9 An Introduction to Elliptic Functions

> The form that Jacobi had given to the theory of elliptic functions was far from perfection; its flaws are obvious. At the base we find three fundamental functions sn, cn and dn. These functions do not have the same periods...
>
> In Weierstrass' system, instead of three fundamental functions, there is only one, $\wp(u)$, and it is the simplest of all having the same periods. It has only one double infinity; and finally its definition is so that it does not change when one replaces one system of periods by another equivalent system.
>
> *H. Poincaré,* 1899

The theory of elliptic functions, which is of interest in several parts of mathematics, initially grew out of the study of elliptic integrals. These can be described generally as integrals of the form $\int R(x, \sqrt{P(x)})\,dx$, where R is a rational function and P a polynomial of degree three or four.[1] These integrals arose in computing the arc-length of an ellipse, or of a lemniscate, and in a variety of other problems. Their early study was centered on their special transformation properties and on the discovery of an inherent double-periodicity. We have seen an example of this latter phenomenon in the mapping function of the half-plane to a rectangle taken up in Section 4.5 of the previous chapter.

It was Jacobi who transformed the subject by initiating the systematic study of doubly-periodic functions (called elliptic functions). In this theory, the theta functions he introduced played a decisive role. Weierstrass after him developed another approach, which in its initial steps is simpler and more elegant. It is based on his $\wp$ function, and in this chapter we shall sketch the beginnings of that theory. We will go as far as to glimpse a possible connection with number theory, by considering the Eisenstein series and their expression involving divisor functions. A number of more direct links with combinatorics and number theory arise from the theta

[1] The case when P is a quadratic polynomial is essentially that of "circular functions", and can be reduced to the trigonometric functions $\sin x$, $\cos x$, etc.

functions, which we will take up in the next chapter. The remarkable facts we shall see there attest to the great interest of these functions in mathematics. As such they ought to soften the harsh opinion expressed above about the imperfection of Jacobi's theory.

1 Elliptic functions

We are interested in meromorphic functions f on $\mathbb{C}$ that have two periods; that is, there are two non-zero complex numbers ω_1 and ω_2 such that

$$f(z+\omega_1) = f(z) \quad \text{and} \quad f(z+\omega_2) = f(z),$$

for all $z \in \mathbb{C}$. A function with two periods is said to be **doubly periodic**.

The case when ω_1 and ω_2 are linearly dependent over $\mathbb{R}$, that is $\omega_2/\omega_1 \in \mathbb{R}$, is uninteresting. Indeed, Exercise 1 shows that in this case f is either periodic with a simple period (if the quotient ω_2/ω_1 is rational) or f is constant (if ω_2/ω_1 is irrational). Therefore, we make the following assumption: the periods ω_1 and ω_2 are linearly independent over $\mathbb{R}$.

We now describe a normalization that we shall use extensively in this chapter. Let $\tau = \omega_2/\omega_1$. Since τ and $1/\tau$ have imaginary parts of opposite signs, and since τ is not real, we may assume (after possibly interchanging the roles of ω_1 and ω_2) that $\mathrm{Im}(\tau) > 0$. Observe now that the function f has periods ω_1 and ω_2 if and only if the function $F(z) = f(\omega_1 z)$ has periods 1 and τ, and moreover, the function f is meromorphic if and only if F is meromorphic. Also the properties of f are immediately deducible from those of F. We may therefore assume, without loss of generality, that f is a meromorphic function on $\mathbb{C}$ with periods 1 and τ where $\mathrm{Im}(\tau) > 0$.

Successive applications of the periodicity conditions yield

$$f(z+n+m\tau) = f(z) \quad \text{for all integers } n, m \text{ and all } z \in \mathbb{C}, \tag{1}$$

and it is therefore natural to consider the lattice in $\mathbb{C}$ defined by

$$\Lambda = \{n + m\tau : \; n, m \in \mathbb{Z}\}.$$

We say that 1 and τ **generate** Λ (see Figure 1).

Equation (1) says that f is constant under translations by elements of Λ. Associated to the lattice Λ is the **fundamental parallelogram** defined by

$$P_0 = \{z \in \mathbb{C} : z = a + b\tau \;\text{ where } 0 \le a < 1 \text{ and } 0 \le b < 1\}.$$

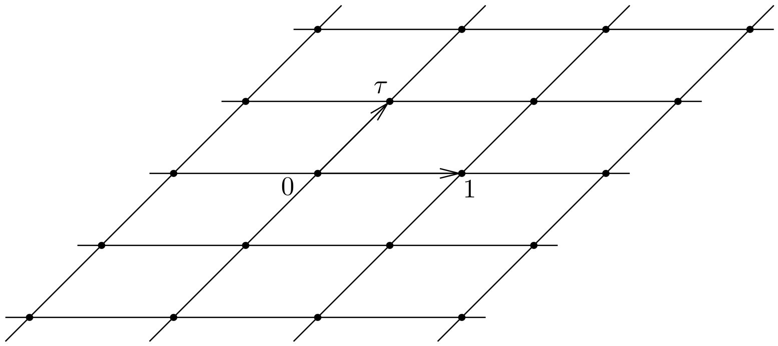

Figure 1. The lattice Λ generated by 1 and τ

The importance of the fundamental parallelogram comes from the fact that f is completely determined by its behavior on P_0. To see this, we need a definition: two complex numbers z and w are **congruent modulo** Λ if

$$z = w + n + m\tau \qquad \text{for some } n, m \in \mathbb{Z},$$

and we write $z \sim w$. In other words, z and w differ by a point in the lattice, $z - w \in \Lambda$. By (1) we conclude that $f(z) = f(w)$ whenever $z \sim w$. If we can show that any point in $z \in \mathbb{C}$ is congruent to a unique point in P_0 then we will have proved that f is completely determined by its values in the fundamental parallelogram. Suppose $z = x + iy$ is given, and write $z = a + b\tau$ where $a, b \in \mathbb{R}$. This is possible since 1 and τ form a basis over the reals of the two-dimensional vector space $\mathbb{C}$. Then choose n and m to be the greatest integers $\leq a$ and $\leq b$, respectively. If we let $w = z - n - m\tau$, then by definition $z \sim w$, and moreover $w = (a - n) + (b - m)\tau$. By construction, it is clear that $w \in P_0$. To prove uniqueness, suppose that w and w' are two points in P_0 that are congruent. If we write $w = a + b\tau$ and $w' = a' + b'\tau$, then $w - w' = (a - a') + (b - b')\tau \in \Lambda$, and therefore both $a - a'$ and $b - b'$ are integers. But since $0 \leq a, a' < 1$, we have $-1 < a - a' < 1$, which then implies $a - a' = 0$. Similarly $b - b' = 0$, and we conclude that $w = w'$.

