Complex Metrics on Spacetime

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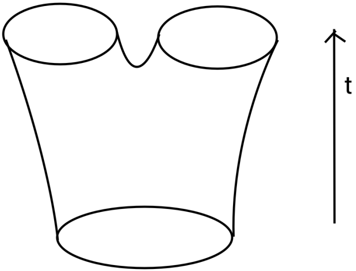
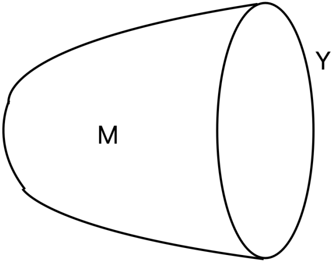
In 1977, Gibbons and Hawking, after using the Euclidean version of the Schwarzschild solution to study the thermodynamics of a Schwarzschild black hole, considered the Kerr solution – a rotating black hole. Upon continuation to imaginary time, the Kerr solution becomes complex – they called it quasi-Euclidean. By computing the action of the quasi-Euclidean solution, they were able to get sensible results for the thermodynamics of a rotating black hole. So this was at least one case in which it appeared appropriate to consider complex solutions of the Einstein equations.

Numerous other possible reasons to consider complex spacetime metrics in quantum gravity have been considered since then. One involves topology change (Louko and Sorkin 1995).



Here there is no smooth Lorentz signature metric, because the "time" would have a stagnation point, but if one slightly regularizes the situation by letting the metric become complex near the critical point of t, one can get what look like sensible results.

Another motivation has to do with the Hartle-Hawking wavefunction of the universe (1983). In D dimensions, let Y be a manifold with metric g_{D-1}. The Hartle-Hawking wavefunction $\Psi_{HH}(g_{D-1})$ is supposed to be computed by summing over all manifolds M with boundary Y, and for each such M one does a path integral over all metrics g on M with boundary value g_{D-1} on Y:



One would do something similar for an ordinary quantum field rather than gravity (with one important difference: in that case one would not sum over M or its metric, but make a particular choice).

For usual quantum fields, to find ground states one uses Euclidean path integrals, so Hartle and Hawking wanted to define a natural state $\Psi_{HH}(g_{D-1})$ for gravity by a Euclidean path integral on M (summed over M).

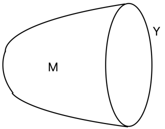
One immediately finds a few strange things about this problem. If the Einstein-Hilbert action were bounded below, then the asymptotic behavior of $\Psi_{HH}(g_{D-1})$ in a semiclassical limit of large volume would be $\exp(-\mathcal{I}(g))$ where $\mathcal{I}(g)$ is the greatest lower bound on the action of any metric (on any M) that satisfies the boundary condition. If the greatest lower bound is positive, we would get exponential decay fo the wavefunction for large volumes. For an ordinary field ϕ with a positive-definite action, the corresponding wavefunction $\Psi_M(\phi) \sim \exp(-\mathcal{I}(\phi))$ decays exponentially for large ϕ for precisely this reason.

We don't want to predict that it is exponentially unlikely to see a large universe, so we need something else to happen. What saves us is that actually, the Einstein-Hilbert action in Euclidean signature is unbounded below so there is no lower bound on the action of a metric that satisfies the boundary condition, But given this, what does the gravitational path integral mean? Gibbons, Hawking, and Perry (1978) suggested that it is a middle-dimensional "contour" integral in the space of complex-valued metrics. But what is the contour? The only concrete idea is to think of it as a sum of "Lefschetz thimbles" associated to critical points. Semiclassically, this just means that one sums over classical solutions (with no clear guidance on what solutions to pick) and for each solution, one evaluates its contribution perturbatively.

