Mathematical structures of non-perturbative topological string theory: from GW to DT invariants

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The partition function of the topological string is of interest both for physics

(effective sugergravity actions, Nekrasov partition functions,...)

and mathematics

(enumerative invariants: Gromov-Witten, Donaldson-Thomas, Gopakumar-Vafa,...)

There are various approaches to its computation

(Topological recursion, holomorphic anomaly, topological vertex,...)

Most of them are perturbative in one way or another, with some exceptions

(matrix model; cf. in particular Marino et. al.)

The problem

The topological string free energy,

$$\mathcal{F}(Q,\lambda) = \mathcal{F}_0(\lambda) + \widetilde{\mathcal{F}}(Q,\lambda)\,,$$

is defined by a divergent series

$$\mathcal{F}_0(\lambda) = \sum_{g\geq 0} \lambda^{2g-2} F_0^g, \qquad F_0^g = \frac{\chi(X)(-1)^{g-1}\, B_{2g}\, B_{2g-2}}{4g(2g-2)\,(2g-2)!}\,, \quad g\geq 2\,,$$

$$\widetilde{\mathcal{F}}(Q,\lambda) = \sum_{g\geq 0}\sum_{\beta\in\Gamma} \lambda^{2g-2} [\mathrm{GW}]_{\beta,g}\, Q^\beta,$$

where $\Gamma = \{\beta \in H_2(X,\mathbb{Z}); \beta \neq 0\}$, $[\mathrm{GW}]_{\beta,g}$: Gromov–Witten invariants.

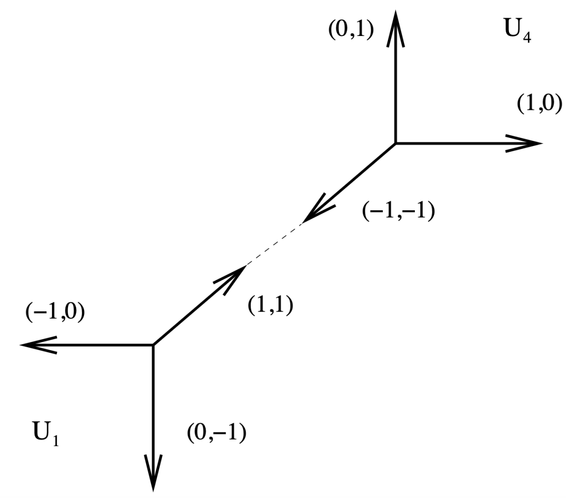
Questions:

- Do there exist analytic functions with asymptotic expansion $\mathcal{F}(Q,\lambda)$?
- If there are several such functions, how are they related to each other?

A special case of the problem

Specialise to case of resolved conifold:

$$\widetilde{\mathcal{F}}(Q_F,t) = \frac{1}{\lambda^2}\mathrm{Li}_3(Q_F) + \frac{B_2}{2}\mathrm{Li}_1(Q_F) - \sum_{g=2}^{\infty} \lambda^{2g-2}\frac{(-1)^g B_{2g}}{2g(2g-2)!}\mathrm{Li}_{3-2g}(Q_F),$$



where $Q_F = e^{2\pi\mathrm{i}\,t}$, t: Kähler modulus.

Gopakumar-Vafa 1998, Faber-Pandharipande 2000

Borel summation

Consider a formal series $\sum_{k=0}^{\infty} a_n \lambda^n$, and assume that the **Borel transform**

$$G(t) := \sum_{k=1}^{\infty} a_k \frac{t^{k-1}}{(k-1)!}$$

is convergent in a neighbourhood of 0, and has an analytic continuation to a set $U \subset \mathbb{C}$ containing a ray $\rho = r\mathbb{R}_+$, $r \in \mathbb{C}$, such that

$$\int_\rho e^{-t/\lambda} G(t) dt$$

converges for λ in a non-empty open set V, then

$$f(\lambda) := a_0 + \int_\rho e^{-t/\lambda} G(t) dt$$

defines a holomorphic in function $f(\lambda)$ on V having asymptotic series $\sum_{k=0}^{\infty} a_k \lambda^k$.

Proof: Taylor-expand $G(t)$ around $t = 0$, and integrate term by term.

