FROM CHABAUTY’S METHOD TO KIM’S NON-ABELIAN
CHABAUTY’S METHOD

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This is a draft, so comments (especially on the exposition) and corrections are welcome!

In 1922, Mordell conjectured that every hyperbolic curve has finitely many rational points. The first major progress on this conjecture came from Chabauty in 1941 ([Cha41]). His method, based on an earlier method of Skolem for integral points on non-proper curves, used $p$-adic analysis to prove finiteness for all curves satisfying a somewhat restrictive condition. While Faltings proved Mordell’s conjecture in general in 1983, his proof was not effective. At the same time, Chabauty’s method gained a newfound importance when Coleman ([Col85]) showed how to make it effective using his newly developed theory of $p$-adic integration. This allowed him and others to provably compute the set of rational points on specific curves, and prove general bounds for the number of rational points.

At the same time, Coleman’s method worked only for curves satisfying Chabauty’s condition. In 2004, Minhyong Kim ([Kim05]) showed how to extend Chabauty’s method to prove Siegel’s

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theorem for $S$-integral points on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ by making use of non-abelian quotients of its fundamental group. Kim’s new method points to a general method for attempting to provably finding the set of rational points on higher genus curves, and some progress has already been made (see especially [BDM'17], the first instance in which Kim’s method led to a previously unknown concrete result).

In this article, we explain how Kim’s method is a natural generalization of the classical Chabauty-Skolem method.

**Notation.** For a scheme $Y$, we let $\mathcal{O}(Y)$ denote its coordinate ring. If $R$ is a ring, we let $Y \otimes R$ or $Y_R$ denote the product (or ‘base-change’) $Y \times \text{Spec } R$. If $Y$ and $\text{Spec } R$ are over an implicit base scheme $S$ (often $\text{Spec } \mathbb{Q}$), we take the product over $S$. Similarly, if $M$ is a linear object (such as a module, an algebra, a Lie algebra, or a Hopf algebra), then $M_R$ denotes $M \otimes R$ (again, with the tensor product taken over an implicit base ring, usually $\mathbb{Z}$ or $\mathbb{Z}_p$).

If $K$ is a number field, we let $\Sigma_K$ denote the set of places of $K$. If $v$ is a place of $K$, then $K_v$ denotes the completion of $K$ at $v$, and if $v$ is a finite place, then $\mathcal{O}_v$ denotes the integer ring of $K_v$. We say that $Z$ is a ring of $S$-integers if there is some elements $\alpha \in \mathcal{O}_K$ for which $Z = \mathcal{O}_K[1/\alpha]$. If $v$ is a finite place and not dividing $\alpha$, we write $Z_v = \mathcal{O}_v$.

1. **Classical Chabauty-Coleman-Skolem**

1.1. **Chabauty’s Method.** We recommend [MP12] as a great introduction to Chabauty’s method and Coleman’s effective version of it. Nonetheless, we give a shorter introduction here, both for completeness and to set some notation.

Let $X$ be a smooth proper hyperbolic curve over a number field $K$, and let $p$ be a finite place of $K$ that is totally split over $\mathbb{Q}$ and such that $X$ has good reduction at $p$. Then $K_p \cong \mathbb{Q}_p$, and $X$ admits a smooth proper model over $\mathbb{Z}_p \cong \mathbb{Z}_p$. We let $J$ be the Jacobian of $X$, and we suppose we have a point $O \in X(K)$, giving an embedding $X \hookrightarrow J$ sending $O$ to the identity of $J$.

If $J(K)$ is finite, then it follows that $X(K)$ must be finite, and it is not hard to determine $X(K)$ (c.f. [MP12] §2). However, $J(K)$, unlike $X(K)$, is not expected to be finite in general; rather, it is proven to be finitely generated as an abelian group. Nonetheless, Chabauty observed that even when $J(K)$ is infinite, one might use $J(K)$ to prove finiteness of $X(K)$ as long as the rank of $J(K)$ as an abelian group is not too large.

More specifically, Chabauty proved that if

$$r := \text{rank}_\mathbb{Z} J(K)$$

is less than the genus $g$ of $X$, then the intersection

$$X(K_p) \cap J(K)$$

of $X(K_p)$ with the $p$-adic closure of $J(K)$ in $J(K_p)$ is finite. Later, Coleman showed how to compute this intersection, using his newly developed theory of $p$-adic integration. The basic intuition is that $J(K)$ should be a $p$-adic manifold of dimension at most $r$, inside the $p$-adic manifold $J(K_p)$ of dimension $g$; then, if $r < g$, its intersection with the one-dimensional $p$-adic manifold $X(K_p)$ should be discrete, and since $J(K_p)$ is compact, this should be finite.

To make this more precise, one needs a description of the structure of $J(K_p)$. For this, as described in §4.1 of loc. cit., we consider $\omega_J \in H^0(J_{K_p}, \Omega^1)$, the $g$-dimensional vector space
of regular one-forms on $J_{K_p}$. Then there is an integration map

$$\eta_J: J(K_p) \to \mathbb{Q}_p$$

$$Q \mapsto \int_Q^O \omega_J$$

characterized uniquely by the fact that it is a homomorphism, and by the fact that for $Q$ sufficiently close to $O$, it is given by formally taking an anti-derivative of $\omega_J$ that vanishes at $O$ and evaluating it at the coordinates of $Q$.

Letting $T$ denote the dual of $H^0(J_{K_p}, \Omega^1)$, or equivalently the tangent space to $J_{K_p}$ at the identity, this gives a homomorphism

$$\log: J(K_p) \to T,$$

which is easily seen to be a local diffeomorphism, hence finite-to-one.

Put together, we now have the diagram:

$$(A) \quad \begin{array}{ccc}
X(K) & \longrightarrow & X(K_p) \\
\downarrow & & \downarrow^f \\
J(K) & \longrightarrow & J(K_p) \quad \eta_J \quad \log \\
& & \downarrow \quad \downarrow \\
& & T
\end{array}$$

The “basic intuition” mentioned above about $J(Z)$ can be made precise by noting that the dimension of $\log J(K)$ is just the $\mathbb{Z}_p$-rank of the $\mathbb{Z}_p$-span of $\log J(K)$, which must be at most $r$ because $J(K)/(\text{torsion})$ can be generated by $r = \text{rank}_{\mathbb{Z}} J(K)$ elements.

1.1.1. Coleman Integration. Finally, the theory of Coleman ([Col85]) allows one to explicitly compute the integration maps in the definition of $\log$. In particular, the diagonal arrow labeled ‘∫’ in Diagram A can be expressed as Coleman integration on the $p$-adic space $X(K_p)$. Therefore, one may explicitly compute a $p$-adic analytic function on $X(K_p)$ that vanishes on $X(K)$, as follows:

1. Choose bases of $T$ and $J(K)$
2. Integrate to find the image of $J(K)$ under $\log$ relative to these bases
3. Find a nonzero element $\omega_J$ of $T^\vee = H^0(J_{K_p}, \Omega^1)$ vanishing on $\log J(K)$
4. Compute the restriction of $\eta_J$ to $X(K_p)$

The commutativity of (A) implies that this function vanishes on $X(K)$. Coleman’s theory shows that this function is locally analytic and has finitely many zeroes. One may in fact compute local power series expansion for this function, and then use the theory of Newton polygons to approximate the locations of its zeroes.

Remark 1.1. Notice that in applying Chabauty’s method, we only care about the span of the image of $J(K)$ in a $\mathbb{Q}_p$-vector space. Therefore, the method doesn’t really need knowledge of $J(K)$ itself, but of its tensorization $J(K)_{\mathbb{Q}_p} = J(K) \otimes_{\mathbb{Z}} \mathbb{Q}_p$. This will be important in Section 3.
1.2. Non-Proper Curves and Siegel’s Theorem. We would like to explain how the content of Section 1.1, both Faltings’ Theorem and Chabauty’s method, fits into a more general fact about integral points on smooth curves of negative Euler characteristic.

Convention: When we say “integral points,” we mean points with values in a fixed open subscheme of $\text{Spec} \mathcal{O}_K[1/S]$; that is, all finiteness results apply equally well to $S$-integral points.

In 1929, Siegel proved that all affine curves of positive genus have finitely many integral points. For $g > 1$, this is just a corollary of Faltings’ Theorem (although it was historically proved many years earlier). But for $g = 1$, this gives a new result, namely that punctured elliptic curves have finitely many integral points (in concrete terms, this implies that a Weierstrass model has finitely many integral solutions). Furthermore, Siegel also proved finiteness of $S$-integral points for $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, and as a corollary, for any curve of genus 0 with at least three punctures.

Note that for a proper curve, integral points are the same as rational points, by the valuative criterion of properness. It follows that Faltings’ and Siegel’s theorems may be jointly summarized by saying that there are finitely many $S$-integral points on

1. curves of genus at least 2,
2. curves of genus 1 with at least one puncture, and
3. curves of genus 0 with at least three punctures.

In fact, such curves have an important common property that distinguishes them. They are precisely the curves that are hyperbolic, which is equivalent to saying that the topological Euler characteristic of their complex points is negative. More importantly for us, they are precisely the smooth curves whose topological fundamental group is non-abelian. We summarize the theorems of Faltings and Siegel as one:

**Theorem 1.2 (Faltings-Siegel).** Let $X$ be a smooth curve over a number field $K$, let $Z$ be an open subscheme of $\text{Spec} \mathcal{O}_K$, and let $X \rightarrow Z$ be a regular minimal model of $X$. If the fundamental group of the Betti topological space of $X_Z$ is nonabelian (or equivalently if the Euler characteristic is negative) then the set $X(Z)$ of integral points of $X$ is finite.

**Remark 1.3.** Just as Siegel’s and Faltings’ Theorems should be seen as one theorem, there are two other theorems that deserve to be combined in a similar way. Those are Dirichlet’s $S$-Unit Theorem and the Mordell-Weil Theorem. A generalized version of Dirichlet’s $S$-Unit Theorem says that the group of integral points on an algebraic torus is finitely generated. A common generalization of these two theorems then says that

**Theorem 1.4 (Dirichlet-Mordell-Weil).** The group of integral points on a semi-abelian scheme is finitely generated.

If one wants to make this about curves, one may write it as a statement about the generalized Jacobian of an arbitrary smooth curve (like the one used for Skolem’s method in the following section), which is a semi-abelian variety.