To try to actually compute $\Psi_{HH}(g_{D-1})$ in examples, we need to pick some classical solutions. Here again we run into what looks like an obstacle but perhaps is really a benefit: in simple cases there are (apparently) no real (Euclidean signature) solutions, For instance, consider Einstein's theory with a positive cosmological constant

$$I = \int \mathrm{d}^D x \sqrt{g} \left(-\frac{1}{16\pi G} R + 2\Lambda \right).$$

And let us take the $D-1$-manifold Y to be a sphere S^{D-1} with a round metric of large radius (compared to $1/\sqrt{G\Lambda}$, the length scale set by the cosmological constant). Then it is believed that in this problem



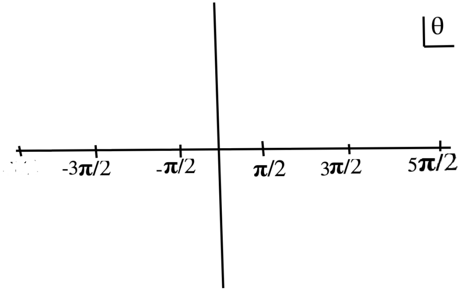
there is no real (Euclidean) classical solution that satisfies the boundary conditions.

If we broaden our horizons a little, however, and allow *complex* solutions of the Einstein equations, then there is one that gives a sensible answer – at least at the level of detail that I will offer today. (In more detail there are unresolved puzzles.) To explain how to make complex solutions, let us start with the metric of a D-sphere

$$\mathrm{d}s^2 = \rho^2 \left(\mathrm{d}\theta^2 + \cos^2\theta \mathrm{d}\Omega^2\right).$$

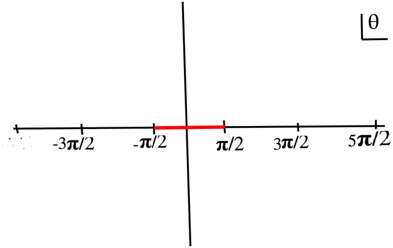
We get a sphere if θ runs on the real axis from $-\pi/2$ to $\pi/2$. But there are a lof of other things that we can make by considering a curve $\theta(u)$ in the complex θ-plane (where u is a real variable).

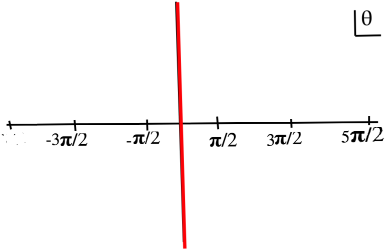
I've drawn the θ plane showing some of the zeroes of the function $\cos\theta$:



The reason the zeroes are important is that if the curve $\theta(u)$ is going to have an endpoint, the endpoint has to be one of the zeroes, or the manifold will have a boundary.

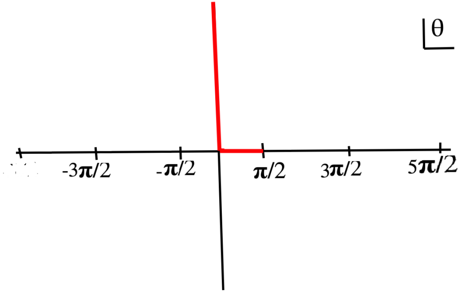
Here are some of the solutions we can make with curves in this picture. First real ones





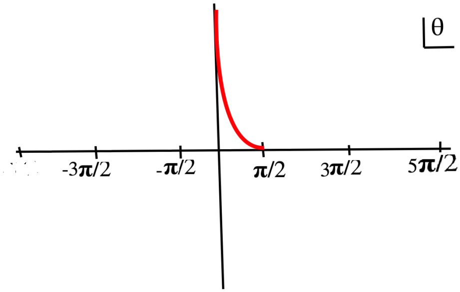
On the left, a sphere, on the right de Sitter space in Lorentz signature (the metric being $\mathrm{d}s^2 = \rho^2(-\mathrm{d}u^2 + \cosh^2 u\, \mathrm{d}\Omega^2)$). These are both considered physically sensible solutions.

Here is another solution that is considered important:



The spacetime is a hemisphere (Euclidean signature) glued onto half of de SItter spacetime (Lorentz signature). It describes creation from nothing of a closed universe that then expands exponentially fast.

A smoother version of the same picture:



Now we have a smooth complex metric, which is asymptotically Lorentzian in the future.

