$T^* \operatorname{Gr}(k, n)$ as a Coulomb branch

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PERIMETER INSTITUT
INSTITUTE FOR THEORETICAL PHYSICS
PÉRIMÈTRE DE PHYSIQUE THÉORIQUE

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How did I get here?

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After that, things get a little blurry, but then I ended up here.

Hanany and Witten study configurations of D5, NS5 and D3 branes in type IIB string theory.

	0	1	2	3	4	5	6	7	8	9
D3	X	X	X				X			
D5	X	X	X					X	X	X
NS5	X	X	X	X	X	X				

They consider how this theory localizes on the plane of the D3's and the result they obtain is a quiver gauge theory for a linear quiver (or a cyclic quiver if we make the x^6 direction a circle).

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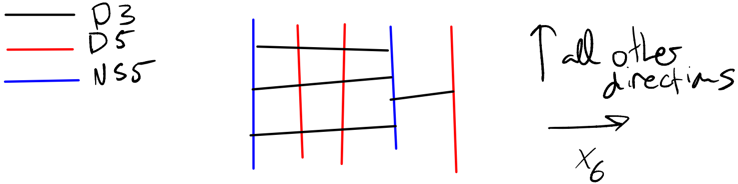

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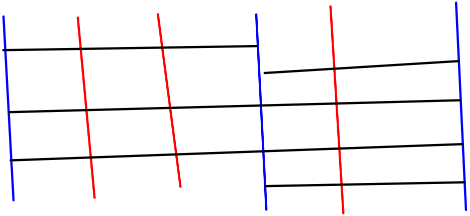
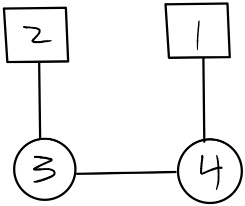
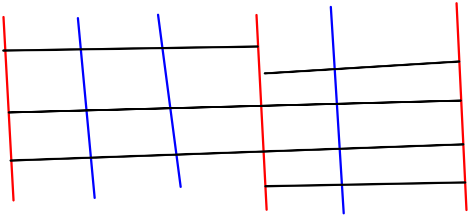
- If all D3 branes end on NS5 branes then we can describe 3d theory as a quiver gauge theory (this is called *cobalanced*), where:
  - nodes correspond to gaps between NS5 branes
  - rank of $U(v_i)$ is # of D3 branes joining consecutive pairs of NS5's.
  - matter is a bifundamental for each pair of consecutive branes, and a fundamental for each D5 between NS5's (w_i=# D5's).

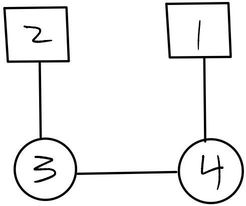




- If all D3 branes end on D5 branes then we can describe 3d theory as a quiver gauge theory (this is called *balanced*), where:
  - nodes correspond to gaps between D5 branes
  - rank of $U(v_i)$ is # of D3 branes joining consecutive pairs of D5's.
  - matter is a bifundamental for each pair of consecutive branes, and a fundamental for each NS5 between D5's (w_i=# NS5's).

but the supersymmetry acts differently!





In a 3d quantum field theory with $\mathcal{N} = 4$ supersymmetry, there are two topological twists, often called the A- and B-twists.

Definition

The local operators in:

1. the A-twist are called the **Coulomb branch chiral ring** A_{Coulomb}
2. the B-twist are called the **Higgs branch chiral ring** A_{Higgs}

From the perspective of the whole theory, these are two natural classes of $\frac{1}{2}$BPS operators.

The spaces of the Coulomb and Higgs branch are $\mathfrak{M}_{\mathrm{Coulomb}} = \mathrm{Spec}(A_{\mathrm{Coulomb}})$ and $\mathfrak{M}_{\mathrm{Higgs}} = \mathrm{Spec}(A_{\mathrm{Higgs}})$

If you swap D5 and NS5 in the Hanany-Witten picture, you get the same theory, but with supersymmetry changed so that A- and B-twists switch.

This is an example of 3-dimensional mirror symmetry/S-duality.

In the balanced case, we have a nice description of the Coulomb branch: it is the Nakajima quiver variety of this quiver gauge theory.

