A Strong Adjoint Functor Theorem

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1 Introduction

Here we discuss a sufficient and necessary condition for a presheaf to be representable. Let's briefly recall the notion of representability:

1.1 Definition. $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ is representable if there is an object $c \in \mathsf{C}$ such that

$$F \simeq \mathsf{C}(-, c).$$

Equivalently, suppose $y : \mathsf{C} \to \widehat{\mathsf{C}}$ is the Yoneda embedding. Then F is representable if F is in the essential image of y.

The notion of representability is very important: many theorems reduce to this notion of functors being representable, such as the adjoint functor theorem. Theorems like the Brown representability theorem and Artin representability also fall under this umbrella of stating that certain functors are representable.

Notice that this notion depends on sizes of categories. Typically C will be locally small, and Set or Spc (or whatever the desired presheaves should be taken in) is the category of all small sets or spaces. In this case, $\widehat{\mathsf{C}}$ is *not* the small cocompletion of C, and it is in fact larger than the small cocompletion. The small cocompletion of C consists of the small colimits of representable functors, which is exactly those that are accessible functors $\mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$. However this result works despite the relative sizes of the two categories C and Set.

I've first learned the following theorem from Kelly's book [Kel05] on enriched categories.

1.2 Theorem. *Given a $\mathcal{V}$-enriched C and a presheaf $F : \mathsf{C}^{\mathrm{op}} \to \mathcal{V}$. Then F is representable if and only if*

1. *The weighted colimit $LF := \mathrm{colim}_F 1_{\mathsf{C}}$ exists in C.*
2. *F sends this weighted colimit to a weighted limit. In other words, the natural comparison map*

$$F(r) \to \lim_F F$$

is an equivalence in $\mathcal{V}$.

He goes on to use his characterization of representability to understand representability and adjoint functor theorems and solution set axioms. Notice that adjoint functor theorems follow from representability ones as given $F : \mathsf{C} \to \mathsf{D}$: it has a right adjoint if and only if $\mathsf{D}(F-, d)$ is a representable presheaf for all $d \in \mathsf{D}$.

These representability theorems usually consist of two parts: a colimit/limit preservation piece, and a solution set axiom. The insight is that several of the solution set axioms are used in order to construct the perhaps large colimit $\mathrm{colim}_F 1_{\mathsf{C}}$, as this is clearly a large colimit if C itself is large.

The second piece, that the presheaf works well with limits/colimits guarentees the second piece of the theorem: that it turns this colimit $\mathrm{colim}_F 1_{\mathsf{C}}$ to a limit.

It is interesting to consider whether there are representability/adjoint functor theorems in greater generality. For $(\infty, 1)$-categories, Nguyen, Raptis, and Schrade have a direct generalization of the general adjoint functor theorem [NRS20]. Further, Bourke, Lack, and Vokřínek have done work in this direction for model theoretic enrichments, which covers the case of usual strict enrichments, but I wonder if work can be done with homotopy coherent enriched categories for example using Hinich's definition [Hin23].

We work in the setting of (ordinary) enriched categories, where $\mathcal{V}$ is a closed symmetric monoidal category, and our categories C are enriched over $\mathcal{V}$. The following arguments also work using $(\infty, 1)$-categories, for example in the style of Lurie/Joyal (for example see [Lur09]), which is not quite enrichment over Spc, but requires a weak sort of enrichment.

2 Proof of the theorem

The argument presented here is slightly different then the one given by Kelly in [Kel05],and it also works for $(\infty,1)$-categories. This of course also covers the case of ordinary Set-enriched categories (or "unenriched" categories).

2.1 Theorem. *Given a $\mathcal{V}$-enriched or $(\infty,1)$-category C and a presheaf $F : \mathsf{C}^{\mathrm{op}} \to \mathcal{V}$. Then F is representable if and only if*

1. *The weighted colimit $LF := \mathrm{colim}_F 1_{\mathsf{C}}$ exists in C.*
2. *F sends this weighted colimit to a weighted limit. In other words, the natural comparison map*

$$F(r) \to \lim_F F$$

is an equivalence in $\mathcal{V}$.

In more detail, so long as the colimit LF exists, we have a natural comparison $\eta_F : F \to \mathsf{C}(-, LF)$, which is an equivalence if the second point holds.

Proof. Notice that the construction of $LF := \mathrm{colim}_F 1_{\mathsf{C}}$ is exactly asking for a local left adjoint on F for the Yoneda embedding $y : \mathsf{C} \to \widehat{\mathsf{C}}$. Observe that by the very definition of the colimit, we have the following equivalence

$$\widehat{\mathsf{C}}(F, yc) \simeq \mathsf{C}(LF, c) \simeq \widehat{\mathsf{C}}(y(LF), yc) \tag{2.2}$$

where of course the second equivalence follows from the fully faithfulness of the Yoneda lemma. This shows that indeed if $y : \mathsf{C} \to \widehat{\mathsf{C}}$ were to have a left adjoint, it must send F to $\mathrm{colim}_F 1_C$.