Stokes phenomena

Pasquetti-Schiappa found a formula for the Borel-transform of $\widetilde{\mathcal{F}}(Q,\lambda)$.
We found a useful alternative expression:

$$G(\xi,t)=\frac{1}{(2\pi)^2}\sum_{m\in\mathbb{Z}\setminus\{0\}}\frac{1}{m^3}\frac{1}{2\xi}\frac{\partial}{\partial\xi}\left(\frac{\xi^2}{1-e^{-2\pi\mathrm{i}t+\xi/m}}-\frac{\xi^2}{1-e^{-2\pi\mathrm{i}t-\xi/m}}\right).$$

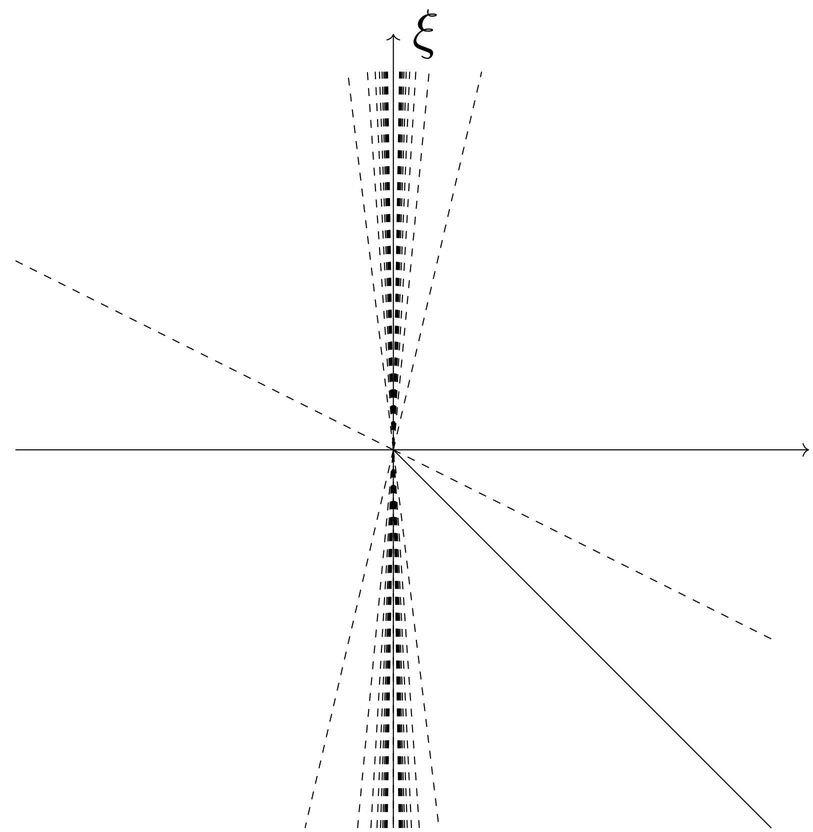
The poles are located on lines l_k, $k\in\mathbb{Z}$, through origin $\Rightarrow$ can take any ray ρ_k in wedges between lines l_k and l_{k-1} as integration contour $\rightsquigarrow$ Collection of functions $F_{\rho_k}(\lambda,t)$.

Borel summations **jump** across lines l_k:

$$\begin{aligned}\phi_{\pm l_k}(\lambda,t):&=F_{\pm\rho_{k+1}}(\lambda,t)-F_{\pm\rho_k}(\lambda,t)\\&=\frac{1}{2\pi\mathrm{i}}\partial_{\check\lambda}\Big(\check\lambda\,\mathrm{Li}_2\big(e^{\pm 2\pi\mathrm{i}(t+k)/\check\lambda}\big)\Big)\,.\end{aligned}$$

Of special interest turns out to be

$$F_{\mathbb{R}_{>0}}(\lambda,t)=-\int_{\mathbb{R}+\mathrm{i}0^+}\frac{\mathrm{d}u}{8u}\frac{e^{u(t-1/2)}}{\sinh(u/2)(\sinh(\check\lambda u/2))^2}\,.$$



Resulting picture

$$F_\rho(\lambda,t) = F_{\rm GV}(\lambda,t) + F_{\rm D}(\lambda,t;\rho), \qquad F_{\rm GV}(\lambda,t) := \sum_{k=1}^{\infty} \frac{e^{2\pi \mathrm{i} kt}}{k\big(2\sin\left(\frac{\lambda k}{2}\right)\big)^2},$$

- $F_{\rm GV}(\lambda,t)$: Re-organisation of **perturbative** corrections a la Gopakumar-Vafa,
- $F_{\rm D}(\lambda,t;\rho)$: **Non-perturbative** part, functions of $Q' = e^{4\pi^2 \mathrm{i}t/\lambda}$, $q' = e^{4\pi^2\mathrm{i}/\lambda}$.