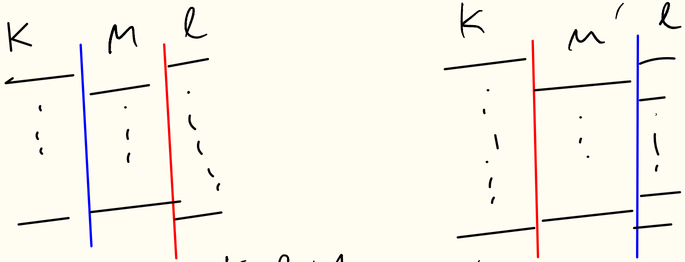
Dually, in the cobalanced case, the same Nakajima quiver variety is the *Higgs branch*.

More generally, for any good brane configuration, we can write both the Higgs and Coulomb branches as **bow varieties** which generalize quiver varieties.

This observation becomes much more powerful if we exploit the existence of Hanany-Witten transitions, which allow us to write a single theory in terms of both balanced and cobalanced brane configurations.

Theorem

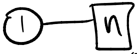
When we swap the order of an NS5 and D5 brane, the number of D3 branes joining them in the new and old configurations are related by:



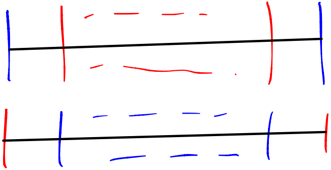
$$k + l + 1 = m + m'$$

First example: $U(1)$ with n hypers has Higgs branch

$$X_0^{(1,n)} = \{A \in M_{n\times n}(\mathbb{C}) \mid A^2 = 0, \mathrm{rk}(A) \leq 1\} \longleftarrow X^{(1,n)} = T^*\mathbb{CP}^{n-1}.$$

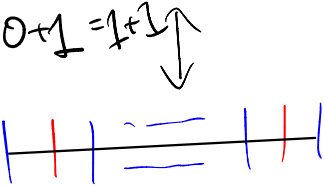
Higgs of 

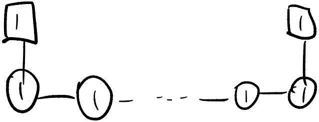
$U(1)$ $T^*\mathbb{C}^n$

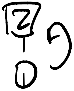


Coulomb of

$1+0+1 = 1+1$



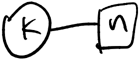




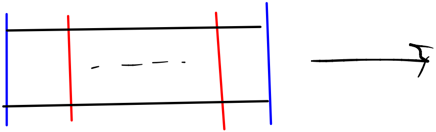
$n=2$ $n=2$ 

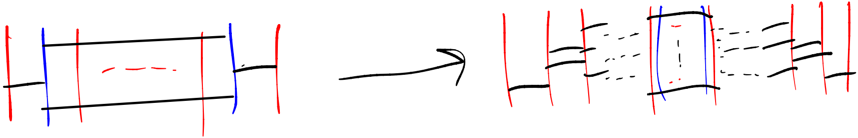
Main example: $U(k)$ with n fundamentals ($k \leq n/2$) has Higgs branch

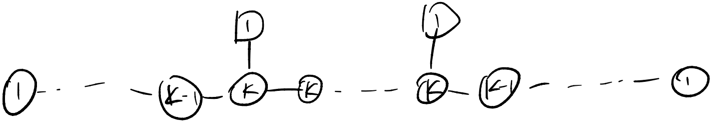
$$X_0^{(k,n)} = \{A \in M_{n\times n}(\mathbb{C}) \mid A^2 = 0, \mathrm{rk}(A) \leq k\} \longleftarrow X^{(k,n)} = T^*\,\mathrm{Gr}(k,n).$$



$k+0+1=k+1$

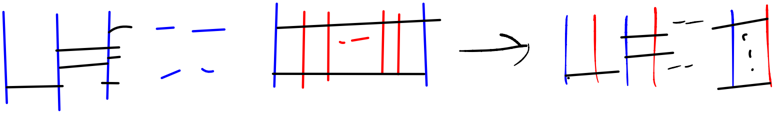


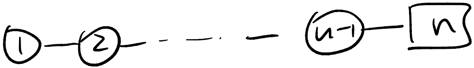




For funsies: we can also get

$$X_0 = \{A \in M_{n\times n}(\mathbb{C}) \mid A^n = 0\} \longleftarrow X = T^*\mathrm{Fl}_n!$$





Self-dual!

Let $X = T^*\mathrm{Fl}_n$ be the cotangent bundle of the flag variety $X_0 = \mathrm{Fl}_n$ over a field $\Bbbk$ of characteristic $p \geq 0$.