1.3. Skolem’s Method. It turns out that Chabauty’s method may be applied to non-proper hyperbolic curves just as well as to proper hyperbolic curves, if one phrases it correctly. In fact, its use for non-proper curves historically predates Chabauty and is known as the method of Skolem, who applied it to Thue equations of the form $f(x, y) = c$ for $f$ a homogeneous
binary form of degree at least 3. We briefly describe the more general Chabauty-Skolem method here, as some of the original applications of non-abelian Chabauty involve non-proper curves.

We let $X$ denote a smooth proper curve over a ring $Z = \mathcal{O}_K[1/S]$ of $S$-integers. While Chabauty-Skolem is very similar to Chabauty, the most subtle point is how to choose $J$; one wants an embedding from $X$ into a semi-abelian scheme $J$ over $Z$, such that the embedding is an isomorphism on (geometric) first homology. This may be achieved via a generalized Jacobian. As an example, consider $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$, defined as $\text{Spec } \mathbb{Z}[x, y, x^{-1}, y^{-1}]/(x + y - 1)$. Then $J = \mathbb{G}_m \times \mathbb{G}_m$, and the embedding sends $(x, y)$ satisfying $x + y = 1$ to $(x, y) \in \mathbb{G}_m \times \mathbb{G}_m$.

More generally, if $X$ is the complement of a non-split closed reduced subscheme of $\mathbb{A}^1$, then $J$ is a non-split torus of dimension equal to the degree of the subscheme.

To apply Chabauty-Skolem, one chooses a closed point $p$ of $Z$ whose completed local ring $Z_p$ is isomorphic to $\mathbb{Z}_p$. We assume we have a point $O \in X(Z)$, mapping to the identity of $J$. We let $T$ be the tangent space to $J_{K_p}$ at $O$, which is dual to the space of all translation-invariant differential 1-forms (not the space of all holomorphic 1-forms unless $J$ is proper). We then have the diagram

$$
\begin{array}{ccc}
X(Z) & \longrightarrow & X(Z_p) \\
\downarrow & & \downarrow \\
J(Z) & \longrightarrow & J(Z_p) \\
\downarrow & & \downarrow \log \\
T & \longrightarrow & T
\end{array}
$$

Then $J(Z)$ is a finitely-generated abelian group (c.f. Theorem 1.4), and the method applies as in the proper case, as long as $\text{rank}_Z J(Z) < \dim J$.

**Remark 1.5.** In the case that $X$ and $J$ are proper, we have $X(Z) = X(K)$, $J(Z) = J(K)$, $X(Z_p) = X(K_p)$, and $J(Z_p) = J(K_p)$, so the methods of Section 1.1 and of this section are equivalent.

**2. The Problem with Abelian Fundamental Groups**

This method often does not work, because $J(Z)$ can be too large. Philosophically, the reason why $J(Z)$ can be large while $X(Z)$ remains finite is because the geometric fundamental group of $J$ is abelian, while the fundamental group of $X$ is non-abelian, even center-free.\footnote{This is not to say that all varieties with abelian fundamental group have infinitely many integral points, but that in the context of the distinction between a curve and its Jacobian, this principle applies.}

The groundbreaking work of Minhyong Kim (\cite{Kim05}) gets around this fact.

Recall the key properties of $J$, mentioned in passing in Section 1.3:

1. $J$ is a semi-abelian scheme over $Z$.
2. There is an embedding $X \hookrightarrow J$ that is an isomorphism on (geometric) first homology.

As a clarification about the meaning of “geometric” in (2), note that we can simply require it to be an isomorphism on integral Betti homology for some embedding $Z \hookrightarrow \mathbb{C}$; it then follows that this is the case for Betti cohomology over all embeddings, for algebraic de Rham cohomology over $K$, and for $\ell$-adic cohomology over $\overline{K}$.

Properties (1) and (2) together imply that the embedding induces the abelianization map on fundamental groups (whether Betti, de Rham, crystalline, or geometric étale), because...
a semi-abelian scheme is in particular a group scheme, so its fundamental group is abelian, hence

\[ \pi_1(J) = \pi_1(J)_{ab} = H_1(J) = H_1(X) = \pi_1(X)_{ab}. \]

It might then seem natural to hope for an embedding from \( X \) into a variety whose fundamental group is not abelian but almost abelian. More specifically, letting \( \pi_1(X) \) denote some version (Betti, de Rham, crystalline, or geometric étale) of the fundamental group of \( X \) based at \( O \), we define the descending central series filtration of \( \pi_1(X) \) by

\[ \pi_1(X)^1 := \pi_1(X) \]
\[ \pi_1(X)^n := [\pi_1(X)^{n-1}, \pi_1(X)] \]

and the corresponding quotients

\[ \pi_1(X)_n := \pi_1(X)/\pi_1(X)^{n+1}, \]

so that \( \pi_1(X)_1 = \pi_1(X)_{ab} = \pi_1(J) = H_1(X) \).

We then might hope that for each \( n \), we can find an embedding

\[ X \hookrightarrow J_n \]

whose induced map on fundamental groups \( \pi_1(X) \to \pi_1(J_n) \) is isomorphic to the quotient map \( \pi_1(X) \to \pi_1(X)_n \). However, I know of no candidate for the varieties \( J_n \), at least in general.

To remedy this, the critical insight of Kim was to rewrite the whole of the Chabauty-Skolem method intrinsically in terms of \( X \) and its fundamental group.

When we do that, we will see that the Chabauty-Skolem specifically uses \( \pi_1(X)_1 = H_1(X) \), which is why classical Chabauty-Skolem is considered “abelian.” If the constructions are sufficiently general, one may then replace \( \pi_1(X)_1 \) by \( \pi_1(X)_n \) to arrive at Kim’s non-abelian Chabauty’s method, as explained in Section 4. Section 3 will be devoted to explaining how to rewrite Chabauty-Skolem in terms of \( X \) and its fundamental group.

3. Expressing Chabauty’s Method Intrinsically in Terms of \( X \)

For simplicity, we assume that \( X \) is proper from now until the end of Section 3.2. We then explain the modifications necessary for non-proper \( X \) in Section 3.3.

In some sense, using \( J \) is like using the (abelianization of the) fundamental group of \( X \). From a complex analytic perspective, \( J \) not only has the same first (co)homology as \( X \); it is in fact determined by the first (co)homology of \( X \). The Abel-Jacobi Theorem shows that one may construct \( J \) as the quotient \( H^1(X, \mathbb{R})/H^1(X, \mathbb{Z}) \), where the complex structure on \( H^1(X, \mathbb{R}) \) is determined by the Hodge structure on \( H^1(X, \mathbb{C}) \). That gives us a first hint as to how to write Chabauty’s method in terms of homology.

Of course, for something arithmetic like Chabauty’s method, we need to go beyond the complex analytic realm. More specifically, we need to express all the parts of diagram (B) in terms of \( X \):

\[ \text{If the group has some kind of topology, it is understood that we always take the closure of the commutator.} \]

\[ \text{3Since I began writing this, I became aware of work in progress by Edixhoven, Lido, and Schoof, in which they use the Poincaré torsor of the Jacobian of \( X \) as \( J_2 \) and show that non-abelian Chabauty’s method may be carried out using the geometry of \( J_2 \).} \]
**Goal 3.1.** Express $J(K)$, $J(K_p)$, $T$, and the map $\log$ intrinsically in terms of the first homology of $X$.

**Remark 3.2.** Following Remark 1.1 we really care only about $J(K)_{\mathbb{Q}_p}$ rather than $J(K)$. The same is true of $J(K_p)$ with respect to its rationalization $J(K_p)_{\mathbb{Q}} = J(K_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. 

3.1. **Intrinsic Description of the Mordell-Weil Group.** We first focus on $J(K)$. Recall, for each positive integer $m$, the Kummer exact sequence

$$0 \to J[m](\overline{K}) \to J(\overline{K}) \xrightarrow{m} J(\overline{K}) \to 0,$$

of $G_K$-modules, giving rise to the long exact sequence

$$0 \to J[m](K) \to J(K) \to J(K) \to H^1(K, J[m]) \to H^1(K, J) \xrightarrow{m} H^1(K, J) \to \cdots$$

and hence a short exact sequence

$$0 \to J(K)/mJ(K) \xrightarrow{\kappa_m} H^1(K, J[m]) \to H^1(K, J)[m] \to 0,$$

where $\kappa_m$ is known as the (mod $m$) Kummer map.

The importance of using $J[m]$ is the following: there is a canonical, Galois-equivariant isomorphism

$$J[m] \cong H^1_{\text{ét}}(J_K, \mathbb{Z}/m\mathbb{Z}) \cong H^1_{\text{ét}}(X_K, \mathbb{Z}/m\mathbb{Z}),$$

In other words, the Kummer map is an embedding

$$J(K)/mJ(K) \hookrightarrow H^1(K, H^1_{\text{ét}}(X_K, \mathbb{Z}/m\mathbb{Z})).$$

To get an embedding of $J(K)$ rather than $J(K)/mJ(K)$, one may simply set $m = p^n$ and take an inverse limit. Let

$$T_p := H^1_{\text{ét}}(X_K, \mathbb{Z}_p) \cong H^1_{\text{ét}}(J_K, \mathbb{Z}_p) \cong T_p J(\overline{K})$$

and

$$V_p := T_p \otimes \mathbb{Q} = H^1_{\text{ét}}(X_K, \mathbb{Q}_p),$$

with their associated $G_K$-actions. Then we get embeddings

$$J(K)_{\mathbb{Q}_p} \hookrightarrow H^1(K, T_p)$$

as well as its rational cousin

$$J(K)_{\mathbb{Q}_p} \hookrightarrow H^1(K, V_p).$$

We now have a ‘container’ for $J(K)$ defined intrinsically in terms of $X$, but we need to identify the subspace $J(K)$ inside it (or at least, following Remark 3.2, its $\mathbb{Q}_p$-span) purely in terms of $V_p = H^1_{\text{ét}}(X_K, \mathbb{Q}_p)$ (i.e., without reference to $J$). The vector space $V_p$ has the structure of a $G_K$-representation, so we want a purely Galois-theoretic way to identify the image of $J(K)_{\mathbb{Q}_p}$ in $H^1(K, V_p)$. As we shall see, this is supplied by the theory of Bloch-Kato Selmer groups, developed in [BK90]. In order to explain what a Bloch-Kato Selmer group is, we first recall the classical theory of Selmer groups.