These last two pictures solve the problem of finding a classical solution of Einstein's equations with cosmological constant whose boundary is a round sphere of arbitrarily large radius. The only trick is that the solution is complex. Because it is complex, the "action", which here is a multiple of the volume

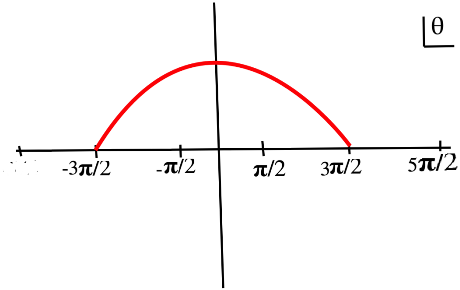
$$\int_M \mathrm{d}\theta \cos^{D-1}\theta \mathrm{d}\Omega$$

is complex. It has the general form $-S/2 + \mathrm{i}\mathcal{I}_{\mathbb{R}}$ with a negative real part $-S/2$ being half the "de Sitter entropy" of Gibbons and Hawking – and an imaginary part $\mathrm{i}\mathcal{I}_{\mathbb{R}}$. So the semiclassical answer is

$$\Psi_{HH}(\rho) \sim \exp(S/2 - \mathrm{i}\mathcal{I}_{\mathbb{R}}(\rho))$$

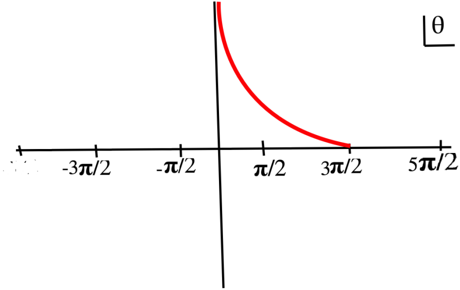
where ρ is the radius of the sphere that we are producing from "nothing." This answer is considered physically sensible, more or less – both the real part and the imaginary part of the exponent.

However, once we start to allow complex metrics, we are opening Pandora's box. We could do lots of other things that will not give physically sensible results. Here is a complex metric on S^D with an action different from the standard value:



A closed loop in the θ plane would instead give a complex classical solution with topology $S^1 \times S^{D-1}$ and zero action.

Here is a complex solution which, if we allow it, gives a contribution to "creation of a universe from nothing" with an even more negative real part of the action:



We can make it worse by starting, say, at $\theta = 5\pi/2$ or $7\pi/2$.

So in short we need a principle that would help us select what complex solutions we consider sensible. Kontsevich and Segal in arXiv:2105.10161 made a suggestion for a distinguished class of "allowable" complex metrics, with the property that ordinary quantum field theory makes sense when coupled to allowable complex metrics. Their motivation was not quantum gravity at all; their goal was to develop an alternative to (some of) the standard axioms of quantum field theory: the new axiom set would assert that quantum field theory can be consistently coupled to allowable metriics. But if there is a good class of complex metrics in which ordinary quantum field theory can be defined, it is natural to think this might be the class one should work in for quantum gravity.

My main observation is that the "good" complex metrics that seem to have given useful results – in the examples that I described and some other examples that we will not really have time for – seem to be allowable, and the obvious "bad" ones, including the ones I've mentioned, and some others that can be constructed similarly, do not seem to be allowable. I should add, before I go on, that Louko and Sorkin, in their work on topology change (1995) that I mentioned before, had a notion of "good" complex metrics that, though not developed as systematically, was somewhat similar to the notion of Kontsevich and Segal.

The idea of Kontsevich and Segal was to require that the theory of a free p-form field A makes sense, for every $p = 0, \cdots, D-1$. Setting $F = \mathrm{d}A$ and $q = p+1$, the action is

$$I_q = \int_M \sqrt{\det g} g^{i_1 j_1} g^{i_2 j_2} \cdots g^{i_q j_q} F_{i_1} F_{i_2 j_2} \cdots F_{i+q j_q}.$$

The requirement that they impose for a metric g to be "allowable" is that $\mathrm{Re}\, I_q > 0$ for every real (nonzero) F and all q. The intuitive idea is that positivity of the real part of the action implies that the path integral of an antisymmetric tensor field coupled to the metric g makes sense. Since the positivity of the real part is supposed to hold for arbitrary real F, it is really a pointwise condition on M: at each point, the quadratic form
$\sqrt{\det g} g^{i_1 j_1} g^{i_2 j_2} \cdots g^{i_q j_q} F_{i_1} F_{i_2 j_2} \cdots F_{i_q j_q}$ on $\wedge^q T^* M$ has positive real part.

