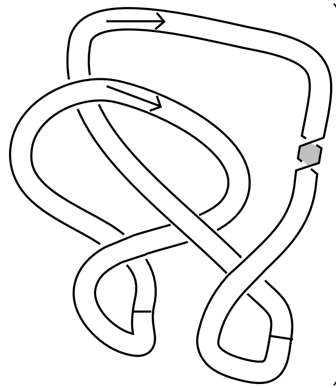
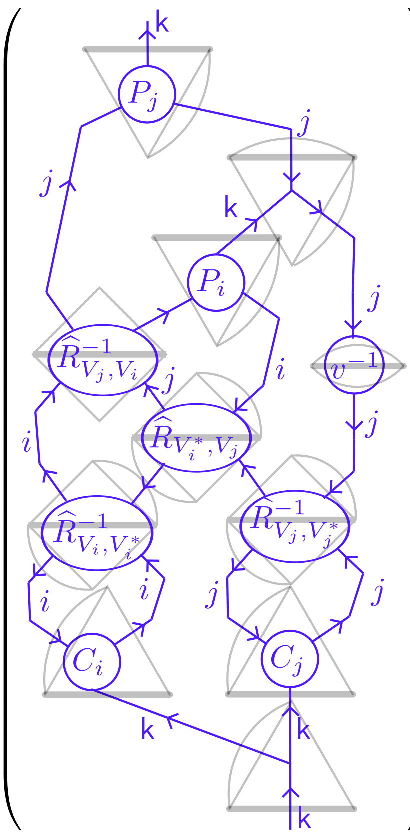
These two disks can be glued together to produce a triangulation of the link inside S^3. In the process of gluing the two disks, one has to insert a copy of $B \times S^1$ to make the arrow orientations match. The pairing between the G-labelings on the disks produces a factor of $|G|^2$ and the three new interior vertices produce a factor of $|G|^{-3}$. Therefore,

$$Z\left(S^3 \setminus \right.$$



$$\left. \right)$$

$$= \frac{1}{|G|} \sum_{i,j} \frac{|G|}{\dim V_i} \frac{|G|}{\dim V_j} \operatorname{tr} \left(\right.$$



$$\left. \right) \pi_i \otimes \pi_j.$$

As before, this identity generalizes to an arbitrary framed directed link $\mathcal{L}$. It follows that the relation between the Dijkgraaf-Witten link invariant and the blue-arrow link invariant $I(\mathcal{L}, \mathbf{i})$ derived from the quasi-Hopf algebra algebra D is:

Theorem 9.6. *Let (K, x_K) be the Δ-complex structure on the complement of a tubular neighborhood of a directed framed link $\mathcal{L}$, as in Example 5.17. Write $I(\mathcal{L}, \mathbf{i})$ for the blue-arrow invariant from obtained from a projection of $\mathcal{L}$ with* *components colored by* $\mathbf{i} = (i_1, \ldots, i_N)$*. Then*

$$I(\mathcal{L}, \mathbf{i}) = \frac{1}{|G|^{N-1}} \langle Z(K, x_K), \chi_{i_1} \otimes \cdots \otimes \chi_{i_N} \rangle.$$

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