Nonabelian DT theory from abelian DT theory

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Preliminaries

Throughout we work on a fixed Calabi-Yau 3-fold X
(smooth complex projective variety with $K_X \cong \mathcal{O}_X$)
with a fixed ample line bundle $\mathcal{O}_X(1)$ and hyperplane class
$H := c_1(\mathcal{O}_X(1)) \in H^2(X, \mathbb{Z})$

satisfying the Bogomolov-Gieseker conjecture of Bayer-Macrì-Toda
(for which, see later) such as a quintic 3-fold (Chunyi Li).

Fix a Chern character $c \in H^{\mathrm{ev}}(X, \mathbb{Q})$
(or a numerical K-theory class $c \in K_{\mathrm{num}}(X) := K(X)/\ker \chi(\,\cdot\,,\,\cdot\,)$).

Consider (semi)stable bundles, or sheaves, or complexes of sheaves
E of class c.

Stability

There are many notions of stability for E.

The ones we consider can be written in terms of some central charge $Z(\mathrm{ch}(E)) \in \mathbb{C}$.

Writing $Z(E) = m(E)\exp(2\pi i\theta(E))$ we let the slope of E be $\mu(E) := \tan\theta(E)$ and say E is (semi)stable if and only if

$$\mu(F)\ (\leq)\ \mu(E/F) \quad \text{for all nonzero} \quad F \subsetneq E.$$

Here $(\leq)$ means $<$ for stability and $\leq$ for semistability. (Definition of $F \subset E$ is tricky, but for now can just take subsheaves of sheaves.)

E.g. $Z(E) = \int_X c_1(E) \cdot H^2 + i\,\mathrm{rank}(E)$ gives $\mu(E) = \frac{\deg(E)}{\mathrm{rank}(E)}$ and *slope stability*.

E.g. $Z(E) = \left[\int_X \mathrm{ch}(E(n)) \cdot \mathrm{td}_X\right]_{\leq 2} + i\,\mathrm{rank}(E)$ for large $n \gg 0$ gives *Gieseker stability*.

DT invariants

Choose c, H so that Gieseker semistability $\implies$ Gieseker stability.

(So all semistable sheaves have only scalar automorphisms.)

Then we can define an invariant $DT(c) \in \mathbb{Z}$ "counting" Gieseker stable bundles or sheaves E of class c.

Moduli space M_c of Gieseker stable sheaves is projective scheme with "*perfect obstruction theory*" of virtual dimension zero.
Obstructions dual to deformations $\mathrm{Ext}^2(E, E)_0 \cong \mathrm{Ext}^1(E, E)_0^*$ by Serre duality, and no higher obstructions $\mathrm{Ext}^3(E, E)_0 = \mathrm{Hom}(E, E)_0^* = 0$.
Therefore it has a 0-dimensional virtual cycle, whose length is $DT(c)$.

(Closely related to holomorphic bundles being the critical points of the holomorphic Chern-Simons functional.)

Can think of it as $(-1)^{\dim M_c} e(M_c)$.

Behrend showed each point $E \in M_c$ can be assigned a multiplicity $\chi^B(E) \in \mathbb{Z}$ such that $DT(c)$ is the weighted Euler characteristic

$$e\big(M_c, \chi^B\big) \;=\; \sum_{i \in \mathbb{Z}} i\, e\big(\{\chi^B = i\}\big).$$

Generalised DT invariants

For general c, H there are *strictly semistable* sheaves of charge c; counting them is much more complicated.

Given stable objects of smaller charge, we can take all their direct sums (and extensions) to get semistable objects of charge c but large automorphism groups.

To invert this process Joyce/Kontsevich-Soibelman took a clever "*plethystic logarithm in the Hall algebra of coherent sheaves of the same slope*" to get more controllable automorphism groups.

Joyce-Song were able to define a generalised invariant $J(c) \in \mathbb{Q}$ which reduces to $DT(c) \in \mathbb{Z}$ when semistable = stable.

Invariant under deformations of X.
Changes via a wall-crossing formula when we change the stability condition.

The simplest wall crossing formula

Suppose a bundle F sits in an exact sequence

$$0 \longrightarrow A \longrightarrow F \longrightarrow B \longrightarrow 0 \qquad (*)$$

with A, B stable, and that we can vary the stability condition so that the slopes of A and B cross.

Just below the wall $\big(\mu(A) < \mu(B)\big)$ F will be stable.
Just above the wall F will be destabilised by $(*)$, but extensions in the opposite direction will become stable.