One might be a little skeptical of attaching so much weight to antisymmetric tensor fields. Why not symmetric tensors, for instance? According to a result by Weinberg and me (1980 - with an assist from Sidney Coleman), the only massless fields that have local energy-momentum tensors, and therefore can potentially be defined in curved spacetime, are the antisymmetric tensors (and their nonlinear versions for $p = 0, 1$, which are nonlinear sigma-models and nonabelian gauge fields). So actually, by including antisymmetric tensor fields of all ranks, one is essentially including all quantum field theories that can be derived from underlying classical field theories. So it is well-motivated to consider this class of theories.

Kontsevich and Segal give a rather explicit description of the allowable metrics. If g is *allowable*, then there is a real basis of the tangent space at any given point in which the metric is diagonal

$$g_{ij} = \delta_{ij}\lambda_i$$

where

$$\sum_i |\mathrm{Arg}\,\lambda_i| < \pi.$$

Conversely, such a metric is allowable.

The proof goes as follows. The $q = 1$ case of the condition of allowability says that the matrix $\sqrt{g}g^{ij}$ has positive real part. Writing this matrix as $A + \mathrm{i}B$, it follows that A and B can be simultaneously diagonalized by a real linear transformation. (First diagonalize A and use the remaining $SO(D)$ symmetry to also diagonalize B.) So there is a real basis that makes $\sqrt{g}g^{ij}$ diagonal. In the same basis, the inverse matrix $(\sqrt{g})^{-1}g$ is diagonal, and, multiplying by a complex scalar, so is g.

Once we write

$$g_{ij} = \lambda_i \delta_{ij}$$

the condition for the quadratic form $\sqrt{\det g} g^{i_1 j_1} g^{i_2 j_2} \cdots g^{i_q j_q} F_{i_1} F_{i_2 j_2} \cdots F_{i_q j_q}$ to be positive at a given point, for all q, is that for any subset S of the set $\{1, 2, \cdots, q\}$,

$$\mathrm{Re}\left((\prod_i \lambda_i)^{1/2} \prod_{j \in S} \lambda_j^{-1} \right) > 0.$$

The condition for this to be true for all S is

$$\sum_i |\mathrm{Arg}\, \lambda_i| < \pi.$$

A corollary, noted by Kontsevich and Segal, is that the space of allowable metrics is contractible onto the space of Euclidean metrics. After writing $g_{ij} = \lambda_i \delta_{ij}$, since the allowability condition says that the λ_i are not negative, we can in a unique way rotate the λ_i in the complex plane to make them positive, thus contacting the space of allowable metrics onto the space of Euclidean ones. Thus for example, in two dimensions the Gauss-Bonnet integral $\int_M \mathrm{d}^2x\sqrt{g}R/4\pi$ can be defined for a large class of not necessarily allowable complex metrics. As noted by Louko and Sorkin, in general it is a topological invariant (invariant under continuous deformation of g in the space of complex metrrics) but not equal to its usual value. However, for an allowable complex metric, the Gauss-Bonnet integral has its standard value.

If g is allowable, then the volume of M, namely $\int_M \mathrm{d}^D x \sqrt{g}$, has positive real part, since this is the $q = 0$ case of allowability. But Kontsevich and Segal prove that if M has an allowable metric, then the induced metric on any submanifold N of M is also allowable. Therefore the real part of the volume of N is also positive. We can take that as an indication that perturbative string theory, and brane theory, are well-defined on such an M.

The condition for allowability

$$\sum_i |\mathrm{Arg}\,\lambda_i| < \pi$$

shows that a Lorentz signature metric, for instance

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}\vec{x}^2,$$

is not quite allowable. It is on the border of the space of allowable metrics. It can be perturbed in either of two ways to make it allowable:

$$\mathrm{d}s^2 = -(1 \mp \mathrm{i}\epsilon)\mathrm{d}t^2 + \mathrm{d}\vec{x}^2.$$