Also notice that if $F = \mathsf{C}(-, r)$ is already representable, then notice that by 2.2, we see that

$$\mathsf{C}(LF, -) \simeq \widehat{\mathsf{C}}(F, y-) \simeq \mathsf{C}(r, -),$$

hence $LF \simeq r$. So we see that if F is representable, then clearly the colimit LF exists and is calculated by the representing object r.

Further if F is representable by r, then notice that clearly again by 2.2, we see

$$F(LF) \simeq \mathsf{C}(LF, r) \simeq \lim_F \mathsf{C}(-, r) \simeq \lim_F F$$

showing that F turns the colimit to a limit. Hence these points prove the forward direction.

Now we show the other direction. Suppose that LF exists in C, and that F turns this colimit into a limit.

First off, notice that the comparison morphism of colimit to limit

$$G(LF) \to \lim_F G \tag{2.3}$$

(which is natural in G) induced a natural morphism

$$\widehat{\mathsf{C}}(y(LF), G) \to \widehat{\mathsf{C}}(F, G). \tag{2.4}$$

This is because the domain $G(LF)$ is equivalent to $\widehat{\mathsf{C}}(y(LF), G)$ by the Yoneda lemma. Then, $\lim_F G \simeq \widehat{\mathsf{C}}(F, G)$ because we have the co-Yoneda lemma: $F \simeq \mathrm{colim}_F\, y$, so

$$\widehat{\mathsf{C}}(F, G) \simeq \widehat{\mathsf{C}}(\mathrm{colim}_F\, y, G) \simeq \lim_F \widehat{\mathsf{C}}(y, G) \simeq \lim_F G.$$

Now 2.4 induces a natural comparison morphism

$$\eta_F : F \to y(LF). \tag{2.5}$$

This morphism corresponds to the unit of the adjunction $y \dashv L$ even for L only locally defined, as the unit is also induced by the comparison of colimits

$$F \simeq \mathrm{colim}_F\, y \to y(\mathrm{colim}_F\, 1_C),$$

and one can see that by taking hom sets into G of this comparison map, we get exactly the other comparison map 2.4. We need to show that this morphism is an equivalence. But this follows from a categorical lemma:

2.6 Lemma. *Let* C *be a* $\mathcal{V}$*-enriched or* $(\infty,1)$*-category and* $f : c \to d$ *a morphism. Then assume that*

$$f^* : \mathsf{C}(d,c) \to \mathsf{C}(c,c)$$

is an equivalence. Then f *is an equivalence.*

We notice that we have this exact situation for the morphism η_F as f. The fact that precomposition by η_F is an equivalence on

$$\widehat{\mathsf{C}}(y(LF),F) \to \widehat{\mathsf{C}}(F,F)$$

is exactly our second hypothesis, that F turns the colimit LF into a limit, using the same argument that 2.3 and 2.4 are equivalent morphisms.

Now let's prove the lemma. Suppose we're given $f : c \to d$ as in the lemma. We first notice that f has a left inverse because

$$f^*\mathsf{C}(d,c) \to \mathsf{C}(c,c)$$

is an equivalence. We get this inverse $e : d \to c$ as the inverse image of the identity 1_c, hence $ef \simeq 1_c$.

Next we draw the natural squares

$$\begin{tikzcd}
\mathsf{C}(d,d) \arrow[r, "f^*"] \arrow[d, "f_*"] & \mathsf{C}(c,d) \arrow[d, "f_*"] \\
\mathsf{C}(d,c) \arrow[r, "f^*"] \arrow[d, "e_*"] & \mathsf{C}(c,c) \arrow[d, "e_*"] \\
\mathsf{C}(d,d) \arrow[r, "f^*"] & \mathsf{C}(c,d)
\end{tikzcd}$$

from the fact that pre-composition by f commutes with post-composition. Notice that now the top and bottom morphism $f^* : \mathsf{C}(d,d) \to \mathsf{C}(c,d)$ is an equivalence as it is a retract of the middle morphism, which is an equivalence! This is because the vertical compositions are equivalences since $ef \simeq 1_c$.

Now since $f^* : \mathsf{C}(d,d) \to \mathsf{C}(c,d)$ is invertible, we see that f must have a right inverse! Hence f has a left and right inverse and is thus an equivalence itself.

□

3 Adjoint functor theorem

Our main representability theorem 2.1 gives a related adjoint functor theorem. Once again, I learned this formulation from Kelly [Kel05].