Let $\mathsf{Coh}_0(X)$ denote the abelian category of coherent sheaves on X which are (set-theoretically) supported on Fl_n.

Consider the algebra $A = U\mathfrak{gl}_n(\Bbbk)$. Let $\mathfrak{U}\text{-mod}_0$ be the principal block of the category of finite dimensional modules with central character.

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Theorem (Bezrukavnikov-Mirkovič)

If $p \gg 0$, there is an equivalence of derived categories

$$D^b(\mathsf{Coh}_0(X)) \cong D^b(\mathfrak{U}).$$

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Conjecture (Bezrukavnikov-Mirkovič)

There is an equivalence of derived categories

$$D^b(\mathsf{Coh}_0(X)) \cong D^b(\mathcal{U}).$$

Bezrukavnikov calls this a “non-commutative counterpart of the Springer resolution.”

This equivalence looks very strange, but that’s because you’ve been thinking about $X = T^*\mathrm{Fl}_n$ too Higgsily. It’s very natural when you use the Coulomb perspective.

$\mathsf{Coh}_0(X)$	$\mathfrak{U}\text{-mod}_0$
geometry	representation theory

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$$\begin{array}{ccc} \mathsf{Coh}_0(X) & & \mathfrak{U}\text{-mod}_0 \\ \text{geometry} & & \text{representation theory} \\ & \nwarrow\!\!\searrow \quad \swarrow\!\!\nearrow & \\ & \mathring{R}\text{-mod}_0 & \\ & \text{combinatorial algebra} & \end{array}$$

The balanced Higgs branch/cobalanced Coulomb branch also has a mathematical description:

Theorem (Braverman–Finkelberg–Nakajima)

The Coulomb branch chiral ring of the sigma model with matter N gauged by G is the Borel-Moore homology

$$A_{\text{Coulomb}} = H_*^{BM}\left(\frac{N[\![t]\!]}{G[\![t]\!]} \times_{\frac{N(\!(t)\!)}{G(\!(t)\!)}} \frac{N[\![t]\!]}{G[\![t]\!]}\right) \qquad \mathfrak{M}_0 = \operatorname{Spec}(A_{\text{Coulomb}})$$

You can think of this computation in the category of lines in the A-twisted TQFT, which we can interpret mathematically as the D-modules on the loop space $N(\!(t)\!)/G(\!(t)\!)$.

$$\frac{*_1}{G[[t]]} \times \frac{*}{G((t))} \quad \frac{*}{G[t]} = G(\mathbb{Q}) \backslash G(K) / G(\mathcal{O})$$

$G(\mathbb{Q})$

$= \mathrm{Gr}$

BFN Coulomb branches

In our case, G and N define a quiver gauge theory for a linear quiver

$U(v_i)$

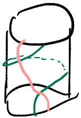
$$G = \prod_{i=1}^{r} GL(v_i) \qquad N = \bigoplus_{i=1}^{r-1} \mathrm{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{v_{i+1}}) \oplus \bigoplus_{i=1}^{r} \mathrm{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{w_i})$$

In this case, we can think of $N[\![t]\!]/G[\![t]\!]$ as the moduli space of quiver representations with $\mathbb{C}[\![t]\!]$ coefficients and $N(\!(t)\!)/G(\!(t)\!)$ with $\mathbb{C}(\!(t)\!)$ coefficients.

This algebra is generated by scalars in $(\operatorname{Sym}\mathfrak{t}^*)^W$ and monopole operators indexed by dominant coweights.

You can think of dominant coweights as paths in T/W and scalars as coupons sitting on these paths. Remarkably, the relations of A_{Coulomb} become simple and local if you write them this way.

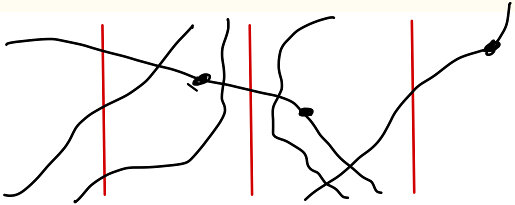
In the quiver case, we can think of $U(1)^{n\;v_i}/S_{n\;v_i}$ as an unordered n-tuple in S^1, and a path in this space as a diagram drawn on the cylinder.



Definition

A (planar) KLRW diagram is a generic collection of curves in $\mathbb{R} \times [0,1]$ which are of the form $\{(\pi(t), t) \mid t \in [0,1]\}$ for $\pi \colon [0,1] \to \mathbb{R}$.