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4For those not familiar with an intrinsic definition of étale homology (of which there are multiple), one may define it simply as the dual of étale cohomology with respect to the chosen coefficients, at least when the cohomology is free.
3.1.1. Finite Selmer Groups. We let $\Sigma_K$ denote the set of all places of $K$. We consider the short exact sequence $\prod$ both over $K$ and over all completions $K_v$ of $K$ to obtain a diagram: (C)

$$
0 \longrightarrow J(K)/mJ(K) \xrightarrow{\kappa_m} H^1(K, J[m]) \xrightarrow{\alpha} H^1(K, J)[m] \xrightarrow{\delta} 0
$$

The Selmer group $\text{Sel}_m(J)$ is defined to be the inverse under $\alpha$ of the image of $\beta$. As the bottom row is exact, this is just the kernel of $\gamma \circ \alpha = \delta$, implying we have a short exact sequence

$$
0 \rightarrow J(K)/mJ(K) \rightarrow \text{Sel}_m(J) \rightarrow \text{III}_J[m] \rightarrow 0,
$$

where $\text{III}_J := \ker(H^1(K, J) \rightarrow \prod_{v \in \Sigma_K} H^1(K_v, J))$.

We also have the following conjecture:

**Conjecture 3.3** (Tate-Shafarevich). *For any abelian variety $J$, the group $\text{III}_J$ is finite.*

3.1.2. $p$-adic Selmer Groups. The map $\kappa_m$ is unfortunately non-injective when viewed as a map from $J(K)$ to $\text{Sel}_m(J)$. To solve this problem, and to make use of the fact that $\text{III}_J$ is finite, we pass to $p$-adic Selmer groups.

More precisely, let $T_p \text{III}_J$ be the $p$-adic Tate module of $\text{III}_J$, namely the inverse limit of the groups $\text{III}_J[p^n]$ taken over the multiplication by $p$ map. Let

$$
\text{Sel}_{p^\infty}(J) := \varprojlim \text{Sel}_{p^n}(J).
$$

As $J(K)$ is finitely generated, we have $\varprojlim J(K)/p^nJ(K) = J(K)_{\mathbb{Z}_p}$, so we have a short exact sequence

$$
0 \rightarrow J(K)_{\mathbb{Z}_p} \rightarrow \text{Sel}_{p^\infty}(J) \rightarrow T_p \text{III}_J \rightarrow 0.
$$

In particular, if, as conjectured, $\text{III}_J$ is finite, then $T_p \text{III}_J = 0$. In fact, as noted in [Sto07, §2], this would follow simply from the weaker claim that the $p$-divisible part of $\text{III}_J$ is trivial. By Remark 3.2 Conjecture 3.3 suggests that we may replace $J(K)$ by $\text{Sel}_{p^\infty}(J)$ in Chabauty’s method. In practice, one may make the replacement without assuming the conjecture: the standard way to compute $J(K)$ is by computing Selmer groups, and if the $\mathbb{Z}_p$-rank of $\text{Sel}_{p^\infty}(J)$ is less than $g$, then Chabauty’s method works. The conjecture simply shows that we do not expect to lose anything by passing from $J(K)_{\mathbb{Z}_p}$ to $\text{Sel}_{p^\infty}(J)$.

Notice that $\text{Sel}_{p^n}(J) \subseteq H^1(K, H^1_{\ell}(X_K, \mathbb{Z}/p^n\mathbb{Z}))$, so

$$
\text{Sel}_{p^\infty}(J) \subseteq \varprojlim H^1(K, H^1_{\ell}(X_K, \mathbb{Z}/p^n\mathbb{Z})) = H^1(K, T_p)
$$

**Remark 3.4.** On the infinite level, the Selmer group is defined by the same kind of local condition as is the finite version. First, we need a definition:

**Definition 3.5.** For an abelian group $M$, we define the $p$-adic completion

$$\widehat{M} := \varprojlim M/p^nM.$$

When $M$ is finitely generated, this is the same as $M \otimes \mathbb{Z}_p$. When $M$ is profinite, it surjects onto its $p$-adic completion.
To understand the local condition for p-adic Selmer groups, we draw the diagram

(D)  \[
\begin{array}{cccccc}
0 & \rightarrow & J(K)_{\mathbb{Z}_p} & \overset{\kappa_p}{\rightarrow} & H^1(K, T_p) & \overset{\alpha}{\rightarrow} & H^1(K, J)[p^\infty] & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \prod_{v \in \Sigma_v} \widehat{J(K_v)} & \overset{\beta}{\rightarrow} & \prod_{v \in \Sigma_K} H^1(K_v, T_p) & \overset{\gamma}{\rightarrow} & \prod_{v \in \Sigma_K} H^1(K_v, J)[p^\infty] & \rightarrow & 0.
\end{array}
\]

Then the Selmer group $\text{Sel}_{p^\infty}(J)$ is just the inverse under $\alpha$ of the image of $\beta$.

**Remark 3.6.** Following Remark 3.2, we only really care about these things $\mathbb{Q}_p$-linearly. In other words, we care about $J(K)_{\mathbb{Q}_p}$, which is conjecturally isomorphic to

$$\text{Sel}_{p^\infty}(J)_\mathbb{Q} := \text{Sel}_{p^\infty}(J) \otimes_{\mathbb{Z}} \mathbb{Q} = \text{Sel}_{p^\infty}(J) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$ 

We may draw the diagram

(D')  \[
\begin{array}{cccccc}
0 & \rightarrow & J(K)_{\mathbb{Q}_p} & \overset{\kappa_p}{\rightarrow} & H^1(K, V_p) & & \rightarrow & & \\
& & \downarrow & & \downarrow & & & & \\
0 & \rightarrow & \prod_{v \in \Sigma_v} \widehat{J(K_v)}_{\mathbb{Q}} & \overset{\beta}{\rightarrow} & \prod_{v \in \Sigma_K} H^1(K_v, V_p). & & & &
\end{array}
\]

Then by exactness of tensoring with $\mathbb{Q}$, the $\mathbb{Q}_p$-Selmer group $\text{Sel}_{p^\infty}(J)_\mathbb{Q}$ is just the inverse under $\alpha$ of the image of $\beta$ in Diagram D’.

**Remark 3.7.** For a given $v \in \Sigma_K$, let $\ell$ denote its residue characteristic. Then $J(K_v)$ is a compact abelian $\ell$-adic Lie group of dimension $g$, so it’s isomorphic to a finite group times $\mathbb{Z}_\ell^g$.

It follows that $J(K_v)$ surjects onto its $p$-adic completion, so we may replace $\prod_{v \in \Sigma_v} \widehat{J(K_v)}$ by $\prod_{v \in \Sigma_v} J(K_v)$ in the diagram without changing the image of $\beta$. As explained in Section 3.3, this is no longer always true when $X$ is not compact.

**Remark 3.8.** When $\ell \neq p$, the $p$-adic completion of $J(K_v)$ is finite, so the image of $\beta$ is torsion and therefore zero when working rationally. In particular, its image has an intrinsic definition. In fact, the entire group $H^1(K_v, V_p)$ is trivial (c.f. [Fen16, Example 2.4] or [Bel09, Exercise 2.9]), so one may ignore the local conditions for $\ell \neq p$.

However, when $X$ and $J$ are non-compact, this is not always the case, and one may need the local conditions for $\ell \neq p$, as explained in Section 3.3. This is fundamentally related to the fact that integral points and rational points may be different on non-proper varieties.

We now want a way to determine the image of $J(K)_{\mathbb{Q}_p}$ in $H^1(K, V_p)$ in terms of the Galois representation $V_p$. We are partway there, as we may replace $J(K)_{\mathbb{Q}_p}$ by $\text{Sel}_{p^\infty}(J)_\mathbb{Q}$; the latter seems more intrinsic, as it is a subgroup of $H^1(K, V_p)$ defined by local conditions.

The problem is that this collection of “local conditions” (by which we mean a certain subgroup of $\prod_{v \in \Sigma_K} H^1(K_v, V_p)$) is still defined using the geometry of $J$, rather than something intrinsic to the representation $T_p$. Bloch and Kato solved this problem.

\[\text{Sel}_{p^\infty}(J)_\mathbb{Q} := \text{Sel}_{p^\infty}(J) \otimes_{\mathbb{Z}} \mathbb{Q} = \text{Sel}_{p^\infty}(J) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.\]
3.1.3. **Bloch-Kato Selmer Groups.** References for this section include [BK90 §3], [Bel09 §2], [Fen16].

Bloch and Kato fixed this problem by observing in [BK90] that one could define the image of $\beta$ by using $p$-adic Hodge theory, and therefore define $\text{Sel}_{p^\infty}(J)_{\mathbb{Q}}$ intrinsically in terms of the Galois action on $V_p$. Their original motivation was to extend the notions of Selmer group and Tate-Shafarevich group (and hence the Birch and Swinnerton-Dyer Conjecture) to motives other than those arising from the $H^1$ of an abelian variety. But their methods are invaluable in Kim’s work, as they will eventually allow us to extend the notion of Selmer group from $T_p = H^1_{\text{ét}}(X, \mathbb{Z}_p) \cong \pi_1^{\text{ét}}(X_{\overline{k}}) \otimes \mathbb{Z}_p$ to certain non-abelian pro-$p$ quotients of $\pi_1^{\text{ét}}(X)_{\overline{k}}$.

By Remark 3.8 because we are assuming $p$ is proper, we need to consider local conditions only for $v$ of residue characteristic $p$, so we need to determine the image of

$$\kappa_v : J(K_v)_{\mathbb{Q}} \to H^1(K_v, V_p).$$

Notice that by considering the case where $v$ is the chosen place $p$, this will also solve the second part of Goal 3.1; that is, writing $J(K_p)$ intrinsically.

