In the homework problems, we looked at calculating the probability of some 5 card poker hands after drawing 5 cards. In this worksheet, we'll look at the more popular (and more interesting) poker variant Texas Hold-Em. Assume we have a standard 52 card deck, with 13 cards (2 through 10, Jack, Queen, King and Ace) of each of 4 different suits (no jokers!). Each player gets two of their own cards (hole cards) and then another 5 are shared (common cards).

We will calculate the odds from the perspective of one player of drawing different hands on the river. The distinction between hole and common cards no longer matters, so this is just the odds of getting each hand after 7 cards. The difficulty is that drawing 7 cards, our hand is still the best possible 5 card hand.


**Problem 1. Find the Probability of getting:**

1. A straight flush
2. 4 of a kind
3. A full house
4. A flush
5. A straight
6. 3 of a kind
7. Two pair
8. One pair
9. A High Card (i.e. none of the above)

**Problem 2. Omaha**

Repeat the above for a different Poker variant - Omaha. The rules: You get 4 cards of your own and 5 common cards, now the distinction matters. To make a 5 card hand, you have to use exactly two of your four hole cards and three of the 5 common cards. This changes the odds, significantly.
Here are some selected solutions.

Solution 1. **Find the Probability of getting: A Full House**

One of the main difficulties of these calculations is that if a set of 7 cards has one made hand, it may also have another and so we need to make sure to count only Full House’s that are not also 4 of a kind or Straight Flushes. Thankfully, a moment’s thought shows it is impossible for 7 cards to have both a Full House and Straight - let alone a straight flush. (Why is this?) We then break down the three different possible structures a Full House could take. Let $X, Y, Z, W$ be arbitrary different ranks from the 13 different possible ranks.

1. **XXXYYZW** The first possibility is that the 7 cards have 3 X’s, 2 Y’s and one Z and one W. The number of ways to do this is as follows. First there are $\binom{13}{1}$ ways to choose $X$, $\binom{12}{1}$ ways to choose $Y$ and $\binom{11}{2}$ ways to choose $Z$ and $W$ - note the symmetry inherent in the last choice.

Then, our hand has 3 X’s out of the 4 total in the deck, which can be chosen $\binom{4}{3}$ ways. The 2 Y’s can be chosen $\binom{4}{2}$ ways, and both the Z and W chosen $\binom{4}{1}$ way. Thus the total number of 7 card hands with this structure is

$$\binom{13}{1} \binom{12}{1} \binom{11}{2} \binom{4}{3} \binom{4}{2} \binom{4}{1} = 3294720.$$  

2. **XXXYYZZ** The number of such hands can be calculated similarly to above as

$$\binom{13}{1} \binom{12}{2} \binom{4}{3} \binom{4}{2} \binom{4}{1} = 123552.$$  

3. **XXXYYYZ** Similarly this can be calculated as

$$\binom{13}{2} \binom{11}{1} \binom{4}{3} \binom{4}{1} = 54912.$$  

Note that by construction, these possibilities exclude any 4 of a kinds. The total number of possibilities is then the sum of these, which is 3,473,184. There are a total of

$$\binom{52}{7} = 133784560$$

possible 7 card hands, giving the probability of a Full House as

$$\frac{3473184}{133784560} \approx 0.02596$$

or 2.59%.

Solution 2. **Find the Probability of getting: Three of a Kind**

For a 7 card to have 3 of a kind but no 4 of a kind and no full house, it must clearly have 3 of one rank and then 4 other distinct ranks. In terms of rank structure used earlier, this looks like $XXXYZWV$
There are \( \binom{13}{5} \) to choose these 5 ranks, however 10 of these choices create straights. Then there are \( \binom{5}{1} \binom{4}{4} \) ways to divide up the ranks, choosing one to be X and then the other 4 as Y, Z, W, and V. Thus there are

\[
\left( \binom{13}{5} - 10 \right) \binom{5}{1} = 6385
\]

choices of ranks.

Next, we have to choose suits so as to avoid flushes. For the three X’s, there are \( \binom{4}{3} \) ways to choose the 3 suits represented. There are now \( \binom{4}{4} \) choices of suits for the other 4 cards, 3 of which - one for each suit represented in XXX - make a flush. Thus there are

\[
\binom{4}{3} \left( \binom{4}{1} - 3 \right) = 1012
\]

choices of suit.

In conclusion, there are

\[
6385 \times 1012 = 6461620
\]

ways to get three of a kind, giving probability \( \approx 4.82\% \).