So on crossing the wall we lose a $\mathbb{P}(\mathrm{Ext}^1(B, A))$ of extensions $(*)$ and gain a $\mathbb{P}(\mathrm{Ext}^1(A, B))$.

So the Euler characteristic changes by $-ext^1(B, A) + ext^1(A, B)$ $= -ext^1(B, A) + ext^2(B, A) = \chi(B, A)$ by Serre duality. WCF is

$$J_+[F] \;=\; J_-[F] + (-1)^{\chi(B,A)-1}\chi(B, A) J[A] J[B].$$

The rough idea

Fix $n \gg 0$ so that $H^{\geq 1}(E(n)) = 0$ for all semistable E of charge c.

Now replace E by the cokernel F of a section $s \in H^0(E(n))$,

$$0 \longrightarrow \mathcal{O}(-n) \xrightarrow{s} E \longrightarrow F \longrightarrow 0.$$

Then $\operatorname{rank}(F) = \operatorname{rank}(E) - 1$ and $\operatorname{ch}(F) = c_n := c - e^{-nH}$.

To a first approximation, suppose all such E, F are stable for $s \neq 0$.

Then we find all Fs come from an (E, s), so M_{c_n} is a $\mathbb{P}^{N-1}$-bundle over M_c $\left(N := \chi(E(n)) = \int_X c \cdot e^{nH} \cdot \mathrm{td}_X\right)$, so

$$J(c_n) \ = \ (-1)^{N-1} \cdot N \cdot J(c).$$

Now wall-cross to handle stability and get the correct formula....

An example: rank 1 from rank 0

The rough idea actually works perfectly when rank $= 1$.

Here M_c is a moduli space of ideal sheaves $E = \mathcal{I}_Z$, where $Z \subset X$ is a subscheme of dimension ≤ 1. (Possibly tensored by a line bundle.)

Then $s \in H^0(\mathcal{I}_Z(n)) \hookrightarrow H^0(\mathcal{O}(n))$ cuts out divisor $\iota\colon D \hookrightarrow X$ and

$$F \;=\; \operatorname{coker} s \;=\; \iota_*(I_Z)$$

is a torsion sheaf supported on D. (*"D4-D2-D0 brane."*)

In this case E, F are Gieseker *stable* and slope stable and are the only stable sheaves are of this form,

$$M_{c_n} \longrightarrow M_c \;\text{ is a } \mathbb{P}^{N-1}\text{-bundle}, \quad N = \chi(c(n)),$$

and $J(c_n) = (-1)^{N-1} \cdot N \cdot J(c)$.

(Eg rank 2 bundles supported on $D \in \left|\frac{n}{2}H\right|$ with $\mathrm{ch} = c_n$ are unstable.)

GW invariants

$J(c_n) = (-1)^{N-1} \cdot N \cdot J(c)$.

The abelian DT invariants $J(c)$ count curves (and points) in X and — by the MNOP conjecture — can be written in terms of the Gromov-Witten invariants of X.
(Maulik-Nekrasov-Okounkov-Pandharipande conjecture now proved for most Calabi-Yau 3-folds by Pandharipande-Pixton.)

The generating series of D4-D2-D0 counts $J(c_n)$ are conjectured by "*S-duality*" to be vector-valued mock modular forms.
([MSW97, dBCDMV06, GSY07, DM11, AMP19]; possibly need further wall-crossing to reach attractor stability)

If there's time at the end we'll discuss how to try to see this modularity from a Noether-Lefschetz point of view, following Maulik-Pandharipande.

Rank r from rank 0

In higher rank $r \geq 1$ there are corrections to the "*rough idea*". They mean we can write rank r invariants in terms of rank $r-1$, $r-2, \ldots, 0$ invariants. Inductively we get to rank 0.

Theorem (arXiv:2103.02915)

For fixed c of rank ≥ 1,

$$J(c) \;=\; F\big(J(\alpha_1), J(\alpha_2), \ldots\big)$$

is a universal polynomial in invariants $J(\alpha_i)$, with all α_i of rank 0 and pure dimension 2.

So to express everything in terms of rank 1 ("*abelian*" theory) what's left is to express rank 0 in terms of rank 1. (See later.)

Weak stability conditions

We use the *weak stability conditions* of Bayer-Macrì-Toda.

Pick $b, w \in \mathbb{R}$ with $w > \frac{1}{2}b^2$.