To explain how Bloch-Kato identified the appropriate subgroup of $H^1(K_v, V_p)$, we need to recall a bit of $p$-adic Hodge theory. This theory identifies a subcategory

$$\{\text{crystalline representations}\} \subseteq \{\text{all continuous } \mathbb{Q}_p\text{-representations of } G_{K_v}\}.$$ 

Crystalline representations are the $\ell = p$ analogue of unramified representations in the $\ell \neq p$ case. More specifically, just as the $\ell$-adic cohomology of a variety with good reduction at $p \neq \ell$ is unramified, the $p$-adic cohomology of such a variety is always crystalline, albeit technically ramified.

We can apply this notion to the group $H^1(K_v, V_p)$, which is the group of extensions of $\mathbb{Q}_p$ (with trivial $G_{K_v}$-action) by $V_p$, in the category of continuous $p$-adic representations of $G_{K_v}$. That is, every element $\alpha \in H^1(K_v, V_p)$ may be represented as an extension

$$0 \to V_p \to E_\alpha \to \mathbb{Q}_p \to 0.$$

**Definition 3.9.** We say that $\alpha \in H^1(K_v, V_p)$ is crystalline if the representation $E_\alpha$ is crystalline as a representation of $G_{K_v}$.

**Definition 3.10.** The subgroup of all crystalline elements of $H^1(K_v, V_p)$ is denoted $H^1_f(K_v, V_p)$ and is known as the **local Bloch-Kato Selmer group at $v$**.

Just as varieties with good reduction at $v$ give rise to crystalline representations, extensions coming from integral points (elements of $J(\mathcal{O}_v)$) are crystalline. But as $J$ is proper, we have $J(K_v) = J(\mathcal{O}_v)$, so the image of $J(K_v)$ lands in $H^1_f(K_v, V_p)$. The key theorem of Bloch-Kato in this context is:

**Theorem 3.11 (Bloch-Kato).** The image of the Kummer map

$$\kappa_v : J(K_v)_{\mathbb{Q}} \to H^1(K_v, V_p)$$

is $H^1_f(K_v, V_p)$.

The proof of the theorem proceeds by noting that the map is injective and then showing that the dimension of the right side is $g(K_v : \mathbb{Q}_p)$, which is the dimension of $J(K_v)$ as a $p$-adic Lie group, and hence of $J(K_v)_{\mathbb{Q}}$ as a $\mathbb{Q}_p$-vector space.

Finally, we recall what happens globally; i.e., over $K$. 

---

*This content is a direct transcription of the given text. Any additional information or context not explicitly included here has been omitted.*
**Definition 3.12.** We let $H^1_f(K,V_p)$ denote the subset of $\alpha \in H^1(K,V_p)$ whose image in $H^1(K_v,V_p)$ lies in $H^1_f(K_v,V_p)$ for each $v | p$. This is known as the (global) Bloch-Kato Selmer group.

We then have:

**Corollary 3.13** (of Theorem 3.11). *For any abelian variety $J$, the group $\text{Sel}_{p^\infty}(J)_{\mathbb{Q}}$ is naturally identified with $H^1_f(K,V_p)$.***

**Proof.** As explained in Remark 3.4, the group $\text{Sel}_{p^\infty}(J)_{\mathbb{Q}}$ is the subset of $H^1(K,V_p)$ that is locally in the image of $J(K_v)_{\mathbb{Q}}$ for each $v$. By Remark 3.8, the only relevant places are those dividing $p$, so the result follows by Theorem 3.11. \hfill \square

Although the equality $J(K)_{\mathbb{Q}} = H^1_f(K,V_p)$ is only conjectural (as it follows from Conjecture 3.3), we still have a natural map $X(K) \to H^1_f(K,V_p)$, which means that we may rewrite Chabauty’s diagram in the following way:

\[(A') \quad \begin{array}{ccc}
X(K) & \xrightarrow{\kappa} & X(K_p) \\
\downarrow & & \downarrow \kappa_p \\
H^1_f(K,V_p) & \longrightarrow & H^1_f(K_p,V_p) \xrightarrow{\log} T
\end{array}\]

We have partially achieved our goal, as we have rewritten the diagram in a way that should be equally amenable as Diagram A to proving finiteness of $X(K)$ (that is, as long as Conjecture 3.3 is true), and in which $J(K)$ and $J(K_p)$ were replaced by objects defined purely in terms of $X$.

However, for the moment, our only way to define log is to compose the logarithm (tensored with $\mathbb{Q}$) associated to the $p$-adic Lie group $J(K_p)$ with the inverse of $\kappa_p$ in Theorem 3.11. This reliance on $J$ means that we do not yet have an intrinsic definition of log (or of its target $T$, for that matter). We now explain how Bloch and Kato’s work defines log intrinsically.

### 3.2. Intrinsic Definition of the Logarithm

Before defining log intrinsically, we must define its target $T$.

**3.2.1. De Rham Homology and its Hodge Filtration.** Recall that, as long as $X$ is proper, $T$ is the dual of $H^0(J_{K_p}, \Omega^1)$. But $H^0(J_{K_p}, \Omega^1)$ has a description in terms of cohomology; more precisely, in terms of the Hodge filtration on the de Rham cohomology of $J_{K_p}$. There is a decreasing filtration $F^i$ on $H^1_{dR}(J_{K_p})$, with

$$0 = F^2H^1_{dR}(J_{K_p}) \subseteq H^0(J_{K_p}, \Omega^1) = F^1H^1_{dR}(J_{K_p}) \subseteq F^0H^1_{dR}(J_{K_p}) = H^1_{dR}(J_{K_p}).$$

It is also true that $\text{Gr}^0H^1_{dR}(J_{K_p}) \cong H^1(J_{K_p}, \mathcal{O}_{J_{K_p}})$, but we will not need this fact. Furthermore, when $X$ is not proper, it is still true that $F^1H^1_{dR}(J_{K_p})$ is isomorphic to the space of translation-invariant holomorphic differentials on $J$.

The de Rham homology $H^1_{dR}(J_{K_p})$ is isomorphic to the dual of $H^1_{dR}(J_{K_p})$ (as with étale homology, one can give an intrinsic definition, but this is not needed). It has a decreasing Hodge filtration dual to that of $H^1_{dR}(J_{K_p})$, defined as an annihilator

$$F^iH^1_{dR}(J_{K_p}) := \text{Ann}(F^{-i+1}H^1_{dR}(J_{K_p})).$$
The funny-looking index $-i+1$ ensures that $\text{Gr}^i H^1_{\text{dR}}(J_{K_p})$ is dual to $\text{Gr}^{-i} H^1_{\text{dR}}(J_{K_p})$.

The upshot is that $T$, isomorphic to the dual of $H^0(J_{K_p}, \Omega^1) = \text{Gr}^1 H^1_{\text{dR}}(J_{K_p})$, is just $\text{Gr}^{-1} H^1_{\text{dR}}(J_{K_p}) = H^1_{\text{dR}}(J_{K_p})/F^0 H^1_{\text{dR}}(J_{K_p})$. In fact, this is intrinsic to $X$, because the inclusion $X \hookrightarrow J$ induces an isomorphism on first homology and cohomology, so we have

$$T = H^1_{\text{dR}}(X_{K_p})/F^0 H^1_{\text{dR}}(X_{K_p}).$$

This is our desired intrinsic definition of $T$.

Bloch-Kato were furthermore able to define a logarithm map $H^1_{\text{dR}}(K_p, V_p) \to T$ purely in terms of the representation $V_p$ (i.e., without reference to $J$). The reader who wishes to take this on faith may safely skip to Section 3.3.

For the interested reader who wishes to see a sketch of this construction, we have to recall how $p$-adic Hodge theory relates the $p$-adic étale cohomology of $X$ to the de Rham cohomology of $X_{K_p}$.

3.2.2. *More p-adic Hodge Theory*. We work only over $K_p \cong \mathbb{Q}_p$. Fontaine’s theory defines a series of $\mathbb{Q}_p$-algebras with $G_{\mathbb{Q}_p}$-action. The two we will need are

$$B_{\text{crys}} \subseteq B_{\text{dR}}.$$ 

Their actual definitions do not concern us. The important facts are as follows. There is a descending filtration $F^i$ on $B_{\text{dR}}$, for which $B^+_i := F^i B_{\text{dR}}$ is a DVR with fraction field $B_{\text{dR}}$ and residue field $\mathbb{C}_p$, the $p$-adic completion of $\mathbb{Q}_p$. There is a Frobenius $\phi$ acting on $B_{\text{crys}}$, whose fixed subring is denoted $B^0_{\text{crys}}$. We also have $B_{\text{crys}}^G = B_{\text{dR}}^G = K_p$, and for a continuous $p$-adic representation $V$ of $G_{\mathbb{Q}_p}$, we define

$$D_{\text{dR}}(V) = (B_{\text{dR}} \otimes \mathbb{Q}_p V)^G_{K_p},$$

$$D^+_{\text{dR}}(V) = (B^+_i \otimes \mathbb{Q}_p V)^G_{K_p},$$

$$D_{\text{crys}}(V) = (B_{\text{crys}} \otimes \mathbb{Q}_p V)^G_{K_p}.$$ 

The decreasing filtration on $B_{\text{dR}}$ induces a filtration on $D_{\text{dR}}(V)$, known as the *Hodge filtration*. If $Y$ is a smooth variety over $\mathbb{Q}_p$, and $V = H^i_{\text{dR}}(Y_{\overline{\mathbb{Q}_p}}, \mathbb{Q}_p)$, then an important theorem in $p$-adic Hodge theory tells us that there is a natural isomorphism

$$D_{\text{dR}}(V) \to H^i_{\text{dR}}(Y)$$

respecting the Hodge filtrations on each side. In addition, $D^+_{\text{dR}}(V)$ is naturally identified with $F^0 D_{\text{dR}}(V)$.

Similarly, the Frobenius on $B_{\text{crys}}$ induces a Frobenius map on $D_{\text{crys}}(V)$, and if $V = H^i_{\text{dR}}(Y_{\overline{\mathbb{Q}_p}}, \mathbb{Q}_p)$, and $Y$ has good reduction with special fiber $Y_{\mathbb{F}_p}$, then there is a natural isomorphism

$$D_{\text{crys}}(V) \to H^i_{\text{crys}}(Y_{\mathbb{F}_p}, \mathbb{Q}_p)$$

respecting the Frobenius morphisms on each side. Note that if $X$ is not proper, we may need to replace crystalline cohomology by log-crystalline or rigid cohomology. Finally, note that $D_{\text{dR}}$ and $D_{\text{crys}}$ commute with taking duals, so the aforementioned results apply equal well to homology.