More generally, a **period parallelogram** P is any translate of the fundamental parallelogram, $P = P_0 + h$ with $h \in \mathbb{C}$ (see Figure 2).

Since we can apply the lemma to $z - h$, we conclude that every point in $\mathbb{C}$ is congruent to a unique point in a given period parallelogram. Therefore, f is uniquely determined by its behavior on any period parallelogram.

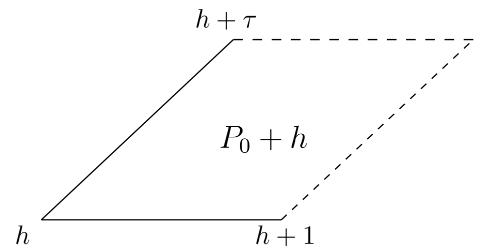

Figure 2. A period parallelogram

Finally, note that Λ and P_0 give rise to a covering (or tiling) of the complex plane

$$\mathbb{C} = \bigcup_{n,m\in\mathbb{Z}} (n + m\tau + P_0), \tag{2}$$

and moreover, this union is disjoint. This is immediate from the facts we just collected and the definition of P_0. We summarize what we have seen so far.

Proposition 1.1 *Suppose f is a meromorphic function with two periods 1 and τ which generate the lattice Λ. Then:*

(i) *Every point in $\mathbb{C}$ is congruent to a unique point in the fundamental parallelogram.*

(ii) *Every point in $\mathbb{C}$ is congruent to a unique point in any given period parallelogram.*

(iii) *The lattice Λ provides a disjoint covering of the complex plane, in the sense of* (2).

(iv) *The function f is completely determined by its values in any period parallelogram.*

1.1 Liouville's theorems

We can now see why we assumed from the beginning that f is meromorphic rather than just holomorphic.

Theorem 1.2 *An entire doubly periodic function is constant.*

Proof. The function is completely determined by its values on P_0 and since the closure of P_0 is compact, we conclude that the function is bounded on $\mathbb{C}$, hence constant by Liouville's theorem in Chapter 2.

A non-constant doubly periodic meromorphic function is called an **elliptic function**. Since a meromorphic function can have only finitely many zeros and poles in any large disc, we see that an elliptic function will have only finitely many zeros and poles in any given period parallelogram, and in particular, this is true in the fundamental parallelogram. Of course, nothing excludes f from having a pole or zero on the boundary of P_0.

As usual, we count poles and zeros with multiplicities. Keeping this in mind we can prove the following theorem.

Theorem 1.3 *The total number of poles of an elliptic function in* P_0 *is always* ≥ 2.

In other words, f cannot have only one simple pole. It must have at least two poles, and this does not exclude the case of a single pole of multiplicity ≥ 2.

Proof. Suppose first that f has no poles on the boundary ∂P_0 of the fundamental parallelogram. By the residue theorem we have

$$\int_{\partial P_0} f(z)\, dz = 2\pi i \sum \operatorname{res} f,$$

and we contend that the integral is 0. To see this, we simply use the periodicity of f. Note that

$$\int_{\partial P_0} f(z)\, dz = \int_0^1 f(z)\, dz + \int_1^{1+\tau} f(z)\, dz + \int_{1+\tau}^{\tau} f(z)\, dz + \int_{\tau}^{0} f(z)\, dz,$$

and the integrals over opposite sides cancel out. For instance

$$\begin{aligned} \int_0^1 f(z)\, dz + \int_{1+\tau}^{\tau} f(z)\, dz &= \int_0^1 f(z)\, dz + \int_1^0 f(s+\tau)\, ds \\ &= \int_0^1 f(z)\, dz + \int_1^0 f(s)\, ds \\ &= \int_0^1 f(z)\, dz - \int_0^1 f(z)\, dz \\ &= 0, \end{aligned}$$

and similarly for the other pair of sides. Hence $\int_{\partial P_0} f = 0$ and $\sum \operatorname{res} f = 0$. Therefore f must have at least two poles in P_0.

If f has a pole on ∂P_0 choose a small $h \in \mathbb{C}$ so that if $P = h + P_0$, then f has no poles on ∂P. Arguing as before, we find that f must have at least two poles in P, and therefore the same conclusion holds for P_0.

The total number of poles (counted according to their multiplicities) of an elliptic function is called its **order**. The next theorem says that elliptic functions have as many zeros as they have poles, if the zeros are counted with their multiplicities.

Theorem 1.4 *Every elliptic function of order m has m zeros in P_0.*

Proof. Assuming first that f has no zeros or poles on the boundary of P_0, we know by the argument principle in Chapter 3 that

$$\int_{\partial P_0} \frac{f'(z)}{f(z)}\, dz = 2\pi i(\mathcal{N}_{\mathfrak{z}} - \mathcal{N}_{\mathfrak{p}})$$

where $\mathcal{N}_{\mathfrak{z}}$ and $\mathcal{N}_{\mathfrak{p}}$ denote the number of zeros and poles of f in P_0, respectively. By periodicity, we can argue as in the proof of the previous theorem to find that $\int_{\partial P_0} f'/f = 0$, and therefore $\mathcal{N}_{\mathfrak{z}} = \mathcal{N}_{\mathfrak{p}}$.

In the case when a pole or zero of f lies on ∂P_0 it suffices to apply the argument to a translate of P.

