Worksheet for Week 2 Tuesday Solutions

1. Absolute Value
   (a) Please draw the possible values of \( t \) on a number line such that \(|t| < 2\).
   **Solution:** \(-2 < t < 2\). The drawing of the number line is omitted here. :) 

   (b) Find \( x \) so that \(|2x - 1| < 5\)
   **Solution:**
   \[-5 < 2x - 1 < 5\]
   \[-4 < 2x < 6\]
   \[-2 < x < 3\]

   (c) Find \( x \) so that \(|x^2 - 2x - 1| = 2\)
   **Solution:**
   \[x^2 - 2x - 1 = 2 \text{ or } x^2 - 2x - 1 = -2\]
   \[x^2 - 2x - 3 = 0 \text{ or } x^2 - 2x + 1 = 0\]
   \[(x - 3)(x + 1) = 0 \text{ or } (x - 1)^2 = 0\]
   \[x = 3, -1, 1\]

2. Secant Line and Tangent Line
   \(f(x) = \frac{2x+1}{x}\).
   (a) What is the slope of the secant line through the points \((1, f(1))\) and \((1 + \Delta x, f(1 + \Delta x))\)?
   (b) Write down a limit expressing the slope of the tangent line at \(x = 1\).
   (c) Find the slope of the tangent line to \(y = f(x)\) at \(x = 1\).
   (d) If \(f(x) = \frac{2x+1}{x} + 2016\) what are the answers to the part (a), (b) and (c)?
   (e) What if \(f(x) = \frac{2x+1}{x^2}\)? Can you tell the answer without any calculation?
   (f) \(\text{[Optional]}\) What is the slope of the tangent line to \(y = \frac{2x+1}{x}\) at \(x = a\). (assume \(a \neq 0\))
   **Solution:**
   (a)

   Slope of secant line \[= \frac{f(1 + \Delta x) - f(1)}{1 + \Delta x - 1}\]
   \[= \frac{\frac{3 + 2\Delta x}{1 + \Delta x} - 3}{\Delta x}\]
   \[= \frac{\frac{3 + 2\Delta x}{1 + \Delta x} - \frac{3 + 3\Delta x}{1 + \Delta x}}{\Delta x}\]
   \[= \frac{-\Delta x}{1 + \Delta x}\]
   \[= -\frac{1}{1 + \Delta x}\]

   (b) & (c)

   Slope of tangent line \[= \lim_{\Delta x \to 0} \frac{f(1 + \Delta x) - f(1)}{1 + \Delta x - 1}\]
   \[= \lim_{\Delta x \to 0} \frac{1}{1 + \Delta x}\]
   \[= -1\]
(d) The answers do not change. Since we are just moving the function up for 2016, and this procedure
does not change the slope of secant or tangent line.

In light of this, we could also compute (a)-(c) in a slightly easier way. Notice \( f(x) = 2 + \frac{1}{x} \)

(e) Notice \( f(x) = \frac{2x+1}{2x} = \frac{1}{2} \frac{2x+1}{x} \). So the slope of secant line is \(-\frac{1}{2(1+\Delta x)}\), and the slope of tangent line is \(-\frac{1}{2}\).

(f) This problem takes quite some computation.

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\begin{align*}
\lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x} &= \lim_{\Delta x \to 0} \frac{2a + 2 \Delta x + 1 - 2a}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{2a + 2 \Delta x + 1 - (2a + 1)(a + \Delta x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{-\Delta x}{(a + \Delta x)(a + \Delta x)} \\
&= \lim_{\Delta x \to 0} \frac{-\Delta x}{(a + \Delta x)^2} \\
&= -\frac{1}{a^2}
\end{align*}
\]

Another way:

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\begin{align*}
\lim_{x \to a} \frac{f(x) - f(a)}{x - a} &= \lim_{x \to a} \frac{2x + 1 - 2a}{x - a} \\
&= \lim_{x \to a} \frac{2ax + a - 2ax - x}{x - a} \\
&= \lim_{x \to a} \frac{a}{ax} \\
&= \lim_{x \to a} \frac{1}{ax} \\
&= \frac{1}{a^2}
\end{align*}
\]