Instead of $\mathrm{Coh}(X) \subset D(X)$ we work in the abelian category

$$\mathcal{A}_b \ := \ \big\{E^{-1} \xrightarrow{d} E^0 \ : \ \mu_H^+(\ker d) \le b\,,\ \mu_H^-(\operatorname{coker} d) > b\big\}.$$

$\mu^+(F)$ is the maximum slope of a subsheaf of F,
$\mu^-(F)$ is the minimum slope of a quotient sheaf of F.

On this we use the central charge
$Z(E) = \big[\mathrm{ch}_1(E).H^2 - b\,\mathrm{ch}_0(E)H^3\big] + i\big[\mathrm{ch}_2(E).H - w\,\mathrm{ch}_0(E)H^3\big]$,
i.e. the slope function

$$\nu_{b,w}(F) \ = \ \begin{cases} \frac{\mathrm{ch}_2(E).H - w\,\mathrm{ch}_0(E)H^3}{\mathrm{ch}_1(E).H^2 - b\,\mathrm{ch}_0(E)H^3} & \text{if } \mathrm{ch}_1(E).H^2 - b\,\mathrm{ch}_0(E)H^3 \neq 0, \\ +\infty & \text{if } \mathrm{ch}_1(E).H^2 - b\,\mathrm{ch}_0(E)H^3 = 0. \end{cases}$$

Bogomolov-Gieseker conjecture

We assume the *Bogomolov-Gieseker conjecture* of Bayer-Macrì-Toda: a certain upper bound on ch_3 for $\nu_{b,w}$-semistable objects E. Setting $C_i := \mathrm{ch}_i(E).H^{3-i}$, it is

$$\big(C_1^2 - 2C_0C_2\big)w + \big(3C_0C_3 - C_1C_2\big)b + \big(2C_2^2 - 3C_1C_3\big) \ \geq \ 0,$$

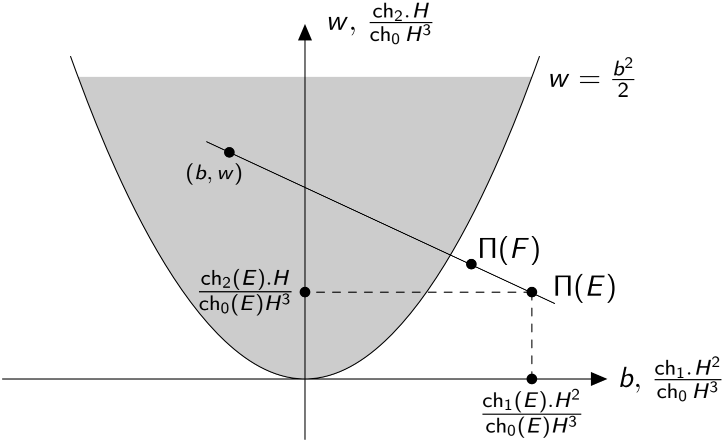
It is a sufficient condition for the existence of *Bridgeland stability conditions* on X, and has now been proved for some Calabi-Yau 3-folds.

For instance Chunyi Li proved it for many (b, w) (enough for our applications) on quintic 3-folds X.

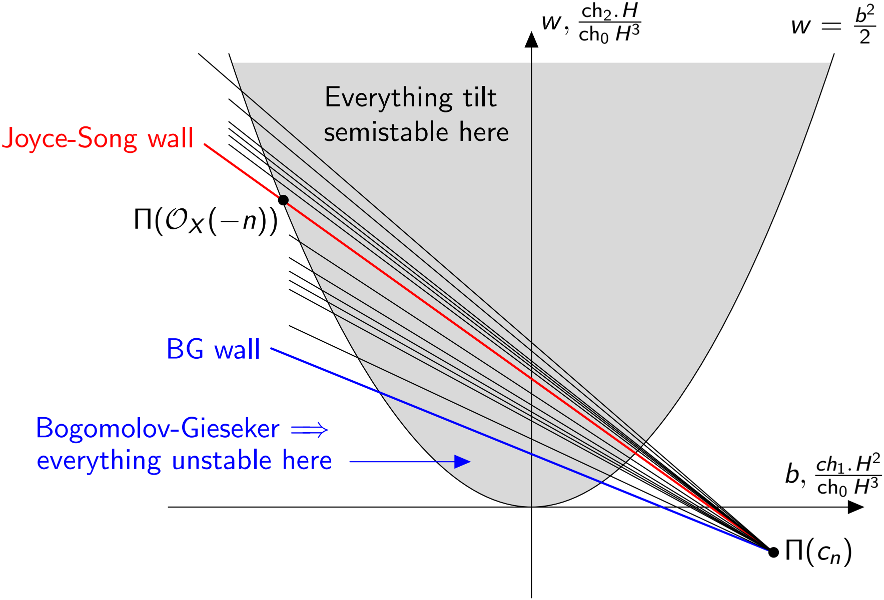
Weak stability conditions II

Plot $\Pi(E) := \left(\frac{\mathrm{ch}_1(E).H^2}{\mathrm{ch}_0(E)H^3}, \frac{\mathrm{ch}_2(E).H^2}{\mathrm{ch}_0(E)H^3}\right)$ on the same axes as (b, w).