Let us apply all of that to $Y = X_{K_p}$ and first homology. Once again, let $V_p = H^1_{\text{dR}}(X_{K_p}, \mathbb{Q}_p)$. Then

$$H^1_{\text{dR}}(X_{K_p}) \cong D_{\text{dR}}(V_p) \cong D_{\text{crys}}(V_p),$$

12
and by compatibility with the Hodge filtration, we also have
\[ T \cong D_{\text{dR}}(V_p)/D_{\text{dR}}^+(V_p). \]

Remark 3.14. It turns out that we can now give a precise definition of the crystalline condition (as well as the de Rham condition introduced below in Section 3.3.2), although we left out this definition out earlier because we did not explicitly use it. For a general \( p \)-adic representation \( V \) of \( G_{K_p} \), we always have
\[ \dim D_{\text{crys}}(V) \leq \dim D_{\text{dR}}(V) \leq \dim V. \]

We say that \( V \) is crystalline if
\[ \dim D_{\text{crys}}(V) = \dim V \]
and de Rham if
\[ \dim D_{\text{dR}}(V) = \dim V. \]

The statements above about the cohomology of smooth varieties \( Y/\mathbb{Q}_p \) imply that the \( p \)-adic cohomology of any variety is de Rham as a representation of \( G_{\mathbb{Q}_p} \), and crystalline if the variety has good reduction at \( p \).

3.2.3. The Bloch-Kato Exponential Map. Armed with all the background from Section 3.2.2, we are now able to sketch how Bloch-Kato defined the map
\[ \log H^1_f(K_p, V_p) \rightarrow T \]
purely in terms of the representation \( V_p \).

For this, we note that by \[ \text{[BK90, 1.17]}, \] there is a short exact sequence
\[ 0 \rightarrow \mathbb{Q}_p \xrightarrow{\alpha} B^{\phi=1}_{\text{crys}} \oplus B_{\text{dR}}^+ \xrightarrow{\beta} B_{\text{dR}} \rightarrow 0, \]
defined by \( \alpha(x) = (x, x) \) and \( \beta(x, y) = x - y \).

Upon tensoring with \( V \) over \( \mathbb{Q}_p \) and taking Galois cohomology, as well as noting by \[ \text{[BK90, Lemma 3.8.1]} \] that the map \( H^1(K_p, V \otimes B_{\text{dR}}^+) \rightarrow H^1(K_p, B_{\text{dR}}) \) is injective, we get a long exact sequence
\[ 0 \rightarrow V^{G_{K_p}} \rightarrow D_{\text{crys}}(V_p)^{\phi=1} \oplus D_{\text{dR}}^+(V_p) \rightarrow D_{\text{dR}}(V_p) \rightarrow H^1(K_p, V_p) \rightarrow H^1(K_p, V_p \otimes B^{\phi=1}_{\text{crys}}) \]

Bloch-Kato also prove that the kernel
\[ \ker(H^1(K_p, V_p) \rightarrow H^1(K_p, V_p \otimes B^{\phi=1}_{\text{crys}})), \]
which they denote \( H^1_e(K_p, V_p) \), is equal to \( H^1_f(K_p, V_p) \). From this, we get a surjective map
\[ T = D_{\text{dR}}(V_p)/D_{\text{dR}}^+(V_p) \rightarrow H^1_f(K_p, V_p), \]
known as the Bloch-Kato exponential map. Bloch-Kato (\[ \text{[BK90, Example 3.10.1-3.11]} \]) show that this coincides with the ordinary exponential map in the case of a \( p \)-adic formal Lie group and of an abelian variety, respectively.

The kernel of this map is \( D_{\text{crys}}(V_p)^{\phi=1}/V^{G_{K_p}} \), which is 0 when \( V_p \) is the Tate module of an abelian (or even semi-abelian) variety, essentially by the Weil conjectures (which imply that \( \phi \) has no eigenvalues equal to 1). In that case, it has an inverse, which is our intrinsically-defined logarithm.
3.3. **Bloch-Kato for Non-Proper** $X$. The reader wishing to see Kim’s method more quickly may take on faith that for a general semi-abelian variety $J$:

1. There is a notion of $p$-adic selmer group

   $$\text{Sel}_{p^\infty}(J) \subseteq H^1(K, T_p)$$

   relative to an integer ring $Z = \mathcal{O}_K[1/S]$,

2. The image of

   $$\kappa: J(Z)_{Q_p} \rightarrow H^1(K, V_p)$$

   is contained in $\text{Sel}_{p^\infty}(J)_{Q_p}$, and they coincide as long as Conjecture 3.3 is true, and

3. The definition of $\text{Sel}_{p^\infty}(J)_{Q_p}$ is intrinsic to the Galois representation $V_p = H^\text{et}_1(J, \mathbb{Q}_p) = H^\text{et}_1(X, \mathbb{Q}_p)$,

and skip to Section 3.4. We have already explained this when $J$ is an abelian variety; the reader wishing to see the more general case of a semi-abelian variety may continue below.

We suppose we’re looking at $Z$-points for $Z = \mathcal{O}_K[1/S]$, with $S$ a finite set of places not containing $p$. The basic modification to Diagram D is that for $v \notin S$, we want to consider the image under $\beta$ not of $J(K_v)$ (or its $\mathbb{Q}_p$-completion) but of $J(O_v)$. In other words, we will define $\text{Sel}_{p^\infty}(J)_{Q_p}$ as the preimage under $\alpha$ of the image of $\beta$ in the diagram:

$$(D'')$$

$$J(Z)_{Q_p} \xrightarrow{\kappa_{p^\infty}} H^1(K, V_p)$$

$$\downarrow$$

$$\prod_{v \in S} J(O_v)_{Q_p} \times \prod_{v \in S} J(K_v)_{Q_p} \xrightarrow{\beta} \prod_{v \in \Sigma_K} H^1(K_v, V_p),$$

This establishes (1). As well, (2) follows by combining the case of an abelian variety with Kummer theory for algebraic tori (essentially following the Kummer theory discussed in 3.3.1 with $K$ in place of $K_v$).

Now, we deal with (3). Write $\beta = \prod_{v \in \Sigma_K} \kappa_v$, where for $v \in S$, we have

$$\kappa_v: J(K_v)_{Q_p} \rightarrow H^1(K_v, V_p),$$

and for $v \notin S$,

$$\kappa_v: J(O_v)_{Q_p} \rightarrow H^1(K_v, V_p).$$

**Definition 3.15.** For each place $v$ of $K$, we refer to the image of $\kappa_v$ as the local Bloch-Kato Selmer group at $v$.

In this terminology, the group $\text{Sel}_{p^\infty}(J)_{Q_p}$ is the preimage under $\alpha$ of the product of all the local Bloch-Kato Selmer groups.

Our goal for the rest of Section 3.3 is to explain why the local Bloch-Kato Selmer group can be defined intrinsically in terms of the representation $V_p$. This will in turn give an intrinsic definition of $\text{Sel}_{p^\infty}(J)_{Q_p}$.

Recall that in Remark 3.8 we mentioned that when $X$ is proper, we may ignore $v$ of residue characteristic $\ell$ different from $p$. This is no longer true if we are considering non-proper $X$. More specifically, we will need local conditions for all $v \notin S$.

---

6Note that this definition of local Bloch-Kato Selmer groups agrees a posteriori with the standard one, although it is not a priori the same. Usually, one first gives the intrinsic definition of local Bloch-Kato Selmer groups via $p$-adic Hodge theory, and then proves as a theorem that they are the same as our definition.
As well, Remark 3.7 is no longer quite true. For example, for \( J = \mathbb{G}_m \), the group \( J(K_v) \) is product of \( \mathbb{Z} \) with a compact \( \ell \)-adic Lie group, so it does not surject onto its \( p \)-adic completion (whether or not \( \ell = p \)). Therefore, the \( p \)-adic completions in Diagram D” are necessary.

For the previous two reasons, we have to be more careful for non-proper \( X \) and \( J \).

### 3.3.1 Bloch-Kato for \( \mathbb{G}_m \)

For simplicity, we first explain what happens when \( J = \mathbb{G}_m \). The case of a general semi-abelian variety will then be a simple combination of what we know for \( \mathbb{G}_m \) and what we know for abelian varieties. We let

\[
T_p = H^1_{\text{ét}}(J_{\overline{\kappa}}, \mathbb{Z}_p) = \mathbb{Z}_p(1).
\]

By Kummer theory, there is an isomorphism

\[
\kappa_v : \widehat{K_v}^\times \sim H^1(K_v, T_p).
\]

Therefore, for \( v \notin S \), the local Bloch-Kato Selmer group is the whole group \( H^1(K_v, V_p) \).

We now cover \( v \in S \). We let \( \ell \) denote the residue characteristic of \( v \).

When \( \ell = p \), the group \( K_v^\times \) is the product of a compact \( p \)-adic Lie group of dimension \([K_v : \mathbb{Q}_p]\) with \( \mathbb{Z} \), so \( \widehat{K_v}^\times \cong H^1(K_v, T_p) \) is a \( \mathbb{Z}_p \)-module of rank \( 1 + [K_v : \mathbb{Q}_p] \). The group \( \widehat{\mathcal{O}_v}^\times \subseteq \widehat{K_v}^\times \) has rank \( [K_v : \mathbb{Q}_p] \), and its image in \( H^1(K_v, T_p) \) consists entirely of crystalline classes. Bloch-Kato show that the group \( H^1_f(K_v, T_p) \) of crystalline classes has rank \( [K_v : \mathbb{Q}_p] \), which implies that the image of

\[
\kappa_v : \widehat{\mathcal{O}_v}^\times \otimes \mathbb{Q} \to H^1(K_v, V_p)
\]

is precisely \( H^1_f(K_v, V_p) \).