1. Each strand is labeled from $[1, r]$ and is colored red or black with v_i black strands and w_i red strands with label i.
2. Red strands must be vertical at fixed, distinct x-values (for example, $x = 1/n, 2/n, \ldots, 1$).
3. We place dots at a finite number of points on black strands, avoiding crossings.



Definition

A cylindrical KLRW diagram is a generic collection of curves in $\mathbb{R}/\mathbb{Z} \times [0,1]$ which are of the form $\{(\pi(t), t) \mid t \in [0,1]\}$ for $\pi\colon [0,1] \to \mathbb{R}/\mathbb{Z}$.

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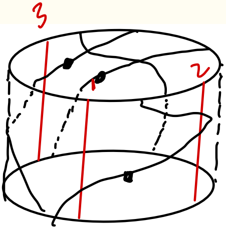
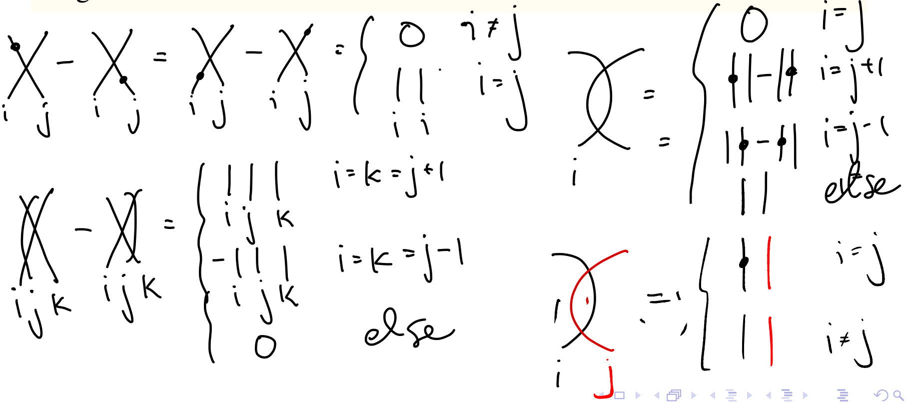


We can compose KLRW diagrams by stacking, if the labels on the bottom of one and top of the other match up to isotopy (never moving red strands).

Definition

The (planar) KLRW algebra R is the formal $\Bbbk$-span of planar KLRW diagrams modulo the local relations below.



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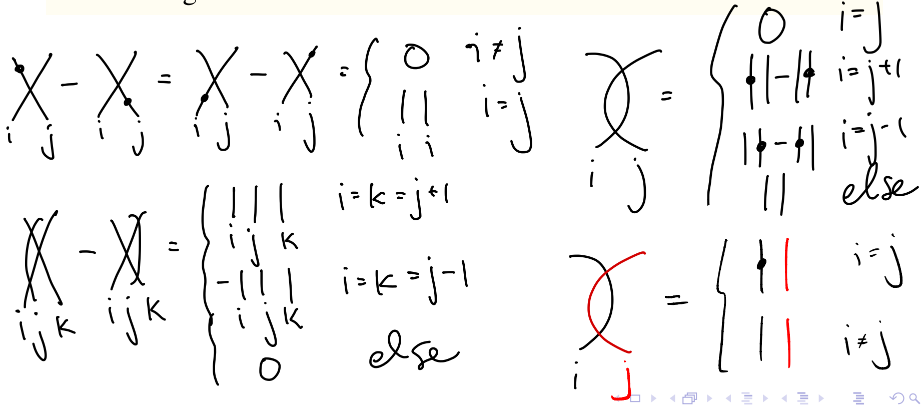
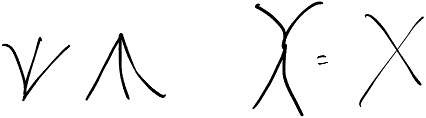
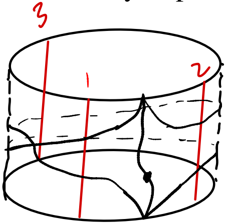
Definition

The cylindrical KLRW algebra $\mathring{R}$ is the formal $\Bbbk$-span of cylindrical KLRW diagrams modulo the local relations below.



To get the algebra A_{Coulomb}, we have to include the ability to pinch together strands.