Then walls of instability for E become straight lines through $\Pi(E)$ and $\Pi(F)$, where F is a destabilising sub- or quotient- object.



Walls of instability for c_n



Some aspects of the proof

- The Joyce-Song wall is where the $\nu_{b,w}$-slopes of F (of charge c_n) and $\mathcal{O}(-n)[1]$ coincide.
- Rotating the exact sequence $0 \longrightarrow \mathcal{O}(-n) \xrightarrow{s} E \longrightarrow F \longrightarrow 0$ in $D(X)$ gives the destabilising exact triangle

$$E \longrightarrow F \longrightarrow \mathcal{O}(-n)[1].$$

- Below the wall F is destabilised by this, above the wall it is stable iff E is $\nu_{b,w}$-semistable and s does not factor through any semi-destabilising subsheaf.
- Gives wall-crossing formula

$$J_{b,w_+}(c_n) \;=\; J_{b,w_-}(c_n) + (-1)^{N-1} \cdot N \cdot J_{b,w}(c) + \cdots ,$$

where $N = \chi(E(n))$. Lower order terms from sections of destabilising subsheaves of E (lower rank, so can induct on rank).
- Now wall cross **second term down to below the BG wall**, and all other terms **up to large volume chamber**.

Some more aspects of the proof

- All these further wall crossings involve only sheaves – no more complexes of sheaves, nor shifts like $\mathcal{O}(-n)[1]$.
- These wall crossings spit out destabilising pieces which we also wall-cross up to the large volume chamber. Their wall-crossing also involves only sheaves. (So rank never increases.)
- At each stage the discriminant $\Delta_H = \left(\mathrm{ch}_1.H^2\right)^2 - 2(\mathrm{ch}_2.H)\,\mathrm{ch}_0\,H^3$ decreases and cannot drop below 0.
- So a double induction on rank and Δ_H turns $J_{b,w_+}(c_n) = J_{b,w_-}(c_n) + (-1)^{N-1} \cdot N \cdot J_{b,w}(c) + \cdots$ into $J_{b,\infty}(c_n) = 0 + (-1)^{N-1} \cdot N \cdot J_{b,\infty}(c) + \cdots$ with $\cdots$ of the form $F(J_{b,\infty}(\alpha_i)),\ \mathrm{rank}(\alpha_i) \leq r-1$
- A further wall-crossing passes from $J_{b,\infty}$ to J.
- Thus have written $J(c)$ in terms of J of lower rank sheaves.

Rank 0 to rank -1

Now suppose c has rank 0. We go one step further to rank -1.

Fix $n \gg 0$ so that $H^{\geq 1}(E(n)) = 0$ for all semistable E of charge c.

For a section $s \in H^0(E(n))$, again replace E by the rank -1 *complex of sheaves* $F \in D(X)$

$$F \ := \ \left\{\mathcal{O}(-n) \xrightarrow{s} E\right\}.$$

Since s is neither injective nor surjective F is no longer quasi-isomorphic to a sheaf (unlike when $\mathrm{rank}(E) > 0$).

So we study $\nu_{b,w}$-semistable rank -1 complexes of charge $\mathrm{ch}(F) = c_n := c - e^{-nH}$. Joyce-Song wall gives relation of $J_{b,w}(c)$ to $J_{b,w}(c_n)$ as before.

Over other walls we show destabilising factors also rank -1 complexes and rank 0 sheaves with strictly smaller degree $\mathrm{ch}_1.H^2 < c.H^2$ allowing us to set up an induction on this degree (in place of rank used earlier). Magic before was that all semistable factors (except JS) were subsheaves – therefore of lower rank; now

Rank -1 to rank 1

The shift by [1] of the *derived dual* of F

$$F^{\vee}[1] \ := \ \left\{E^{\vee} \xrightarrow{s} \mathcal{O}(n)\right\}$$

has rank 1, and after wall crossing becomes a stable pair. After a further, older wall-crossing (Bridgeland, Toda) it becomes an ideal sheaf, recovering the MNOP (or GW) invariants again.