As a consequence, if f is elliptic then the equation $f(z) = c$ has as many solutions as the order of f for every $c \in \mathbb{C}$, simply because $f - c$ is elliptic and has as many poles as f.

Despite the rather simple nature of the theorems above, there remains the question of showing that elliptic functions exist. We now turn to a constructive solution of this problem.

1.2 The Weierstrass $\wp$ function

An elliptic function of order two

This section is devoted to the basic example of an elliptic function. As we have seen above, any elliptic function must have at least two poles; we shall in fact construct one whose only singularity will be a double pole at the points of the lattice generated by the periods.

Before looking at the case of doubly-periodic functions, let us first consider briefly functions with only a single period. If one wished to construct a function with period 1 and poles at all the integers, a simple choice would be the sum

$$F(z) = \sum_{n=-\infty}^{\infty} \frac{1}{z+n}.$$

Note that the sum remains unchanged if we replace z by $z + 1$, and the poles are at the integers. However, the series defining F is not absolutely convergent, and to remedy this problem, we sum symmetrically, that is, we define

$$F(z) = \lim_{N\to\infty} \sum_{|n|\le N} \frac{1}{z+n} = \frac{1}{z} + \sum_{n=1}^{\infty}\left[\frac{1}{z+n} + \frac{1}{z-n}\right].$$

On the far right-hand side, we have paired up the terms corresponding to n and $-n$, a trick which makes the quantity in brackets $O(1/n^2)$, and hence the last sum is absolutely convergent. As a consequence, F is meromorphic with poles precisely at the integers. In fact, we proved earlier in Chapter 5 that $F(z) = \pi\cot\pi z$.

There is a second way to deal with the series $\sum_{-\infty}^{\infty} 1/(z+n)$, which is to write it as

$$\frac{1}{z} + \sum_{n\neq 0}\left[\frac{1}{z+n} - \frac{1}{n}\right],$$

where the sum is taken over all non-zero integers. Notice that $1/(z+n) - 1/n = O(1/n^2)$, which makes this series absolutely convergent. Moreover, since

$$\frac{1}{z+n} + \frac{1}{z-n} = \left(\frac{1}{z+n} - \frac{1}{n}\right) + \left(\frac{1}{z-n} - \frac{1}{-n}\right),$$

we get the same sum as before.

In analogy to this, the idea is to mimic the above to produce our first example of an elliptic function. We would like to write it as

$$\sum_{\omega\in\Lambda} \frac{1}{(z+\omega)^2},$$

but again this series does not converge absolutely. There are several approaches to try to make sense of this series (see Problem 1), but the simplest is to follow the second way we dealt with the cotangent series.

To overcome the non-absolute convergence of the series, let Λ^* denote the lattice minus the origin, that is, $\Lambda^* = \Lambda - \{(0,0)\}$, and consider instead the following series:

$$\frac{1}{z^2} + \sum_{\omega\in\Lambda^*}\left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2}\right],$$

where we have subtracted the factor $1/\omega^2$ to make the sum converge. The term in brackets is now

$$\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} = \frac{-z^2 - 2z\omega}{(z+\omega)^2\omega^2} = O\left(\frac{1}{\omega^3}\right) \qquad \text{as } |\omega|\to\infty,$$

and the new series will define a meromorphic function with the desired poles once we have proved the following lemma.

Lemma 1.5 *The two series*

$$\sum_{(n,m)\neq(0,0)} \frac{1}{(|n|+|m|)^r} \qquad \textit{and} \qquad \sum_{n+m\tau\in\Lambda^*} \frac{1}{|n+m\tau|^r}$$

converge if $r>2$.

Recall that according to the Note at the end of Chapter 7, the question whether a double series converges absolutely is independent of the order of summation. In the present case, we shall first sum in m and then in n.

For the first series, the usual integral comparison can be applied.[2] For each $n \neq 0$

$$\begin{aligned}\sum_{m\in\mathbb{Z}} \frac{1}{(|n|+|m|)^r} &= \frac{1}{|n|^r} + 2\sum_{m\geq 1} \frac{1}{(|n|+|m|)^r} \\ &= \frac{1}{|n|^r} + 2\sum_{k\geq |n|+1} \frac{1}{k^r} \\ &\leq \frac{1}{|n|^r} + 2\int_{|n|}^{\infty} \frac{dx}{x^r} \\ &\leq \frac{1}{|n|^r} + C\frac{1}{|n|^{r-1}}.\end{aligned}$$

Therefore, $r>2$ implies

$$\begin{aligned}\sum_{(n,m)\neq(0,0)} \frac{1}{(|n|+|m|)^r} &= \sum_{|m|\neq 0} \frac{1}{|m|^r} + \sum_{|n|\neq 0}\sum_{m\in\mathbb{Z}} \frac{1}{(|n|+|m|)^r} \\ &\leq \sum_{|m|\neq 0} \frac{1}{|m|^r} + \sum_{|n|\neq 0}\left(\frac{1}{|n|^r} + C\frac{1}{|n|^{r-1}}\right) \\ &< \infty.\end{aligned}$$

To prove that the second series also converges, it suffices to show that there is a constant c such that

$$|n|+|m| \leq c|n+\tau m| \qquad \text{for all } n,m\in\mathbb{Z}.$$

[2] We simply use $1/k^r \leq 1/x^r$ when $k-1\leq x\leq k$; see also the first figure in Chapter 8, Book I.

We use the notation $x \lesssim y$ if there exists a positive constant a such that $x \leq ay$. We also write $x \approx y$ if both $x \lesssim y$ and $y \lesssim x$ hold. Note that for any two positive numbers A and B, one has

$$(A^2 + B^2)^{1/2} \approx A + B.$$

On the one hand $A \leq (A^2 + B^2)^{1/2}$ and $B \leq (A^2 + B^2)^{1/2}$, so that $A + B \leq 2(A^2 + B^2)^{1/2}$. On the other hand, it suffices to square both sides to see that $(A^2 + B^2)^{1/2} \leq A + B$.