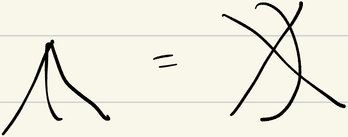


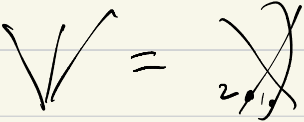


Theorem

The ring $A_{\text{Coulomb}} = e_C \mathring{R} e_C$ is the algebra of KLR diagrams where all strands with the same label are pinched together at the top and bottom (at the same x-value).

e_C = any idempotent w/ all black w/ same label together, take any idempotent pinchin

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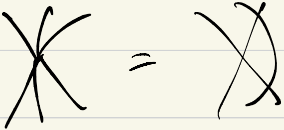
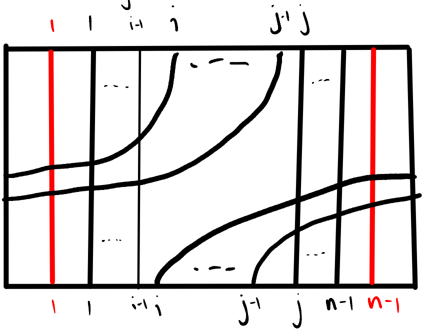


Coulomb from KLR

$\{A \in M_{n\times n} \mid A^2 = 0,\ \mathrm{rk}(A) \leq 1\} \subset M_{n\times n}(\mathbb{C})$

Example: $X_0^{(1,n)}$ as a Coulomb branch.

$e_{ij} \mapsto$



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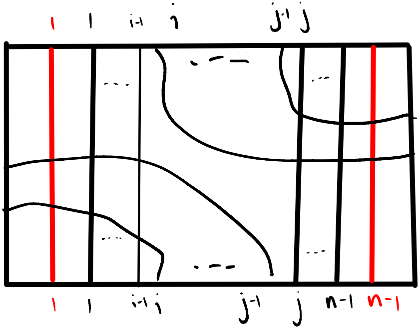
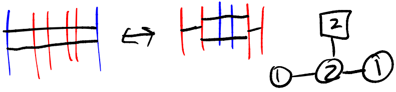


$e_{ii} \mapsto$

Coulomb from KLR

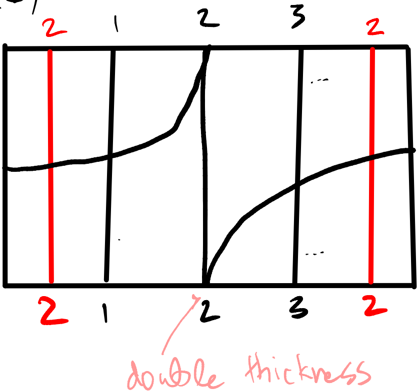
$\{A \mid A^2=0 \quad \mathrm{rk}(A)\le 2\}$

Example: $X_0^{(2,4)}$ as a Coulomb branch.



$M_{4\times4}(\mathbb{C})$

$e_{23} \mapsto$



$i<j$

e_{ij} = grab 1 strand w/ label $i, i+1, \ldots, j-1$ loop around cylinder once

To get the resolution $T^*\mathbb{CP}^{n-1}$, need to find not just functions on $T^*\mathbb{CP}^{n-1}$ but sections $\Gamma(T^*\mathbb{CP}^{n-1}, \mathcal{O}(k))$.

Consider the rings $\tilde{A}_{\mathrm{Coulomb}}, \tilde{\mathring{R}}$ where instead of requiring red strands to be vertical, they wrap around the cylinder some number of times, specified by $\mathbf{m} \in \mathbb{Z}^\ell$. ℓ = # of red strand

This ring is graded by $\mathbb{Z}^\ell$, which induces a $U(1)^\ell$ action on $\mathrm{Spec}(\tilde{A}_{\mathrm{Coulomb}})$, and we can define a partial resolution $\mathfrak{M}^{\mathbf{m}}$ as the GIT quotient for $\mathbf{m}$:

$$\mathfrak{M}^{\mathbf{m}} = \mathrm{Proj}\left(\bigoplus_{k\geq 0} \tilde{A}^{k\mathbf{m}}_{\mathrm{Coulomb}}\right) \qquad \tilde{A}^{k\mathbf{m}}_{\mathrm{Coulomb}} = \Gamma(\mathfrak{M}^{\mathbf{m}}, \mathcal{O}(k)).$$

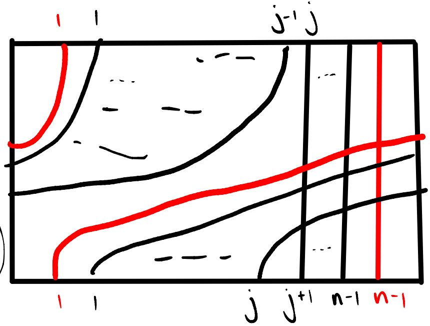
Resolved Coulomb branches

Example: $T^*\mathbb{CP}^{n-1}$ as a resolved Coulomb branch.