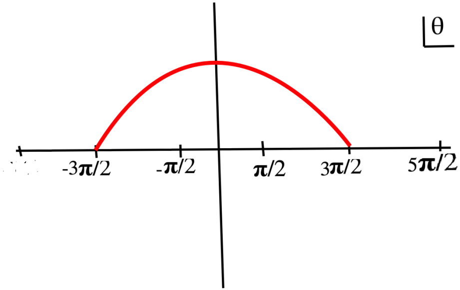
The two choices

$$\mathrm{d}s^2 = -(1 \mp \mathrm{i}\epsilon)\mathrm{d}t^2 + \mathrm{d}\vec{x}^2$$

differ by the sign of $\sqrt{\det g}$. For an allowable metric, $\sqrt{\det g}$ has positive real part, by definition, but when we approach Lorentz signature by taking $\epsilon \to 0$, $\sqrt{\det g}$ approaches the positive or negative imaginary axis, depending on the sign of the $\mathrm{i}\epsilon$. In one case, the limit is a standard Lorentz signature path integral in which the integrand is $e^{\mathrm{i}S}$ (with S being the usual Lorentz signature action). In the other case, we get a complex conjugate Lorentz signature path integral with the integrand being $e^{-\mathrm{i}S}$. In the Schwinger-Keldysh approach to thermal physics, the path integral with $e^{\mathrm{i}S}$ propagates the ket, and the path integral with $e^{-\mathrm{i}S}$ propagates the bra. (Which is which really depends on a convention.) Either way, the ϵ in this formalism is playing a similar role to the usual Feynman $\mathrm{i}\epsilon$: the Lorentz signature evolution is always accompanied by a little bit of Euclidean evolution as a regulator.

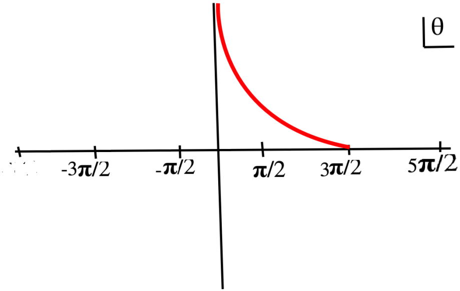
Clearly, we cannot go continuously from $\epsilon > 0$ to $\epsilon < 0$. This helps in understanding what is wrong with some of the oddball metrics that I described before.

The oddball metrics that I described before, and others I don't have time for today, violate the condition $\sum_i |\mathrm{Arg}\,\lambda_i| < \pi$:



An allowable metric cannot cross any of the lines $\mathrm{Re}\,\theta = (n + 1/2)\pi$, $n \in \mathbb{Z}$, on which $\mathrm{Re}\,\cos\theta = 0$ and $\cos^2\theta < 0$. Crossing one of those lines is like crossing from positive to negative ϵ.

The same applies here



There are other examples I would have wanted to talk about given more time. One of the more interesting is why the quasi-Euclidean metrics of Gibbons and Hawking, which I mentioned at the start, turn out to be allowable whenever the black hole is such that one would expect quantum corrections to be well-defined.

This is a little subtle. The thermodynamics of a Schwarzschild black hole is related to a standard thermal ensemble $\mathrm{Tr}\ \exp(-\beta H)$. The thermodynamics of a rotating (Kerr) black hole is related to a more general ensemble $\mathrm{Tr}\ \exp(-\beta(H-\Omega J)$ where J is a conserved angular momentum and Ω is called the angular velocity. However, in asymptotically flat spacetime, this ensemble is unstable because a particle far from the black hole can have a negative value of $H-\Omega J$. Hence the quantum corrections to this ensemble are not well-defined, and it turns out that this is reflected in the fact that the quasi-Euclidean metric is not allowable.

It was noted in the early days of the AdS/CFT correspondence by Hawking and Reall, with further work by Hawking, Hunter, and M. Taylor, that the situation is better in Anti de Sitter space. A black hole that is rotating not too fast (i.e. Ω is sufficiently small) has the property that $H - \Omega J$ is bounded below. This is the case that one can hope that the quantum corrections will make sense. It turns out that precisely when $H - \Omega J$ is bounded below, the quasi-Euclildean metric of Gibbons and Hawking is allowable.

in Lorentz signature, the condition for $H - \Omega J$ to be bounded below for excitations outside the horizon is that the Killing vector field V that generates this conserved charge should be everywhere timelike outside the horizon. Thus when and only when V is everywhere timelike outside the horizon, we can hope that the quasi-Euclidean metric will be allowable.