The proof that the second series in Lemma 1.5 converges is now a consequence of the following observation:

$$|n| + |m| \approx |n + m\tau| \qquad \text{whenever } \tau \in \mathbb{H}.$$

Indeed, if $\tau = s + it$ with $s, t \in \mathbb{R}$ and $t > 0$, then

$$|n + m\tau| = [(n + ms)^2 + (mt)^2]^{1/2} \approx |n + ms| + |mt| \approx |n + ms| + |m|,$$

by the previous observation. Then, $|n + ms| + |m| \approx |n| + |m|$, by considering separately the cases when $|n| \leq 2\,|m|\,|s|$ and $|n| \geq 2\,|m|\,|s|$.

Remark. The proof above shows that when $r > 2$ the series $\sum |n + m\tau|^{-r}$ converges uniformly in every half-plane $\mathrm{Im}(\tau) \geq \delta > 0$. In contrast, when $r = 2$ this series fails to converge (Exercise 3).

With this technical point behind us, we may now return to the definition of the **Weierstrass $\wp$ function**, which is given by the series

$$\begin{aligned}
\wp(z) &= \frac{1}{z^2} + \sum_{\omega \in \Lambda^*} \left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} \right] \\
&= \frac{1}{z^2} + \sum_{(n,m) \neq (0,0)} \left[\frac{1}{(z + n + m\tau)^2} - \frac{1}{(n + m\tau)^2} \right].
\end{aligned}$$

We claim that $\wp$ is a meromorphic function with double poles at the lattice points. To see this, suppose that $|z| < R$, and write

$$\wp(z) = \frac{1}{z^2} + \sum_{|\omega| \leq 2R} \left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} \right] + \sum_{|\omega| > 2R} \left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} \right].$$

The term in the second sum is $O(1/|\omega|^3)$ uniformly for $|z| < R$, so by Lemma 1.5 this second sum defines a holomorphic function in $|z| < R$. Finally, note that the first sum exhibits double poles at the lattice points in the disc $|z| < R$.

Observe that because of the insertion of the terms $-1/\omega^2$, it is no longer obvious whether $\wp$ is doubly periodic. Nevertheless this is true, and $\wp$ has all the properties of an elliptic function of order 2. We gather this result in a theorem.

Theorem 1.6 *The function $\wp$ is an elliptic function that has periods 1 and τ, and double poles at the lattice points.*

Proof. It remains only to prove that $\wp$ is periodic with the correct periods. To do so, note that the derivative is given by differentiating the series for $\wp$ termwise so

$$\wp'(z) = -2 \sum_{n,m\in\mathbb{Z}} \frac{1}{(z+n+m\tau)^3}.$$

This accomplishes two things for us. First, the differentiated series converges absolutely whenever z is not a lattice point, by the case $r = 3$ of Lemma 1.5. Second, the differentiation also eliminates the subtraction term $1/\omega^2$; therefore the series for $\wp'$ is clearly periodic with periods 1 and τ, since it remains unchanged after replacing z by $z+1$ or $z+\tau$.

Hence, there are two constants a and b such that

$$\wp(z+1) = \wp(z) + a \quad \text{and} \quad \wp(z+\tau) = \wp(z) + b.$$

It is clear from the definition, however, that $\wp$ is even, that is, $\wp(z) = \wp(-z)$, since the sum over $\omega \in \Lambda$ can be replaced by the sum over $-\omega \in \Lambda$. Therefore $\wp(-1/2) = \wp(1/2)$ and $\wp(-\tau/2) = \wp(\tau/2)$, and setting $z = -1/2$ and $z = -\tau/2$, respectively, in the two expressions above proves that $a = b = 0$.

A direct proof of the periodicity of $\wp$ can be given without differentiation; see Exercise 4.

Properties of $\wp$

Several remarks are in order. First, we have already observed that $\wp$ is even, and therefore $\wp'$ is odd. Since $\wp'$ is also periodic with periods 1 and τ, we find that

$$\wp'(1/2) = \wp'(\tau/2) = \wp'\left(\frac{1+\tau}{2}\right) = 0.$$

Indeed, one has, for example,

$$\wp'(1/2) = -\wp'(-1/2) = -\wp'(-1/2+1) = -\wp'(1/2).$$

Since $\wp'$ is elliptic and has order 3, the three points $1/2$, $\tau/2$, and $(1+\tau)/2$ (which are called the **half-periods**) are the only roots of $\wp'$ in the fundamental parallelogram, and they have multiplicity 1. Therefore, if we define

$$\wp(1/2) = e_1, \quad \wp(\tau/2) = e_2, \quad \text{and} \quad \wp\left(\frac{1+\tau}{2}\right) = e_3,$$

we conclude that the equation $\wp(z) = e_1$ has a double root at $1/2$. Since $\wp$ has order 2, there are no other solutions to the equation $\wp(z) = e_1$ in the fundamental parallelogram. Similarly the equations $\wp(z) = e_2$ and $\wp(z) = e_3$ have only double roots at $\tau/2$ and $(1+\tau)/2$, respectively. In particular, the three numbers e_1, e_2, and e_3 are distinct, for otherwise $\wp$ would have at least four roots in the fundamental parallelogram, contradicting the fact that $\wp$ has order 2. From these observations we can prove the following theorem.

Theorem 1.7 *The function $(\wp')^2$ is the cubic polynomial in $\wp$*

$$(\wp')^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3).$$

Proof. The only roots of $F(z) = (\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3)$ in the fundamental parallelogram have multiplicity 2 and are at the points $1/2, \tau/2$, and $(1+\tau)/2$. Also, $(\wp')^2$ has double roots at these points. Moreover, F has poles of order 6 at the lattice points, and so does $(\wp')^2$ (because $\wp'$ has poles of order 3 there). Consequently $(\wp')^2/F$ is holomorphic and still doubly-periodic, hence this quotient is constant. To find the value of this constant we note that for z near 0, one has

$$\wp(z) = \frac{1}{z^2} + \cdots \quad \text{and} \quad \wp'(z) = \frac{-2}{z^3} + \cdots,$$

where the dots indicate terms of higher order. Therefore the constant is 4, and the theorem is proved.