$X_j \mapsto$

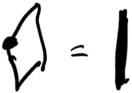
$\cap$

$\Gamma(T^*\mathbb{CP}^{n-1}, \mathcal{O}(1))$

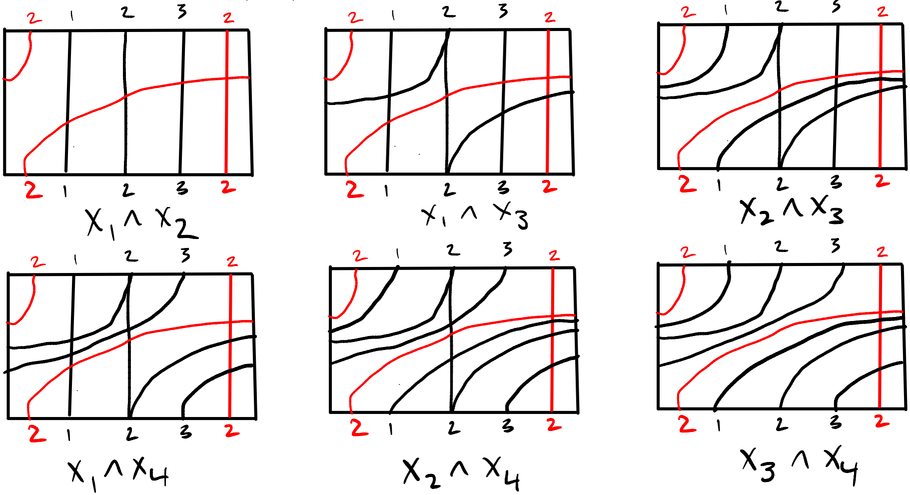


Resolved Coulomb branches

$$\Gamma(\mathrm{Gr}, \mathcal{O}(1)) = \Lambda^2 \mathbb{C}^4$$



Example: $T^*\mathrm{Gr}(2,4)$ as a resolved Coulomb branch.



What new does this tell us about geometry?

Recall that we call a vector bundle T on an algebraic variety X a tilting generator if $\mathbb{R}\mathrm{Hom}(T, -)$ induces an equivalence of derived categories $D^b(\mathsf{Coh}(X)) \cong D^b(\mathrm{End}(T)^{\mathrm{op}}\text{-mod.})$

Theorem (W.)

Over $\mathbb{C}$, there is a tilting generator T on $\mathfrak{M}^{\mathbf{m}}$ such that $\mathrm{End}(T)^{\mathrm{op}} = \mathring{R}$.

$$D^b(\mathsf{Coh}(\mathfrak{M}^{\mathbf{m}})) \cong D^b(\mathring{R}\text{-mod}).$$

In particular, the ring $\mathring{R}$ is a non-commutative crepant resolution of singularities of $\mathfrak{M}_{\circ}$ which is D-equivalent to $\mathfrak{M}^{\mathbf{m}}$.

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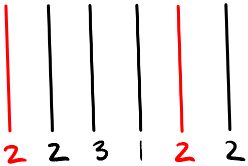
Conjecture (W.)

Over any field, there is a tilting generator T on $\mathfrak{M}^{\mathbf{m}}$ such that $\mathrm{End}(T)^{\mathrm{op}} = \mathring{R}$.

$$D^b(\mathsf{Coh}(X)) \cong D^b(\mathring{R}\text{-mod}).$$

In particular, the ring $\mathring{R}$ is a non-commutative crepant resolution of singularities of $\mathfrak{M}$ which is D-equivalent to $\mathfrak{M}^{\mathbf{m}}$.

To define T, need idempotents where all strands are vertical.



$e(223122)$

There's one of these for each possible order on strands. Can encode this in a word $\mathbf{i}$. Denote by $e(\mathbf{i})$.