So the "*rough idea*" in this case gives a simple universal formula relating rank 0 to rank 1 DT invariants (or D4-D2-D0 counts to curve counts), just as we wanted.

GW invariants

Recall the rank 1 to rank 0 WCF $J(c_n) = (-1)^{N-1} \cdot N \cdot J(c)$ where the rank 1 DT invariants $J(c)$ are equivalent to GW invariants.

The generating series of D4-D2-D0 counts $J(c_n)$ are conjectured by "*S-duality*" to be vector-valued mock modular forms.

Can try to see this modularity from a Noether-Lefschetz point of view, following Maulik-Pandharipande.

For simplicity assume $H^1(\mathcal{O}_X) = 0 = H^2(X, \mathbb{Z})_{\rm tors}$ and $H^2(X, \mathbb{Z}) = \mathbb{Z}.H$.

Digression: modular forms and Noether-Lefschetz theory

Since slope stability invariant under $\otimes\mathcal{O}(1)$,

$$J(0,\, nH,\, \beta - D^2/2,\, m) \;=\; J(0,\, nH,\, nH^2 + \beta - D^2/2,\, m + \dots).$$

Hence all information encoded in the vector of generating series

$$\left(\sum_m J(0,\, nH,\, \beta,\, m)\, q^{m+\dots}\right)_{\beta\,\in\,\frac{H^4(X,\mathbb{Z})}{nH\,\cup\,H^2(X,\mathbb{Z})}}.$$

Finite group by Lefschetz. On support $D \in |nH|$ of one of these sheaves it can be described as

$$\frac{H^4(X,\mathbb{Z})}{nH\,\cup\,H^2(X,\mathbb{Z})} \;=\; \frac{H^2(D,\mathbb{Z})/\Lambda}{H^2(X,\mathbb{Z})\big|_D} \;=\; \frac{\Lambda^*}{\Lambda},$$

where Λ is the *primitive cohomology* of D,

$$\Lambda \;:=\; H^2_{\mathrm{prim}}(D,\mathbb{Z}) \;=\; \big\langle H|_D \big\rangle^{\perp} \;\subset\; H^2(D,\mathbb{Z}).$$

Noether-Lefschetz theory

Our theorem (stable rank 0 sheaves are cokernels of maps from $\mathcal{O}_X(-n)$ to ideal sheaves) shows we're counting sheaves $\iota_*(L \otimes I_Z)$, where

$$(*) \qquad \iota\colon D \hookrightarrow X \text{ is a divisor } D \in |nH| \text{ and } \iota_* c_1(L) = \beta.$$

$Z \subset D$ is zero dimensional; "*counting*" and summing over $q^{m+\cdots}$ $= q^{\text{length}(Z)+\cdots}$ gives modular form $\eta(q)$ by Göttsche's formula.

Leaves counting of classes $(*)$, i.e. counting D in Noether-Lefschetz loci in $|\mathcal{O}(n)|$ for class β.

Each smooth $D \in |\mathcal{O}(n)|$ defines a point in the quotient Q of the period domain of Hodge structures on Λ by $\operatorname{Aut}\big(H^2(D,\mathbb{Z}), H|_D\big)$ $= \ker\big(\operatorname{Aut}\Lambda \to \operatorname{Aut}\left(\Lambda^*/\Lambda\right)\big)$. Gives $|\mathcal{O}(n)| \dashrightarrow Q$.

Want intersection with universal Noether-Lefschetz loci $NL_{d,\beta} \subset Q$ of Hodge structures with an extra $(1,1)$ class $\ell \in \Lambda^*$ of "*coset*" $[\ell] = \beta \in \Lambda^*/\Lambda$ and square $d = m + \ldots$ (equivalent to discriminant of $\langle c_1(L), H|_D\rangle \subset H^2(D,\mathbb{Z})$).

Mock modular forms

Summing these cohomology classes over $d = \ell^2 = m + \dots$

$$\bigoplus_{\beta \in \Lambda^*/\Lambda} \sum_d [NL_{d,\beta}]\, q^d$$

we get modular forms valued in $H^*(Q)$ [Borcherds, Kudla-Millson, Garcia, …]. Pulling back and integrating over $|\mathcal{O}(n)|$ would give the modularity we seek if all $D \in |\mathcal{O}(n)|$ were smooth.

Compactifying the moduli space of Hodge structures Q to allow degenerations of D — especially splittings $D = D_1 + \dots + D_k$ — is expected to lead to corrections (non-holomorphic modular completions made from $k-1$ iterated Eichler integrals involving the contributions of the $D_1, \dots, D_k$) giving vector-valued mock modular forms.