When \( \ell \neq p \), the group \( \mathcal{O}_v^\times \) is a compact \( \ell \)-adic Lie group, so its \( p \)-completion is finite, and its \( \mathbb{Q} \)-tensorization is trivial. Therefore, we may take the local Bloch-Kato Selmer group to be 0.

We summarize the cases as follows:

\[
\begin{array}{|c|c|c|}
\hline
\ell & v \in S & v \notin S \\
\hline
p & H^1(K_v, V_p) & H^1_f(K_v, V_p) \\
\ell \neq p & H^1(K_v, V_p) & 0 \\
\hline
\end{array}
\]

### 3.3.2 Bloch-Kato for Semi-abelian Varieties

Everything in Section 3.3.1 applies whenever \( J \) is an algebraic torus, which follows by Galois descent applied to a product of copies of \( \mathbb{G}_m \). However, when \( J \) is a general semi-abelian variety, we need to combine what we just did for \( \mathbb{G}_m \) with what we did in Section 3.1.3.

The trickiest case is when \( \ell = p \), but \( v \in S \). In that case, we are considering rational points \( J(K_v) \). When \( J \) is an abelian variety, these are cut out by the crystalline condition. But when \( J \) is a torus, the crystalline condition cuts out \( J(\mathcal{O}_v) \), not the full group \( J(K_v) \). We therefore need a new \( p \)-adic Hodge theory condition that corresponds to crystalline classes for an abelian variety but all classes for an algebraic torus.

In addition to the category of crystalline representations, there is an intermediate subcategory

\{crystalline representations\} \subseteq \{de Rham representations\} \subseteq \{all \ \mathbb{Q}_p \text{-representations of } G_{K_v}\},

which is the \( \ell = p \) analogue of all representations in the \( \ell \neq p \). In particular, every representation coming from the étale cohomology of a variety is de Rham, but not all continuous \( \mathbb{Q}_p \)-representation of \( G_{K_v} \) is de Rham.
We recall the notation that if \( \alpha \in H^1(K_v, V_p) \), there is a corresponding extension
\[
0 \rightarrow V_p \rightarrow E_\alpha \rightarrow \mathbb{Q}_p \rightarrow 0
\]
of representations of \( G_{K_v} \).

**Definition 3.16.** We say that \( \alpha \in H^1(K_v, V_p) \) is **de Rham** if the representation \( E_\alpha \) is de Rham as a representation of \( G_{K_v} \).

**Definition 3.17.** Following [BK90], the subgroup of all de Rham elements of \( H^1(K_v, V_p) \) is denoted
\[
H^1_g(K_v, V_p).
\]

It is a theorem that any representation coming from the étale cohomology of a variety is de Rham. Therefore, the image of \( \kappa_v : \hat{K}_v \otimes \mathbb{Q} \cong H^1(K_v, \mathbb{Q}_p(1)) \) consists entirely of de Rham classes, so
\[
H^1_g(K_v, \mathbb{Q}_p(1)) = H^1(K_v, \mathbb{Q}_p(1)).
\]

On the other hand, if \( V_p \) is the (rational) Tate module of an abelian variety, we have
\[
H^1_f(K_v, V_p) = H^1_g(K_v, V_p).
\]

The reader should think of this as a cohomological manifestation of the fact that integral and rational points are the same on a proper variety.

Combining these two cases, it makes sense to say that when \( \ell = p \) and \( v \notin S \), our local Bloch-Kato Selmer group is \( H^1_g(K_v, V_p) \).

In the \( \ell = p \) and \( v \in S \) case, we do not need any modifications, as the image of
\[
\hat{J}(\mathcal{O}_v) \otimes \mathbb{Q} \rightarrow H^1(K_v, V_p)
\]
is just \( H^1_f(K_v, V_p) \), the subgroup of crystalline classes.

For \( \ell \neq p \) and \( v \in S \), we note that
\[
\hat{J}(K_v) \otimes \mathbb{Q} \rightarrow H^1(K_v, V_p)
\]
is an isomorphism; this is an easy combination of Remark 3.8 with the case \( \ell = pv \notin S \) for \( J = \mathbb{G}_m \).

Finally, for \( \ell \neq p \) and \( v \notin S \), we note that \( J(\mathcal{O}_v) \) is a compact \( \ell \)-adic Lie group, so its \( p \)-adic completion is torsion. In this case, the local Bloch-Kato Selmer group is 0.

We summarize the cases as follows:

<table>
<thead>
<tr>
<th>( \ell = p )</th>
<th>( \ell = p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v \in S )</td>
<td>( H^1_g(K_v, V_p) )</td>
</tr>
</tbody>
</table>

3.4. **Summary of Intrinsic Chabauty-Skolem.** We may now rewrite Diagram A in a way that refers only to \( X \) and its first homology:

\[
\begin{align*}
(A'^{v}) & \quad X(\mathbb{Z}) \quad \xrightarrow{f} \quad X(\mathbb{Z}_p) \\
& \quad \xrightarrow{H^1_f(K, H^1_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)} \quad \xrightarrow{\text{log}_{\text{BK}}} \quad H^1_{\text{dR}}(X_{\mathbb{K}_v})/F^0H^1_{\text{dR}}(X_{\mathbb{K}_v})
\end{align*}
\]
Notice that we’ve written out $H_1^{\et}(X_{\overline{K}}, \mathbb{Z}_p)$ rather than the equivalent shorthand $T_p$. This is to emphasize the fact that $T_p$ only depends on the first homology (and therefore fundamental group) of $X$.

Recall the descending central series filtration of $\pi_1(X)$ by

$$\pi_1(X)^{[1]} := \pi_1(X)$$

$$\pi_1(X)^{[n]} := [\pi_1(X)^{[n-1]}, \pi_1(X)],$$

and the corresponding quotients

$$\pi_1(X)_n := \pi_1(X)/\pi_1(X)^{[n+1]},$$

so that $\pi_1(X)^{ab} = \pi_1(X)_1 = H_1(X)$.

Our goal is to think of the homology in Diagram A” as the abelianization $\pi_1(X)^{ab} = \pi_1(X)_1$, and then replace $\pi_1(X)_1$ by $\pi_1(X)_n$ for $n > 1$. In the next section, we explain how to do this.

4. Making Sense of Chabauty-Skolem for Non-Abelian Quotients

****Need to add more detail in some places here, but the basic idea is there.

4.1. Chabauty-Skolem in terms of the Fundamental Group. We just mentioned that we want to replace all instances of first homology of $X$ in Diagram A” by something in the form $\pi_1(X)^{ab} = \pi_1(X)_1$. We must now be precise about what kind of fundamental group we are using.

There are two instances of homology where we must do this. The first instance is the $p$-adic étale homology $H_1^{\et}(X_{\overline{K}}, \mathbb{Z}_p)$. Note that $\pi_1^{\et}(X_{\overline{K}})_1$ is isomorphic to $H_1^{\et}(X_{\overline{K}}, \hat{\mathbb{Z}})$. Letting $\pi_1^{\et}(X_{\overline{K}})^{(p)}$ denote the pro-$p$ completion of $\pi_1^{\et}(X_{\overline{K}})$, we have

$$H_1^{\et}(X_{\overline{K}}, \mathbb{Z}_p) = \pi_1^{\et}(X_{\overline{K}})_1^{(p)}.$$

One may define $\pi_1^{\et}(X_{\overline{K}})^{(p)}_n$ for $n > 1$, but it is a priori a mystery what one means by the tensorization $\pi_1^{\et}(X_{\overline{K}})^{(p)}_n \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. As an approximation, note that

$$\pi_1^{\et}(X_{\overline{K}})^{(p),[n]} / \pi_1^{\et}(X_{\overline{K}})^{(p),[n+1]} = \text{Im}(\pi_1^{\et}(X_{\overline{K}})^{(p),[n]} \to \pi_1^{\et}(X_{\overline{K}})_n^{(p)})$$

is a free $\mathbb{Z}_p$-module of finite rank, so its $\mathbb{Q}_p$-tensorization is a $p$-adic representation of $G_K$. Therefore, whatever it is, the $n$th subquotient of the descending central series filtration of $\pi_1^{\et}(X_{\overline{K}})_n^{(p)} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ should be Galois-equivariantly isomorphic to

$$(\pi_1^{\et}(X_{\overline{K}})^{(p),[n]} / \pi_1^{\et}(X_{\overline{K}})^{(p),[n+1]} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p).$$

The second instance is the de Rham homology $H_1^{dR}(X_{\overline{K}_p})$. There is a notion of de Rham fundamental group $\pi_1^{dR}(X_{\overline{K}_p})$, which we will explain in more detail in Section 4.2.1.

It turns out that each of $\pi_1^{\et}(X_{\overline{K}})_n^{(p)} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and $\pi_1^{dR}(X_{\overline{K}_p})$ has the structure of a pro-unipotent group over $\mathbb{Q}_p$. We therefore now review the notion of pro-unipotent groups and pro-unipotent completion.
4.2. Pro-Unipotent Groups.

**Definition 4.1.** A pro-unipotent group over $\mathbb{Q}_p$ is a group scheme over $\mathbb{Q}_p$ that is a projective limit of unipotent algebraic groups over $\mathbb{Q}_p$.

Any unipotent group over $\mathbb{Q}_p$ is trivially a pro-unipotent group. Furthermore, an abelian unipotent group is the same thing as a vector space over $\mathbb{Q}_p$, under the identification

$$V \mapsto \text{Spec}(\text{Sym} V^\vee).$$

There is a notion of the $\mathbb{Q}_p$-pro-unipotent completion $\pi_1^{\text{ét}}(X_{\mathbb{Q}_p})^\wedge$ of the profinite group $\pi_1^{\text{ét}}(X_{\mathbb{Q}_p})$. One may define the lower central series quotients

$$\pi_1^{\text{ét}}(X_{\mathbb{Q}_p})_{n} = \pi_1^{\text{ét}}(X_{\mathbb{Q}_p}) / \pi_1^{\text{ét}}(X_{\mathbb{Q}_p})^{[n+1]},$$

and in the cases we consider, these quotients are not only pro-unipotent but actually unipotent algebraic groups (i.e., they are finite-dimensional varieties).