This word is really cyclic, but can always start with red at $x = 0$.

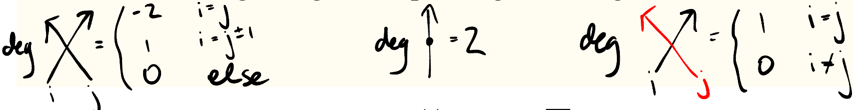
We can think of $\tilde{\mathring{R}}e_C$ as a right module over $\tilde{A}_{\text{Coulomb}} = e_C\tilde{\mathring{R}}e_C$. This has a left $\mathring{R}$-module structure by left multiplication.

Theorem

T is the coherent sheaf obtained on $\mathfrak{M}^{\mathbf{m}}$ by GIT quotient. That is, for $k \gg 0$, we have $\Gamma(\mathfrak{M}^{\mathbf{m}}, T \otimes \mathcal{O}(k)) \cong \tilde{\mathring{R}}^{k\mathbf{m}}e_C$

In physical terms, these come from vortex line operators, that is, natural D-modules on loop space.

There is a natural grading, corresponding to scaling $\mathbb{C}^\times$ action.

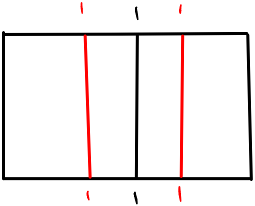


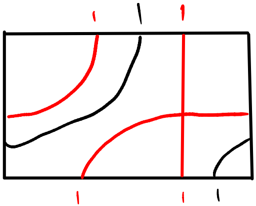
This breaks up into summands $Te(\mathbf{i})$ of rank $\prod v_i!$, the order of Weyl group of G.

What tilting generator do we get on $T^*\mathbb{CP}^{n-1}$? The only possibility is $\mathscr{O}(a) \oplus \cdots \oplus \mathscr{O}(a-n+1)$ for some a.

$n=2$ $\quad \underline{m} = (1,0)$

$| \; | \; | \longrightarrow \mathcal{O}$ $\qquad\qquad$ $| \; | \; | \longrightarrow \mathcal{O}(-1)$



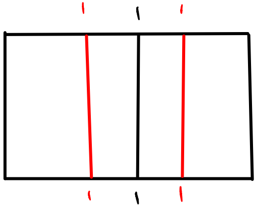


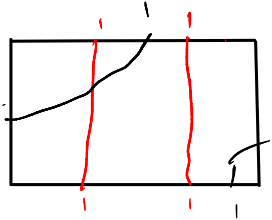
Can get $a = 0, 1, \ldots, n-1$ depending on conventions.

What tilting generator do we get on $T^*\mathbb{CP}^{n-1}$? The only possibility is $\mathscr{O}(a) \oplus \cdots \oplus \mathscr{O}(a-n+1)$ for some a.

$n=2 \quad \underline{m} = (-1,0)$

$| \, | \, | \longrightarrow \mathcal{O} \qquad\qquad | \, | \, | \longrightarrow \mathcal{O}(1)$





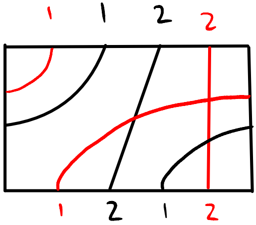
Can get $a = 0, 1, \ldots, n-1$ depending on conventions.

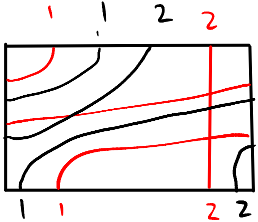
What tilting generator do we get on $T^*\mathbb{CP}^{n-1}$? The only possibility is $\mathscr{O}(a) \oplus \cdots \oplus \mathscr{O}(a-n+1)$ for some a.

$n=3$ 1122 $\longrightarrow$ $\mathcal{O}$

1221 $\longrightarrow$ $\mathcal{O}(-2)$

1212 1212
1221 1122 $\longrightarrow$ $\mathcal{O}(-1)$





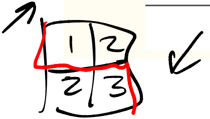
Can get $a = 0, 1, \ldots, n-1$ depending on conventions.

What about $T^*G(2,4)$? First non-projective space, and summands of T have ranks 1 and 2.

Theorem (Suter-W.)