We next demonstrate the universality of $\wp$ by showing that every elliptic function is a simple combination of $\wp$ and $\wp'$.

Theorem 1.8 *Every elliptic function f with periods 1 and τ is a rational function of $\wp$ and $\wp'$.*

The theorem will be an easy consequence of the following version of it.

Lemma 1.9 *Every even elliptic function F with periods 1 and τ is a rational funcion of $\wp$.*

Proof. If F has a zero or pole at the origin it must be of even order, since F is an even function. As a consequence, there exists an integer m so that $F\wp^m$ has no zero or pole at the lattice points. We may therefore assume that F itself has no zero or pole on Λ.

Our immediate goal is to use $\wp$ to construct a doubly-periodic function G with precisely the same zeros and poles as F. To achieve this, we recall that $\wp(z) - \wp(a)$ has a single zero of order 2 if a is a half-period, and two distinct zeros at a and $-a$ otherwise. We must therefore carefully count the zeros and poles of F.

If a is a zero of F, then so is $-a$, since F is even. Moreover, a is congruent to $-a$ if and only if it is a half-period, in which case the zero is of even order. Therefore, if the points $a_1, -a_1, \ldots, a_m, -a_m$ counted with multiplicities[3] describe all the zeros of F, then

$$[\wp(z) - \wp(a_1)] \cdots [\wp(z) - \wp(a_m)]$$

has precisely the same roots as F. A similar argument, where $b_1, -b_1, \ldots, b_m, -b_m$ (with multiplicities) describe all the poles of F, then shows that

$$G(z) = \frac{[\wp(z) - \wp(a_1)] \cdots [\wp(z) - \wp(a_m)]}{[\wp(z) - \wp(b_1)] \cdots [\wp(z) - \wp(b_m)]}$$

is periodic and has the same zeros and poles as F. Therefore, F/G is holomorphic and doubly-periodic, hence constant. This concludes the proof of the lemma.

To prove the theorem, we first recall that $\wp$ is even while $\wp'$ odd. We then write f as a sum of an even and an odd function,

$$f(z) = f_{\text{even}}(z) + f_{\text{odd}}(z),$$

where in fact

$$f_{\text{even}}(z) = \frac{f(z) + f(-z)}{2} \quad \text{and} \quad f_{\text{odd}}(z) = \frac{f(z) - f(-z)}{2}.$$

Then, since $f_{\text{odd}}/\wp'$ is even, it is clear from the lemma applied to f_{even} and $f_{\text{odd}}/\wp'$ that f is a rational function of $\wp$ and $\wp'$.

[3]If a_j is not a half-period, then a_j and $-a_j$ have the multiplicity of F at these points. If a_j is a half-period, then a_j and $-a_j$ are congruent and each has multiplicity half of the multiplicity of F at this point.

2 The modular character of elliptic functions and Eisenstein series

We shall now study the modular character of elliptic functions, that is, their dependence on τ.

Recall the normalization we made at the beginning of the chapter. We started with two periods ω_1 and ω_2 linearly that are independent over $\mathbb{R}$, and we defined $\tau = \omega_2/\omega_1$. We could then assume that $\mathrm{Im}(\tau) > 0$, and also that the two periods are 1 and τ. Next, we considered the lattice generated by 1 and τ and constructed the function $\wp$, which is elliptic of order 2 with periods 1 and τ. Since the construction of $\wp$ depends on τ, we could write $\wp_\tau$ instead. This leads us to change our point of view and think of $\wp_\tau(z)$ primarily as a function of τ. This approach yields many interesting new insights.

Our considerations are guided by the following observations. First, since 1 and τ generate the periods of $\wp_\tau(z)$, and 1 and $\tau+1$ generate the same periods, we can expect a close relationship between $\wp_\tau(z)$ and $\wp_{\tau+1}(z)$. In fact, it is easy to see that they are identical. Second, since $\tau = \omega_2/\omega_1$, by the normalization imposed at the beginning of Section 1, we see that $-1/\tau = -\omega_1/\omega_2$ (with $\mathrm{Im}(-1/\tau) > 0$). This corresponds essentially to an interchange of the two periods ω_1 and ω_2, and thus we can also expect an intimate connection between $\wp_\tau$ and $\wp_{-1/\tau}$. In fact, it is easy to verify that $\wp_{-1/\tau}(z) = \tau^2\wp_\tau(\tau z)$.

So we are led to consider the group of transformations of the upper half-plane $\mathrm{Im}(\tau) > 0$, generated by the two transformations $\tau \mapsto \tau + 1$ and $\tau \mapsto -1/\tau$. This group is called the **modular group**. On the basis of what we said, it can be expected that all quantities intrinsically attached to $\wp_\tau(z)$ reflect the above transformations. We see this clearly when we consider the Eisenstein series.

2.1 Eisenstein series

The **Eisenstein series** of order k is defined by

$$E_k(\tau) = \sum_{(n,m)\neq(0,0)} \frac{1}{(n+m\tau)^k},$$

whenever k is an integer ≥ 3 and τ is a complex number with $\mathrm{Im}(\tau) > 0$. If Λ is the lattice generated by 1 and τ, and if we write $\omega = n + m\tau$, then another expression for the Eisenstein series is $\sum_{\omega\in\Lambda^*} 1/\omega^k$.

Theorem 2.1 *Eisenstein series have the following properties:*

(i) *The series* $E_k(\tau)$ *converges if* $k \geq 3$, *and is holomorphic in the upper half-plane.*

(ii) $E_k(\tau) = 0$ *if* k *is odd.*

(iii) $E_k(\tau)$ *satisfies the following transformation relations:*

$$E_k(\tau + 1) = E_k(\tau) \quad \text{and} \quad E_k(\tau) = \tau^{-k} E_k(-1/\tau).$$

The last property is sometimes referred to as the **modular** character of the Eisenstein series. We shall return to these and other modular identities in the next chapter.

Proof. By Lemma 1.5 and the remark after it, the series $E_k(\tau)$ converges absolutely and uniformly in every half-plane $\mathrm{Im}(\tau) \geq \delta > 0$, whenever $k \geq 3$; hence $E_k(\tau)$ is holomorphic in the upper half-plane $\mathrm{Im}(\tau) > 0$.