3.1 Theorem. *Given a functor* $F : \mathsf{C} \to \mathsf{D}$ *of enriched categories. Then it has a right adjoint if and only it the left Kan extension* $G := \mathrm{Lan}_F 1_{\mathsf{C}}$ *exists and is preserved by* F*.*

Kelly uses the representability theorem 2.1 to prove this, applying it to the functors $\mathsf{D}(F-,d)$. We however give a direct argument that works in any 2-category C, including $(\infty,2)$-categories. For example, I believe it works on Riehl and Verity's notions of ∞-cosmoi for doing higher categories [RV22]. Note however this is just as basic as the representability theorem above; however in a general 2-category one doesn't have the notion of representability necessarily. One can probably just instead use the Yoneda embedding $\mathsf{C} \to [\mathsf{C}^{\mathrm{op}}, \mathsf{Cat}]$ and apply the adjoint functor theorem for Cat, but we give a direct argument for simplicity:

3.2 Theorem. *Let* C *be a* 2*-category (or an* $(\infty,2)$*-category). Then given a morphism* $f : c \to d$*, it has a right adjoint iff* $\mathrm{Lan}_f 1_c$ *exists and* f *preserves this left Kan extension.*

Proof. For the forward direction, notice that if f has a right adjoint g, then clearly g itself computes $\mathrm{Lan}_f 1_c$: we have the following equivalence

$$[c,c](1_c, hf) \simeq [d,c](g,h)$$

by the unit η and counit ε. In particular: given $u : 1_c \to hf$ we construct $\hat{u} : g \to h$ as the composite

$$g \xrightarrow{u \circ g} hfg \xrightarrow{h \circ \varepsilon} h.$$

Without refering to explicit elements, we can write the equivalence

$$[c,c](1_c, hf) \simeq [d,c](g,h)$$

as given by

$$[c,c](1_c, hf) \xrightarrow{g_{1,*}} [c,d](g, hfg) \xrightarrow{h\varepsilon_*} [c,d](g,h)$$

where $g_{1,*}$ is post-composition of 1-cells with g (ie "horizontal composition").

To go the other way, we start with $v : g \to h$ and obtain $\hat{v} : 1 \to hf$ by taking the composite

$$1_c \xrightarrow{\eta} gf \xrightarrow{v \circ f} hf.$$

These two constructions clearly invert each other because of the unit and counit laws.

Again on hom categories, we could write

$$[c,d](g,h) \xrightarrow{f^{1,*}} [c,c](gf, hf) \xrightarrow{\eta^*} [c,c](1_c, hf)$$

where $f^{1,*}$ is again precomposition of 1-cells.

Hence we see the left kan extension $\mathrm{Lan}_f\, 1_c$ exists. Further, we note that f preserves this left kan extension: we have the equivalence

$$[c,d](f, hf) \simeq [c,c](fg, h)$$

by an analogous argument to above, using the unit and counit.

Now for the more interesting direction: let us write $g := \mathrm{Lan}_f\, 1_c$. Then we have a natural unit 2-cell $\eta : 1_c \to gf$ from the equivalence

$$[d,c](g,h) \simeq [c,c](1_c, hf). \tag{3.3}$$

Here η arises from plugging in $h = g$ and tracking the identity of g on the left hand side.

Next, we observe fg is the Left Kan extension of f along f, hence we have an equivalence

$$[c,c](f, hf) \simeq [d,c](fg, h). \tag{3.4}$$

Letting $h = 1_d$ and plugging in the identity of f gives us a counit $\varepsilon : fg \to 1_d$.

We now need to check that these satisfy the triangle identities. Let's begin with $\varepsilon f \circ f\eta \simeq 1_f$. Notice that the fact that f preserves the LKE implies that since $\eta : 1 \to gf$ is the unit for g as a LKE, we have $f\eta : f \to fgf$ is the unit for fg as a LKE.

Hence the adjunct to $f\eta$ is the identity $fg \to fg$. Now observe we have the following square:

$$\begin{tikzcd}
{[c,d](f, fgf)} \arrow[r, "\simeq"] \arrow[d, "(\varepsilon f)_*"] & {[d,d](fg, fg)} \arrow[d, "\varepsilon_*"] \\
{[c,d](f,f)} \arrow[r, "\simeq"] & {[d,d](fg, 1_d).}
\end{tikzcd}$$

It commutes by the naturality of the equivalence 3.4. Notice that if we trace $F\eta$ on the upper left: it goes to the right to 1_{fg} as they are adjuncts, which then maps to ε. The other way, we get the adjunct of $\varepsilon_f \circ f\eta$. Since the adjunct of $\varepsilon_f \circ f\eta$ is ε, it must be 1_f, as this is the definition of η! Hence $\varepsilon_f \circ f\eta \simeq 1_f$.

Now the second triangle identity follows from the first: we want to show that $G\varepsilon \circ \eta G$ is 1_g. To do so, we show that it is adjunct to $\eta : 1 \to gf$.

We calculate the adjunct to be the composite

$$\begin{tikzcd}
1_g \arrow[r, "\eta"] \arrow[d, "\eta"] & gf \arrow[d, "gf\eta"] & \\
gf \arrow[r, "\eta gf"] & gfgf \arrow[r, "g\varepsilon f"] & gf.
\end{tikzcd}$$

The square commutes because it computes the horizontal composition $\eta * \eta$. Notice that the top route then composes to η, using our previous triangle identity! Hence our adjunct is indeed η, and we've shown the second triangle identity. $\square$

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