Let $\mathcal{T}$ be the tautological bundle on $T^*G(2,4)$ and $\mathcal{O}(1) = \bigwedge^2\mathcal{T}^*$. Every summand $Te(\mathbf{i})$ is isomorphic to one of:

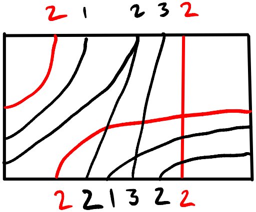
KLRW idempotent	**Grassmanian sheaf**
222231	$\mathcal{O} \oplus \mathcal{O}(-1)$
223122	$\mathcal{O}(-1) \oplus \mathcal{O}$
223221	$\mathcal{T}$
221223	$\mathcal{T}^\perp \subset \mathcal{O}^{\oplus 4}$
223122	$0 \to \mathcal{O}(-2) \to \mathcal{V} \to \mathcal{O} \to 0$
222312	$0 \to \mathcal{O}(-1) \to \mathcal{W} \to \mathcal{O}(1) \to 0$



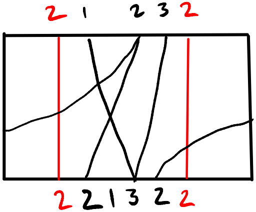
How do we check something like this?

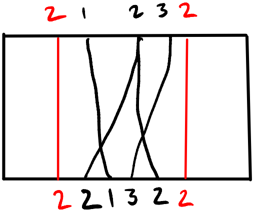
- Elements of $\bigwedge^2\mathbb{C}^4$ give divisors on $T^*G(2,4)$. Vector bundle is trivial when complement is $\mathbb{A}^n$.
- Only need to check that vector bundle is right on open subset of codim ≥ 2, so enough to find transition function between patches.
- Calculate!
- Sneaky trick: Bezrukavnikov says T is GL_4-equivariant, so T must be induced by a representation of $P = \left[\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right] \subset GL_4$.

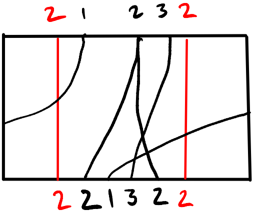
Grassmannians

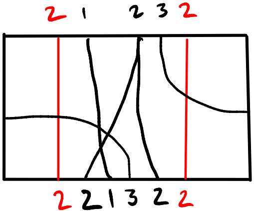


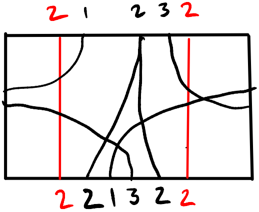
$$\mathcal{O}(-1) \to T_{e_{221322}} \to \mathcal{O}(1)$$

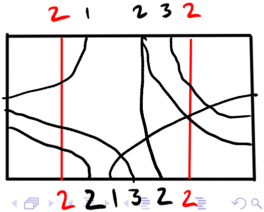








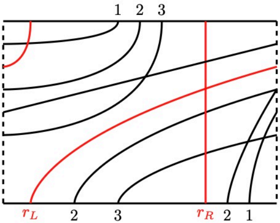




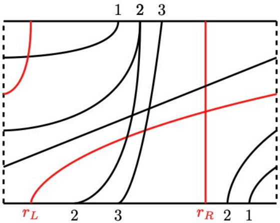
Grassmannians

$(X_1 \wedge X_3)^{-1}$

$\mathcal{Z}^1_{13} =$

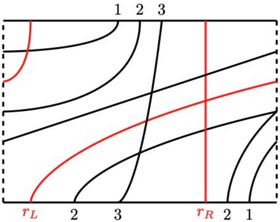


$\mathcal{Z}^2_{13} =$

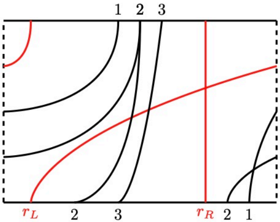


$(X_2 \wedge X_4)^{-1}$

$\mathcal{Z}^1_{24} =$



$\mathcal{Z}^2_{24} =$



$$\mathcal{Z}^1_{13} = \mathcal{D}^{-1}_{24}(\mathcal{D}_{24}\mathcal{Z}^1_{24} + \mathcal{D}_{12}\mathcal{Z}^2_{24}), \qquad \mathcal{Z}^2_{13} = \mathcal{D}^{-1}_{24}(\mathcal{D}_{14}\mathcal{Z}^2_{24} - \mathcal{D}_{34}\mathcal{Z}^1_{24}).$$

Thanks!