By symmetry, replacing n and m by $-n$ and $-m$, we see that whenever k is odd the Eisenstein series is identically zero.

Finally, the fact that $E_k(\tau)$ is periodic of period 1 is clear from the fact that $n + m(\tau + 1) = n + m + m\tau$, and that we can rearrange the sum by replacing $n + m$ by n. Also, we have

$$(n + m(-1/\tau))^k = \tau^{-k}(n\tau - m)^k,$$

and again we can rearrange the sum, this time replacing $(-m, n)$ by (n, m). Conclusion (iii) then follows.

Remark. Because of the second property, some authors define the Eisenstein series of order k to be $\sum_{(n,m)\neq(0,0)} 1/(n + m\tau)^{2k}$, possibly also with a constant factor in front.

The connection of the E_k with the Weierstrass $\wp$ function arises when we investigate the series expansion of $\wp$ near 0.

Theorem 2.2 *For* z *near* 0, *we have*

$$\begin{aligned}\wp(z) &= \frac{1}{z^2} + 3E_4 z^2 + 5E_6 z^4 + \cdots \\ &= \frac{1}{z^2} + \sum_{k=1}^{\infty} (2k+1) E_{2k+2} z^{2k}.\end{aligned}$$

Proof. From the definition of $\wp$, if we note that we may replace ω by $-\omega$ without changing the sum, we have

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda^*} \left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} \right] = \frac{1}{z^2} + \sum_{\omega \in \Lambda^*} \left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right],$$

where $\omega = n + m\tau$. The identity

$$\frac{1}{(1-w)^2} = \sum_{\ell=0}^{\infty} (\ell+1) w^{\ell}, \qquad \text{for } |w| < 1,$$

which follows from differentiating the geometric series, implies that for all small z

$$\frac{1}{(z-\omega)^2} = \frac{1}{\omega^2} \sum_{\ell=0}^{\infty} (\ell+1) \left(\frac{z}{\omega}\right)^{\ell} = \frac{1}{\omega^2} + \frac{1}{\omega^2} \sum_{\ell=1}^{\infty} (\ell+1) \left(\frac{z}{\omega}\right)^{\ell}.$$

Therefore

$$\begin{aligned}
\wp(z) &= \frac{1}{z^2} + \sum_{\omega \in \Lambda^*} \sum_{\ell=1}^{\infty} (\ell+1) \frac{z^{\ell}}{\omega^{\ell+2}} \\
&= \frac{1}{z^2} + \sum_{\ell=1}^{\infty} (\ell+1) \left(\sum_{\omega \in \Lambda^*} \frac{1}{\omega^{\ell+2}} \right) z^{\ell} \\
&= \frac{1}{z^2} + \sum_{\ell=1}^{\infty} (\ell+1) E_{\ell+2} z^{\ell} \\
&= \frac{1}{z^2} + \sum_{k=1}^{\infty} (2k+1) E_{2k+2} z^{2k},
\end{aligned}$$

where we have used the fact that $E_{\ell+2} = 0$ whenever ℓ is odd.

From this theorem, we obtain the following three expansions for z near 0:

$$\begin{aligned}
\wp'(z) &= \frac{-2}{z^3} + 6E_4 z + 20E_6 z^3 + \cdots, \\
(\wp'(z))^2 &= \frac{4}{z^6} - \frac{24E_4}{z^2} - 80E_6 + \cdots, \\
(\wp(z))^3 &= \frac{1}{z^6} + \frac{9E_4}{z^2} + 15E_6 + \cdots.
\end{aligned}$$

From these, one sees that the difference $(\wp'(z))^2 - 4(\wp(z))^3 + 60E_4\wp(z) + 140E_6$ is holomorphic near 0, and in fact equal to 0 at the origin. Since this difference is also doubly periodic, we conclude by Theorem 1.2 that it is constant, and hence identically 0. This proves the following corollary.

Corollary 2.3 *If $g_2 = 60E_4$ and $g_3 = 140E_6$, then*

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3.$$

Note that this identity is another version of Theorem 1.7, and it allows one to express the symmetric functions of the e_j's in terms of the Eisenstein series.

2.2 Eisenstein series and divisor functions

We will describe now the link between Eisenstein series and some number-theoretic quantities. This relation comes about if we consider the Fourier coefficients in the Fourier expansion of the periodic function $E_k(\tau)$. Equivalently, we can write $\mathcal{E}(z) = E_k(\tau)$ with $z = e^{2\pi i\tau}$, and investigate the Laurent expansion of $\mathcal{E}$ as a function of z.

We begin with a lemma.

Lemma 2.4 *If* $k \geq 2$ *and* $\mathrm{Im}(\tau) > 0$*, then*

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+\tau)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{\ell=1}^{\infty} \ell^{k-1} e^{2\pi i\tau\ell}.$$

Proof. This identity follows from applying the Poisson summation formula to $f(z) = 1/(z+\tau)^k$; see Exercise 7 in Chapter 4.

An alternate proof consists of noting that it first suffices to establish the formula for $k = 2$, since the other cases are then obtained by differentiating term by term. To prove this special case, we differentiate the formula for the cotangent derived in Chapter 5

$$\sum_{n=-\infty}^{\infty} \frac{1}{n+\tau} = \pi \cot \pi\tau.$$

This yields

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+\tau)^2} = \frac{\pi^2}{\sin^2(\pi\tau)}.$$

Now use Euler's formula for the sine and the fact that

$$\sum_{r=1}^{\infty} rw^r = \frac{w}{(1-w)^2} \qquad \text{with } w = e^{2\pi i\tau}$$

to obtain the desired result.