With this in mind, let us take a look at a rotating black hole in four dimensions. A general form of the metric is

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + \rho^2(N^\phi\mathrm{d}t + \mathrm{d}\phi)^2 + g_{rr}\mathrm{d}r^2 + g_{\theta\theta}\mathrm{d}\theta^2.$$

Rotations and time translations shift ϕ and t. The functions N, N^ϕ, g_{rr}, $g_{\theta\theta}$ depend only on r, θ, not on t, ϕ. The horizon is the outermost surface with $N^2 = 0$. Part of the definition of "asymptotically flat" or "asymptotically Anti de Sitter" is that $N^\phi \to 0$ at $r \to \infty$. An important fact in constructing the quasi-Euclidean solution is the function N^ϕ is a constant on the horizon. In fact, this constant N_h is none other than the angular velocity Ω that appears in the thermodynamics. In constructing the quasi-Euclidean metric, it is convenient to introduce a new angular coordinate $\widetilde{\phi} = \phi - \Omega t$. The metric is then

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + \rho^2((N^\phi - \Omega)\mathrm{d}t + \mathrm{d}\widetilde{\phi})^2 + g_{rr}\mathrm{d}r^2 + g_{\theta\theta}\mathrm{d}\theta^2.$$

In these coordinates

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + \rho^2((N^\phi - \Omega)\mathrm{d}t + \mathrm{d}\widetilde{\phi})^2 + g_{rr}\mathrm{d}r^2 + g_{\theta\theta}\mathrm{d}\theta^2$$

the vector field that generates $H - \Omega J$ is just $\partial_t|_{\widetilde{\phi}}$, and the condition for it to be everywhere timelike outside the horizon is just $g_{tt} > 0$. Further, in these coordinates, the quasi-Euclidean metric is constructed by discarding the region behind the horizon, forgetting the value of $\widetilde{\phi}$ on the horiizon, and replacing $t \to \mathrm{i}\tau$, to get

$$\mathrm{d}s^2 = -N^2\mathrm{d}\tau^2 + \rho^2(\mathrm{i}(N^\phi - \Omega)\mathrm{d}\tau + \mathrm{d}\widetilde{\phi})^2 + g_{rr}\mathrm{d}r^2 + g_{\theta\theta}\mathrm{d}\theta^2.$$

From the criterion for allowability, it is rather immediate that this four-dimensional metric is allowable if and only if, at fixed r, θ, the two-dimensional metric

$$-N^2\mathrm{d}\tau^2 + \rho^2(\mathrm{i}(N^\phi - \Omega)\mathrm{d}\tau + \mathrm{d}\widetilde{\phi})^2$$

is allowable.

Thus we reduce to understanding the allowability of a two-dimensional metric

$$A \mathrm{d}\tau^2 + 2\pi B \mathrm{d}\tau \mathrm{d}\widetilde{\phi} + C \mathrm{d}\widetilde{\phi}^2$$

for real A, B, C. It turns out that such a metric is allowable if and only if $A, C > 0$. Necessity follows from positivity of $\mathrm{Re}\sqrt{g} g^{-j}$, and once one knows $A, C > 0$, one can put the metric in the form $\sum_{i=1}^{2} \lambda_i \delta_{ij}$ with $|\lambda_1| = |\lambda_2| < \pi/2$.

In the appllication to the black hole, C is always positive, and A is positive everywhere outside the horizon if and only if the ensemble $\mathrm{Tr}\ \exp(-\beta(H-\Omega J))$ associated to the black hole that we are trying to study is well-defined. So in other words, precisely when it is expected that quantum corrections to the black hole will make sense, the quasi-Euclidean metric is allowable.

In short, I've explained a little of why it is motivated to consider complex solutions of Einstein's equations in the context of quantum gravity, why a good class of allowed metrics is needed, and why the class identified by Kontsevich and Segal (and in embryo by Louko and Sorkin) has at least some of the right properties. There is a lot missing. To understand the "Euclidean" path integral of gravity requires much more than a knowledge of what is a good class of complex metrics. Ideally one wants an integration cycle.