Then $\pi_1^{\text{ét}}(X_{\mathbb{Q}_p})_{n}$ is the $\mathbb{Q}_p$-pro-unipotent completion of $\pi_1^{\text{ét}}(X_{\mathbb{Q}_p})_{n}$, and our desired isomorphism

$$\pi_1^{\text{ét}}(X_{\mathbb{Q}_p})_{n} / \pi_1^{\text{ét}}(X_{\mathbb{Q}_p})^{[n+1]} \cong (\pi_1^{\text{ét}}(X_{\mathbb{Q}_p})^{(p)}, [n] / \pi_1^{\text{ét}}(X_{\mathbb{Q}_p})^{(p)}, [n+1]) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

is satisfied. In particular, $\pi_1^{\text{ét}}(X_{\mathbb{Q}_p})$ has the important property that its abelianization is naturally isomorphic to $H_1^{\text{ét}}(X_{\mathbb{Q}_p})$.

Furthermore, the continuous Galois action of $G_K$ on $\pi_1^{\text{ét}}(X_{\mathbb{Q}_p})$ induces a continuous (for the topology of $\mathbb{Q}_p$) action of $G_K$ on $\pi_1^{\text{ét}}(X_{\mathbb{Q}_p})$. The isomorphism (2) is Galois-equivariant for this action.

Notice that $\pi_1^{\text{ét}}(X_{\mathbb{Q}_p})$ is in particular a scheme over $\mathbb{Q}_p$, so its coordinate ring $\mathcal{O}(\pi_1^{\text{ét}}(X_{\mathbb{Q}_p})_{\mathbb{Q}_p})$ is a vector space over $\mathbb{Q}_p$. In particular, the continuous action of $G_K$ on $\pi_1^{\text{ét}}(X_{\mathbb{Q}_p})_{\mathbb{Q}_p}$ is the same as giving the vector space

$$\mathcal{O}(\pi_1^{\text{ét}}(X_{\mathbb{Q}_p})_{\mathbb{Q}_p})$$

the structure of a continuous $\mathbb{Q}_p$-linear representation of $G_K$, for which the multiplication and comultiplication maps are Galois equivariant. Note also that for $v \in \Sigma_K$, we can restrict this to give a representation of $G_{K_v}$.

**Remark 4.2.** Normally, one works with $p$-adic Galois representations on finite-dimensional vector spaces. The theory of finite-dimensional $p$-adic Galois representations still applies to

$$\mathcal{O}(\pi_1^{\text{ét}}(X_{\mathbb{Q}_p})_{\mathbb{Q}_p}),$$

because it is an inductive limit of finite-dimensional $p$-adic Galois representations. For example, we say that it is crystalline or de Rham if it is an inductive limit of crystalline or de Rham representations, and we define $D_{\text{dr}}$ and $D_{\text{crys}}$ so that they commute with inductive limits of finite-dimensional representations.
4.2.1. The de Rham Fundamental Group. Just as the de Rham homology is $D_{dR}$ of the $p$-adic étale homology, we may define the de Rham fundamental group $\pi_1^{dR}(X_{K_p})$ of $X_{K_p}$ as

$$\text{Spec } D_{dR}(\mathcal{O}(\pi_1^{\text{ét}}(X_{\overline{K}})_{\mathbb{Q}_p})).$$

Note that $D_{dR}$ is compatible with tensor products, so the Hopf algebra structure on $\mathcal{O}(\pi_1^{\text{ét}}(X_{\overline{K}})_{\mathbb{Q}_p})$ makes $\pi_1^{dR}(X_{K_p})$

There is even a Bloch-Kato exponential map, which we will discuss in Section 4.3.3. However, in order to compute the composition of the logarithm with $\kappa_p$ (a.k.a. the diagonal arrow of Diagrams A-A$^\flat$), we must express it as some sort of integration, as we did in the abelian case in Section 1.1.1. Therefore, we will need to give a more intrinsic definition of $\pi_1^{\text{dR}}(X_{K_p})$ in terms of differential calculus, rather than in terms of $p$-adic Hodge theory.

The definition turns out to be the Tannakian fundamental group of the category of algebraic vector bundles on $X_{K_p}$ with nilpotent integrable connection. It has a Hodge filtration. It is also the same as

$$\text{Spec } D_{\text{crys}}(\mathcal{O}(\pi_1^{\text{ét}}(X_{\overline{K}})_{\mathbb{Q}_p})), $$

which gives it a Frobenius action. We leave non-abelian Coleman integration (i.e., the explicit description of the diagonal arrow) to Section 4.3.4.

Finally, just as with $\pi_1^{\text{ét}}(X_{\overline{K}})_{\mathbb{Q}_p,n}$, there is a descending central series filtration, along with corresponding quotients $\pi_1^{dR}(X_{K_p})_n$.

This is compatible via $p$-adic Hodge theory with the descending central series filtration on $\pi_1^{\text{ét}}(X_{\overline{K}})_{\mathbb{Q}_p,n}$, in the sense that

$$D_{dR}\mathcal{O}(\pi_1^{\text{ét}}(X_{\overline{K}})_{\mathbb{Q}_p,n}) \cong \mathcal{O}(\pi_1^{dR}(X_{K_p})_n).$$

4.3. Making sense of the Chabauty-Skolem diagram with Unipotent Fundamental Group. Armed with our notion of $\mathbb{Q}_p$-unipotent fundamental group $\pi_1^{\text{ét}}(X_{\overline{K}})_{\mathbb{Q}_p}$, we can make a non-abelian version of Diagram A. We choose some level of nilpotency $n$ and simply replace $H_1^{\text{ét}}(X_{\overline{K}},\mathbb{Q}_p)$ by $\pi_1^{\text{ét}}(X_{\overline{K}})_{\mathbb{Q}_p,n}$, and $H_1^{dR}(X_{K_p})$ by $\pi_1^{dR}(X_{K_p})_n$:

(Non-Abelian A)

$$
\begin{array}{ccc}
X(Z) & \rightarrow & X(Z_p) \\
\downarrow & & \downarrow f \\
H_1^1(K,\pi_1^{\text{ét}}(X_{\overline{K}})_{\mathbb{Q}_p,n}) & \rightarrow & H_1^1(K_p,\pi_1^{\text{ét}}(X_{\overline{K}})_{\mathbb{Q}_p,n}) \\
& \rightarrow_{\text{log-BK}} & \pi_1^{dR}(X_{K_p})_n/F_0^n\pi_1^{dR}(X_{K_p})_n
\end{array}
$$

This diagram is known as Kim’s cutter. We must now describe how to make sense of each part of this diagram.

4.3.1. Non-Abelian Cohomology Varieties. We first mention what we mean by $H_1^1(K,\pi_1^{\text{ét}}(X_{\overline{K}})_{\mathbb{Q}_p,n})$.

For a group $G$ acting on a group $U$, the set $H^1(G,U)$ is defined to be the set of isomorphism classes of $G$-equivariant torsors under $U$.

Therefore, for us, an element of $H_1^1(K,\pi_1^{\text{ét}}(X_{\overline{K}})_{\mathbb{Q}_p,n})$ is a scheme over $\mathbb{Q}_p$ with a continuous action of $G_K$ and a $G_K$-equivariant algebraic action of $\pi_1^{\text{ét}}(X_{\overline{K}})_{\mathbb{Q}_p,n}$ making it into a $\pi_1^{\text{ét}}(X_{\overline{K}})_{\mathbb{Q}_p,n}$-torsor.
Kim defines this in \cite{Kim05} §1 and shows that the short exact sequence
\[ 0 \to \pi_1^{\text{ét}}(X_K)^[n] / \pi_1^{\text{ét}}(X_K)[n+1] \to \pi_1^{\text{ét}}(X_K)_{Q_p,n} \to \pi_1^{\text{ét}}(X_K)_{Q_p,n-1} \to 0 \]
gives rise to a long exact sequence
\[ \cdots \to H^1(K, \pi_1^{\text{ét}}(X_K)^[n] / \pi_1^{\text{ét}}(X_K)[n+1]) \to H^1(K, \pi_1^{\text{ét}}(X_K)_{Q_p,n}) \to H^1(K, \pi_1^{\text{ét}}(X_K)_{Q_p,n-1}) \to \cdots \]
(3)

Note that, since \( \pi_1^{\text{ét}}(X_K)^[n] / \pi_1^{\text{ét}}(X_K)[n+1] \) is an ordinary \( p \)-adic Galois representation, the cohomology group \( H^1(K, \pi_1^{\text{ét}}(X_K)^[n] / \pi_1^{\text{ét}}(X_K)[n+1]) \) has the structure of a \( \mathbb{Q}_p \)-vector space. Because we are considering non-abelian cohomology, there is no natural group structure on \( H^1(K, \pi_1^{\text{ét}}(X_K)_{Q_p,n}) \). But using induction on the long exact sequences (3), Kim \cite{Kim05} gives it the structure of an affine variety over \( \mathbb{Q}_p \) (in fact isomorphic to an affine space).

4.3.2. Bloch-Kato Selmer Conditions. We have to understand what it means for an element \( \alpha \in H^1(K, \pi_1^{\text{ét}}(X_K)_{Q_p,n}) \)
to be crystalline, de Rham, or unramified.

In fact, this is simple. Such an element corresponds to a torsor \( T_\alpha \) under \( \pi_1^{\text{ét}}(X_K)_{Q_p,n} \) whose coordinate ring is a \( p \)-adic representation of \( G_{K_v} \). We say that the element is \( \alpha \) is crystalline, de Rham, or unramified, if the corresponding is true of \( O(T_\alpha) \)
as a \( p \)-adic representation of \( G_{K_v} \).

Kim shows that the subset of
\[ H^1(K, \pi_1^{\text{ét}}(X_K)_{Q_p,n}) \]
cut out by the corresponding local conditions, denoted
\[ H^1_f(K, \pi_1^{\text{ét}}(X_K)_{Q_p,n}), \]
is actually a subvariety, known as the Selmer variety. It is discussed in more detail in \cite{Kim09}, where it is denoted \( \text{Sel}(X/Z)_n \).

We similarly refer to \( H^1_f(K_{p,1}, \pi_1^{\text{ét}}(X_K)_{Q_p,n}) \) as the local Selmer variety and denote it by \( \text{Sel}(X/Z_p)_n \).