As a consequence of this lemma, we can draw a connection between the Eisenstein series, the zeta function, and the divisor functions. The

divisor function $\sigma_\ell(r)$ that arises here is defined as the sum of the ℓ^{th} powers of the divisors of r, that is,

$$\sigma_\ell(r) = \sum_{d|r} d^\ell.$$

Theorem 2.5 *If* $k \geq 4$ *is even, and* $\operatorname{Im}(\tau) > 0$*, then*

$$E_k(\tau) = 2\zeta(k) + \frac{2(-1)^{k/2}(2\pi)^k}{(k-1)!} \sum_{r=1}^{\infty} \sigma_{k-1}(r) e^{2\pi i \tau r}.$$

Proof. First observe that $\sigma_{k-1}(r) \leq r r^{k-1} = r^k$. If $\operatorname{Im}(\tau) = t$, then whenever $t \geq t_0$ we have $|e^{2\pi i r \tau}| \leq e^{-2\pi r t_0}$, and we see that the series in the theorem is absolutely convergent in any half-plane $t \geq t_0$, by comparison with $\sum_{r=1}^{\infty} r^k e^{-2\pi r t_0}$. To establish the formula, we use the definition of E_k, that of ζ, the fact that k is even, and the previous lemma (with τ replaced by $m\tau$) to get successively

$$\begin{aligned}
E_k(\tau) &= \sum_{(n,m)\neq(0,0)} \frac{1}{(n+m\tau)^k} \\
&= \sum_{n\neq 0} \frac{1}{n^k} + \sum_{m\neq 0} \sum_{n=-\infty}^{\infty} \frac{1}{(n+m\tau)^k} \\
&= 2\zeta(k) + \sum_{m\neq 0} \sum_{n=-\infty}^{\infty} \frac{1}{(n+m\tau)^k} \\
&= 2\zeta(k) + 2\sum_{m>0} \sum_{n=-\infty}^{\infty} \frac{1}{(n+m\tau)^k} \\
&= 2\zeta(k) + 2\sum_{m>0} \frac{(-2\pi i)^k}{(k-1)!} \sum_{\ell=1}^{\infty} \ell^{k-1} e^{2\pi i m \tau \ell} \\
&= 2\zeta(k) + \frac{2(-1)^{k/2}(2\pi)^k}{(k-1)!} \sum_{m>0} \sum_{\ell=1}^{\infty} \ell^{k-1} e^{2\pi i \tau m \ell} \\
&= 2\zeta(k) + \frac{2(-1)^{k/2}(2\pi)^k}{(k-1)!} \sum_{r=1}^{\infty} \sigma_{k-1}(r) e^{2\pi i \tau r}.
\end{aligned}$$

This proves the desired formula.

Finally, we turn to the *forbidden* case $k = 2$. The series we have in mind $\sum_{(n,m)\neq(0,0)} 1/(n+m\tau)^2$ no longer converges absolutely, but we seek to give it a meaning anyway. We define

$$F(\tau) = \sum_m \left(\sum_n \frac{1}{(n+m\tau)^2} \right)$$

summed in the indicated order with $(n, m) \neq (0, 0)$. The argument given in the above theorem proves that the double sum converges, and in fact has the expected expression.

Corollary 2.6 *The double sum defining F converges in the indicated order. We have*

$$F(\tau) = 2\zeta(2) - 8\pi^2 \sum_{r=1}^{\infty} \sigma(r) e^{2\pi i r\tau},$$

where $\sigma(r) = \sum_{d|r} d$ is the sum of the divisors of r.

It can be seen that $F(-1/\tau)\tau^{-2}$ does *not* equal $F(\tau)$, and this is the same as saying that the double series for F gives a different value ($\tilde{F}$, the reverse of F) when we sum first in m and then in n. It turns out that nevertheless the **forbidden Eisenstein series** $F(\tau)$ can be used in a crucial way in the proof of the celebrated theorem about representing an integer as the sum of four squares. We turn to these matters in the next chapter.

3 Exercises

1. Suppose that a meromorphic function f has two periods ω_1 and ω_2, with $\omega_2/\omega_1 \in \mathbb{R}$.

(a) Suppose ω_2/ω_1 is rational, say equal to p/q, where p and q are relatively prime integers. Prove that as a result the periodicity assumption is equivalent to the assumption that f is periodic with the simple period $\omega_0 = \frac{1}{q}\omega_1$. [Hint: Since p and q are relatively prime, there exist integers m and n such that $mq + np = 1$ (Corollary 1.3, Chapter 8, Book I).]

(b) If ω_2/ω_1 is irrational, then f is constant. To prove this, use the fact that $\{m - n\tau\}$ is dense in $\mathbb{R}$ whenever τ is irrational and m, n range over the integers.

2. Suppose that $a_1, \ldots, a_r$ and $b_1, \ldots, b_r$ are the zeros and poles, respectively, in the fundamental parallelogram of an elliptic function f. Show that

$$a_1 + \cdots + a_r - b_1 - \cdots - b_r = n\omega_1 + m\omega_2$$

for some integers n and m.

[Hint: If the boundary of the parallelogram contains no zeros or poles, simply integrate $zf'(z)/f(z)$ over that boundary, and observe that the integral of $f'(z)/f(z)$ over a side is an integer multiple of $2\pi i$. If there are zeros or poles on the side of the parallelogram, translate it by a small amount to reduce the problem to the first case.]

3. In contrast with the result in Lemma 1.5, prove that the series

$$\sum_{n+m\tau\in\Lambda^*} \frac{1}{|n+m\tau|^2} \qquad \text{where } \tau \in \mathbb{H}$$

does not converge. In fact, show that

$$\sum_{1\leq n^2+m^2\leq R^2} 1/(n^2+m^2) = 2\pi \log R + O(1) \qquad \text{as } R \to \infty.$$

4. By rearranging the series

$$\frac{1}{z^2} + \sum_{\omega\in\Lambda^*} \left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2}\right],$$

show directly, without differentiation, that $\wp(z+\omega) = \wp(z)$ whenever $\omega \in \Lambda$.

[Hint: For R sufficiently large, note that $\wp(z) = \wp^R(z) + O(1/R)$, where $\wp^R(z) = z^{-2} + \sum_{0<|\omega|<R}((z+\omega)^{-2} - \omega^{-2})$. Next, observe that both $\wp^R(z+1) - \wp^R(z)$ and $\wp^R(z+\tau) - \wp^R(z)$ are $O(\sum_{R-c<|\omega|<R+c} |\omega|^{-2}) = O(1/R)$.]