Solution 3. Find the Probability of getting: Two Pair

We use a similar technique here, breaking down the different possible structures. We’ll consider two different structures of ranks and then the distribution of suits.

1. **XXYYZZW** There are only 4 ranks present so no need to worry about straights or flushes. There are \( \binom{13}{3} \) ways to choose X, Y and Z, and \( \binom{10}{1} \) ways to choose W. Then there are \( \binom{4}{2} \) ways to choose two X’s, similarly for Y and Z, and then \( \binom{4}{1} \) to choose a W. Thus there are

\[
\binom{13}{3} \binom{10}{1} \binom{4}{2} \binom{4}{1} = 2471040
\]

total possibilities.

2. **XXYYZWU** First we choose 5 ranks, which has \( \binom{13}{5} \) possibilities. However, we need to exclude the 10 possible choices that result in straights (check this!). Next there are \( \binom{5}{2} \) ways among the ranks we chose to choose 2 of them to be X and Y. Finally, we need to look at the number of possible distribution of suits.

First, the two pairs XX and YY could have 2 suits amongst them. This occurs \( \binom{4}{2} \) ways. There are then \( \binom{4}{3} \) ways to distribute suits to Z W and U, only 2 ways of which would make a flush.

Next possibility, the two pairs XX and YY have 3 suits amongst them, which occurs in \( \binom{4}{2} \binom{4}{1} \) ways. There are still \( \binom{4}{3} \) ways to distribute suits to Z W and U, and the only way to make a flush is if they are all the same suit as the two of XXYY which match, which occurs in 1 way.

Final possibility, XXYY consists of four different suits. This occurs in \( \binom{4}{1} \binom{2}{2} \) ways. There are again \( \binom{4}{3} \) further choices, none of which results in a flush.

Thus in this case the total is

\[
\left( \binom{13}{5} - 10 \right) \frac{5}{2} \left( \binom{4}{1} \binom{3}{2} - 2 \right) + \frac{4}{2} \frac{4}{1} \left( \binom{4}{3} - 1 \right) + \frac{4}{2} \frac{2}{1} \frac{4}{1} = 28962360.
\]
Therefore there are
\[ 2471040 + 28962360 = 31433400 \]
possible two pairs, giving a probability of 0.2349 or 23.49%.

\[ \square \]

**Solution 4. Find the Probability of getting: High Card**

First, note that for 7 cards to not have any made hand (other than a high card) they must be 1) all different ranks, 2) not a straight 3) not a flush. Since our 7 cards will have no pairs, we can choose the ranks and then suits independently.

First, we look at the number of ways to choose 7 different ranks without making a straight. There are \( \binom{13}{7} \) ways to choose 7 different ranks. To see the number of ways that would make a straight, lets count the number of ways to make each possible straight. First, to make an Ace to 5 straight, where the first five cards are A 2 3 4 5, the only cards we can’t choose for the last two are 6’s (otherwise it would be a 2 to 6 straight). So there are \( \binom{7}{2} \) ways to make an straight from the ace.

The same goes for all other straights except for the 10 J Q K A straight, for which we can fill in the hand with any other two ranks giving \( \binom{8}{2} \) possibilities. Therefore, the total number of choices of ranks without a straight is
\[
\binom{13}{7} - 9 \binom{7}{2} - \binom{8}{2} = 1499.
\]

Next, there are \( \binom{4}{7} \) ways to distribute suits (equiv. pick a card of) each of the 7 ranks. Some of these however make flushes. To see how many, we break it into three cases.

First, for 5 of the 7 being the same suit: there are \( \binom{4}{1} \) suits and \( \binom{7}{5} \) ways to pick 5 cards to make that suit. Then there are \( \binom{3}{1} \) choices of suit for the last two cards. For 6 of the same suit, there are \( \binom{4}{1} \) choices of suit, \( \binom{7}{6} \) choices of cards, and then \( \binom{3}{1} \) choices of suit for the last card. For 7 of the same suit there are \( \binom{4}{1} \) choices of suits and \( \binom{7}{7} \) choice of cards.

So, the total number of ways to get a flush are
\[
\binom{4}{1}\binom{7}{5}\binom{3}{1}^2 + \binom{4}{1}\binom{7}{6}\binom{3}{1} + \binom{4}{1}\binom{7}{7} = 844.
\]

The total number ways to distribute the suits without getting a flush are then
\[
\binom{4}{1}^7 - 844 = 15540.
\]

As the two choices are independent, the total number of high card hands is then
\[
1499 \times 15540 = 23294460,
\]
giving a probability of approximately 17.41%.

\[ \square \]