4.3.3. Non-Abelian Bloch-Kato Exponential. We want to define the Bloch-Kato exponential in the non-abelian setting. To do this, we need to understand how to write the Bloch-Kato logarithm in terms of the torsor description of cohomology classes.

As discussed in Section 4.3.1, the cohomology group
\[ H^1(K_v, V_p) \]
is isomorphic to the set of \( G_{K_v} \)-equivariant torsors under \( V_p \). To see how this relates to the definition of cohomology in terms of extensions, note that to the extension
\[ 0 \to V_p \to E_\alpha \to \mathbb{Q}_p \to 0 \]
is associated the preimage \( T_\alpha \) of \( 1 \in \mathbb{Q}_p \). This preimage has a Galois action, in the sense of a Galois action on its coordinate ring as an affine space. But it does not have the structure of a linear representation - for it is merely a torsor under \( V_p \) - and therefore an affine space rather than a vector space.
By considering coordinate rings, one may give $D_{\text{dr}}(T_\alpha)$ the structure of a torsor under $D_{\text{dr}}(V_p)$. Furthermore, there is still a subspace $D^+_{\text{dr}}(T_\alpha) \subseteq D_{\text{dr}}(T_\alpha)$; but this is an affine subspace, not a vector subspace in any natural way.

By analyzing the long exact sequence in Section 3.2.3, it turns out that $\log_{\text{BK}}$ may be defined as follows. Given $\alpha \in H^1_f(K_v, V_p)$, there is a Frobenius action on the affine space $D_{\text{dr}}(T_\alpha) = D_{\text{crys}}(T_\alpha)$, and it has a unique Frobenius-invariant point. The difference of this element with $D_{\text{dr}}(T_\alpha)$ is a well-defined element of $D_{\text{dr}}(V_p)/D^+_{\text{dr}}(V_p)$ that coincides with $\log_{\text{BK}} \alpha$.

In the non-abelian setting, one may define a Bloch-Kato exponential map

$$H^1_f(K_v, \pi_1^{\text{et}}(X_{K_p}, n)) \to D_{\text{dr}} \pi_1^{\text{et}}(X_{K_p}, n)/D^+_\text{dr} \pi_1^{\text{et}}(X_{K_p}, n) = \pi_1^{\text{dr}}(X_{K_p}, n)/F^0 \pi_1^{\text{dr}}(X_{K_p}, n)$$

analogously, as follows. We find the unique Frobenius-invariant point on $D_{\text{crys}}T_\alpha$ and then take the difference between this and $D^+_\text{dr}T_\alpha$. It is a simple exercise to check that this non-abelian Bloch-Kato logarithm is compatible with exact sequences. It then follows by induction on the exact sequences (3) that the non-abelian Bloch-Kato logarithm is compatible with the ordinary Bloch-Kato logarithm for the representations $\pi_1^{\text{et}}(X_{K_p})[n]/\pi_1^{\text{et}}(X_{K_p})[n+1]$.

4.3.4. Non-Abelian Integration. The down-right diagonal arrow labeled ‘$f$’ may be described by iterated Coleman integration. Since we stated the definition of $\pi_1^{\text{dr}}(X_{K_p})$ in terms of vector bundles with integrable connection, we

Recall that all of our constructions are relative to an implicit basepoint $O \in X(Z)$. In fact, the integration $\int$ map applied to $z \in X(Z_p)$ will be an (iterated) integral $\int_O^z$ from $O$ to $z$.

Recall that $\pi_1^{\text{dr}}(X_{K_p})$ is an algebraic group, and so $\pi_1^{\text{dr}}(X_{K_p})/F^0 \pi_1^{\text{dr}}(X_{K_p})$ is an algebraic variety (all over $\mathbb{Q}_p$). So the map $\int$ should be determined by how it pulls back algebraic functions on $\pi_1^{\text{dr}}(X_{K_p})/F^0 \pi_1^{\text{dr}}(X_{K_p})$ to functions on $X(Z_p)$. In fact, the map $\int$ is a Coleman map, which is the same as saying that it pulls back algebraic functions to Coleman functions.

To specify a function on $\pi_1^{\text{dr}}(X_{K_p})$, we recall that it is the Tannakian fundamental group of the category of algebraic vector bundles on $X_{K_p}$ with nilpotent integrable connection. Functions on $\pi_1^{\text{dr}}(X_{K_p})/F^0 \pi_1^{\text{dr}}(X_{K_p})$ come from trivial vector bundles $M$ with nilpotent connection. As $M$ is trivial, we may identify it with its global sections. A function on $\pi_1^{\text{dr}}(X_{K_p})/F^0 \pi_1^{\text{dr}}(X_{K_p})$ arises from a pair of $v \in M$ and $f \in M'$. We denote the function by $(M, v, f)$.

To explain the pullback of $(M, v, f)$ under $\int$, we do the following. We find the unique global horizontal section of $M$ that agrees with $v$ at $O$. Call this section $s$. This section exists and is unique by existence and uniqueness theorems for unipotent $p$-adic differential equations. Finally, we pair $s$ with $f$ to get a $\mathbb{Q}_p$-valued function on $X(Z_p)$. This function is the pullback.

5. The Motivic Tannakian Theory of Dan-Cohen and Wewers

We hope the reader now understands the theoretical underpinnings of Minhyong Kim’s non-abelian version of Chabauty’s method. We now describe of the methods used to compute integral points using non-abelian Chabauty.
5.1. From Kim’s Method to Mixed Tate Motives. For a place \( p \) of \( K \) over \( p \) and a smooth curve \( X \) over \( Z \), we now have a diagram

\[
X(Z) \xrightarrow{\kappa} X(Z_p) \\
\downarrow \kappa \downarrow \kappa_p \\
\text{Sel}(X/Z)_n \xrightarrow{\text{loc}_n} \text{Sel}(X/Z_p)_n,
\]

which we refer to as Kim’s cutter. Kim (\cite{Kim05}) proves for \( X = \mathbb{P}^1 \setminus \{0, 1, \infty\} \) and \( K = \mathbb{Q} \) that the morphism of schemes \( \text{loc}_n \) is non-dominant for sufficiently large \( n \). The ideal

\[ \mathcal{I}_n^{\mathbb{Z}, \text{Kim}} \]

is then defined as the set of pullbacks under \( \kappa_p \) of functions on \( \text{Sel}(X/Z_p)_n \) vanishing on the image of \( \text{loc}_n \).

In order to compute such functions concretely, one must understand \( \text{Sel}(X/Z)_n \) as well as the morphism \( \text{loc}_n \). Let

\[ U_n = \pi_1^{et}(X_K) \mathbb{Q}_p, n \]

denote the \( n \)th quotient of the pro-unipotent completion of the étale fundamental group of \( X_K \) along the descending central series as in \cite{BDCKW}. The Selmer variety is defined so that its set of \( \mathbb{Q}_p \)-points is the set

\[ H^1_f(G_K; U_n) \]

of cohomology classes of \( G_K \) with coefficients in \( U_n \) that are unramified at closed points of \( Z \) and crystalline at primes over \( p \). Both the group \( G_K \) and the local conditions are hard to understand explicitly.

An important observation is that one only needs to understand the category of continuous \( p \)-adic representations of \( G_K \) that appear in \( U_n \) and its torsors. More specifically, \( U_n \) and its torsors are pro-varieties over \( \mathbb{Q}_p \), and their coordinate rings are (ind-)objects of a certain subcategory of the category of all continuous \( p \)-adic representations of \( G_K \).

This subcategory is the category of mixed Tate \( p \)-adic representations of \( \mathbb{Q}_p \) unramified at closed points of \( Z \) and crystalline at places above \( p \); being “mixed Tate” means that its semi-simplification is a direct sum of tensor powers \( \mathbb{Q}_p(n) \) for \( n \in \mathbb{Z} \) of the \( p \)-adic cyclotomic character. The subcategory of semisimple objects is therefore equivalent to the category of representations of \( G_m \), so the full category is equivalent by the Tannakian formalism to the category of representations of a group \( \pi_1^{\text{MT}}(Z) \) isomorphic to an extension of \( G_m \) by a pro-unipotent group. The pro-unipotent group may be determined by computing the Bloch-Kato Selmer groups \( H^1_f(G_K, \mathbb{Q}_p(n)) \) for each \( n \), and these are known (\cite{Sou79}). In this way, the Selmer variety becomes simply the group cohomology of \( \pi_1^{\text{MT}}(Z) \), with no further local conditions (other than those encoded in the category itself).

In fact, the category of mixed Tate Galois representations with local conditions mentioned above is just the extension of scalars from \( \mathbb{Q} \) to \( \mathbb{Q}_p \) of the category

\[ \text{MT}(Z, \mathbb{Q}) \]

of mixed Tate motives over \( Z \) with coefficients in \( \mathbb{Q} \). This latter category was defined in \cite{DG05}, and its Tannakian fundamental group is denoted \( \pi_1^{\text{MT}}(Z) \). The unipotent de Rham fundamental group \( \pi_1^{un}(X) \) is the Tannakian fundamental group of the category of vector bundles with unipotent integrable connection, and the theory of \cite{DG05} (or the later
theories of [Lev10] and [DCS17]) gives it an action of $\pi_{1}^{\text{MT}}(Z)$. This motivic Selmer variety, as developed in [Had11] and [DCW16], is just the group cohomology of $\pi_{1}^{\text{MT}}(Z)$, with coefficients in (a quotient depending on $n$ of) $\pi_{1}^{\text{un}}(X)$.

More specifically, let $\Pi$ denote a $\pi_{1}^{\text{MT}}(Z)$-equivariant quotient of $\pi_{1}^{\text{un}}(X)$. Then there is a motivic version of Kim’s cutter:

$$
\begin{array}{ccc}
X(Z) & \xrightarrow{\kappa} & X(Z_p) \\
\downarrow{\kappa} & & \downarrow{\kappa_p} \\
H^1(\pi_{1}^{\text{MT}}(Z), \Pi) & \xrightarrow{\text{loc}} & \Pi(Q_p)
\end{array}
$$

The computability of this diagram as opposed to Kim’s original diagram comes from our precise understanding of the abstract structure of the group $\pi_{1}^{\text{MT}}(Z)$ and Goncharov’s study of certain special functions on it.

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