5. Let $\sigma(z)$ be the canonical product

$$\sigma(z) = z \prod_{j=1}^{\infty} E_2(z/\tau_j),$$

where τ_j is an enumeration of the periods $\{n+m\tau\}$ with $(n,m) \neq (0,0)$, and $E_2(z) = (1-z)e^{z+z^2/2}$.

(a) Show that $\sigma(z)$ is an entire function of order 2 that has simple zeros at all the periods $n+m\tau$, and vanishes nowhere else.

(b) Show that

$$\frac{\sigma'(z)}{\sigma(z)} = \frac{1}{z} + \sum_{(n,m)\neq(0,0)} \left[\frac{1}{z-n-m\tau} + \frac{1}{n+m\tau} + \frac{z}{(n+m\tau)^2}\right],$$

and that this series converges whenever z is not a lattice point.

(c) Let $L(z) = -\sigma'(z)/\sigma(z)$. Then

$$L'(z) = \frac{(\sigma'(z))^2 - \sigma(z)\sigma''(z)}{(\sigma(z))^2} = \wp(z).$$

6. Prove that $\wp''$ is a quadratic polynomial in $\wp$.

7. Setting $\tau = 1/2$ in the expression

$$\sum_{m=-\infty}^{\infty} \frac{1}{(m+\tau)^2} = \frac{\pi^2}{\sin^2(\pi\tau)},$$

deduce that

$$\sum_{m\geq 1,\ m \text{ odd}} \frac{1}{m^2} = \frac{\pi^2}{8} \quad \text{and} \quad \sum_{m\geq 1} \frac{1}{m^2} = \frac{\pi^2}{6} = \zeta(2).$$

Similarly, using $\sum 1/(m+\tau)^4$ deduce that

$$\sum_{m\geq 1,\ m \text{ odd}} \frac{1}{m^4} = \frac{\pi^4}{96} \quad \text{and} \quad \sum_{m\geq 1} \frac{1}{m^4} = \frac{\pi^4}{90} = \zeta(4).$$

These results were already obtained using Fourier series in the exercises at the end of Chapters 2 and 3 in Book I.

8. Let

$$E_4(\tau) = \sum_{(n,m)\neq(0,0)} \frac{1}{(n+m\tau)^4}$$

be the Eisenstein series of order 4.

(a) Show that $E_4(\tau) \to \pi^4/45$ as $\mathrm{Im}(\tau) \to \infty$.

(b) More precisely,

$$\left| E_4(\tau) - \frac{\pi^4}{45} \right| \leq ce^{-2\pi t} \qquad \text{if } \tau = x + it \text{ and } t \geq 1.$$

(c) Deduce that

$$\left| E_4(\tau) - \tau^{-4}\frac{\pi^4}{45} \right| \leq ct^{-4}e^{-2\pi/t} \qquad \text{if } \tau = it \text{ and } 0 < t \leq 1.$$

4 Problems

1. Besides the approach in Section 1.2, there are several alternate ways of dealing with the sum $\sum 1/(z+\omega)^2$, where $\omega = n + m\tau$. For example, one may sum either (a) circularly, (b) first in n then in m, (c) or first in m then in n.

(a) Prove that if $z \notin \Lambda$, then

$$\lim_{R\to\infty} \sum_{n^2+m^2\le R^2} \frac{1}{(z+n+m\tau)^2} = S_1(z)$$

exists and $S_1(z) = \wp(z) + c_1$.

(b) Similarly,

$$\sum_m \left(\sum_n \frac{1}{(z+n+m\tau)^2} \right) = S_2(z)$$

exists and $S_2(z) = \wp(z) + c_2$, where $c_2 = F(\tau)$, and F is the forbidden Eisenstein series.

(c) Also

$$\sum_n \left(\sum_m \frac{1}{(z+n+m\tau)^2} \right) = S_3(z)$$

exists with $S_3(z) = \wp(z) + c_3$, and $c_3 = \tilde{F}(\tau)$, the reverse of F.

[Hint: To prove (a), it suffices to show that $\lim_{R\to\infty}$, $\displaystyle\sum_{1\le n^2+m^2\le R^2} 1/(n+m\tau)^2 = c_1$ exists. This is proved by a comparision with $\int_{1\le x^2+y^2\le R^2} \frac{dx}{(x+y\tau)^2} = I(R)$. It can be shown that $I(R) = 0$, which follows because $(x+y\tau)^{-2} = -(\partial/\partial x)(x+y\tau)^{-1}$.]

2. Show that

$$\wp(z) = c + \pi^2 \sum_{m=-\infty}^{\infty} \frac{1}{\sin^2((z+m\tau)\pi)}$$

where c is an appropriate constant. In fact, by part (b) of the previous problem $c = -F(\tau)$.

3.* Suppose Ω is a simply connected domain that excludes the three roots of the polynomial $4z^3 - g_2 z - g_3$. For $\omega_0 \in \Omega$ and ω_0 fixed, define the function I on Ω by

$$I(\omega) = \int_{\omega_0}^{\omega} \frac{dz}{\sqrt{4z^3 - g_2 z - g_3}} \qquad \omega \in \Omega.$$

Then the function I has an inverse given by $\wp(z+\alpha)$ for some constant α; that is,

$$I(\wp(z+\alpha)) = z$$

for appropriate α.

[Hint: Prove that $(I(\wp(z+\alpha)))' = \pm 1$, and use the fact that $\wp$ is even.]

4.* Suppose τ is purely imaginary, say $\tau = it$ with $t > 0$. Consider the division of the complex plane into congruent rectangles obtained by considering the lines $x = n/2$, $y = tm/2$ as n and m range over the integers. (An example is the rectangle whose vertices are $0, 1/2, 1/2 + \tau/2$, and $\tau/2$.)

(a) Show that $\wp$ is real-valued on all these lines, and hence on the boundaries of all these rectangles.

(b) Prove that $\wp$ maps the interior of each rectangle conformally to the upper (or lower) half-plane.