Varying ρ interpolates between $F_{\rm D}(\lambda,t;\mathrm{i}\mathbb{R}_+) = 0$ and

$$F_{\rm D}(\lambda,t;\mathbb{R}_+) = F_{\mathrm{N}S}(\lambda,t), \qquad F_{\rm NS}(g,t) := \frac{1}{2\mathrm{i}} \sum_{k=1}^{\infty} \frac{e^{2\pi\mathrm{i}kt}}{k^2 \sin\left(\frac{gk}{2}\right)},$$

the Nekrasov-Shatshvili limit of the refined version of $F_{\rm GV}(\lambda,t)$.

This proves, in particular, the conjecture of Hatsuda-Okuyama that

$$F_{\mathbb{R}_+}(\lambda,t) = F_{\rm np}(\lambda,t) := F_{\rm GV}(\lambda,t) + F_{\mathrm{N}S}(\lambda,t),$$

with $F_{\rm np}(\lambda,t)$ being the **non-perturbative completion** of the topological string partition function proposed by Hatsuda-Marino-Moriyama-Okuyama.

Relation to BPS-states on resolved conifold

- Charge lattice Γ_c given by the trivial local system $\Gamma_c = \mathbb{Z} \cdot \delta \oplus \mathbb{Z} \cdot \beta$,

- Central charge functions $Z_{n\beta+m\delta}(t) = 2\pi \mathrm{i}(nt+m)$, for $n, m \in \mathbb{Z}$.

- BPS-indices[1]

$$\Omega(\gamma) = \begin{cases} 1 & \text{if } \gamma = \pm\beta + n\delta \quad \text{for} \quad n \in \mathbb{Z}\,, \\ -2 & \text{if } \gamma = k\delta \quad \text{for} \;\; k \in \mathbb{Z} \setminus \{0\}\,, \\ 0 & \text{otherwise.} \end{cases}$$

One-to-one correspondence between **jumps** and **BPS-states**:

$$\phi_{Z_\gamma}(\lambda, t) = \frac{\Omega(\gamma)}{2\pi \mathrm{i}} \frac{\partial}{\partial \lambda}\Big(\lambda \, \mathrm{Li}_2\big(e^{Z_\gamma(t)/\lambda}\big)\Big), \qquad \gamma \in \Gamma_c.$$

Non-perturbative contributions $F_{\mathrm{D}}(\lambda, t; \rho)$:

Sum over all **stable** D-branes with charge γ having positive imaginary part of $Z_\gamma(t)$.

[1] Joyce-Song, Banerjee-Longhi-Romo

Relation to geometry of spaces of stability conditions I

(T. Bridgeland)

Consider the local system of (charge-)lattices Γ with intersection pairing $\langle -,-\rangle$, fibered over the space of stability conditions $\mathcal{B}$ such that

- there is an atlas of local charts for $\mathcal{B}$ admitting local bases for Γ generated by elements denoted as $\gamma_0,\dots,\gamma_d,\check{\gamma}^0,\dots,\check{\gamma}^d$ satisfying $\langle \gamma_r,\check{\gamma}^s\rangle = \delta_r{}^s$.

- A local chart should have local complex coordinates for $\mathcal{B}$ given by the central charge functions $Z=(Z^0,\dots,Z^d)$, $\check{Z}=(\check{Z}_0,\dots,\check{Z}_d)$.

Consider total space $\mathcal{M} := \mathcal{T}\mathcal{B}$ with coordinates $\Theta=(\Theta^0,\dots,\Theta^d)$, $\check{\Theta}=(\check{\Theta}_0,\dots,\check{\Theta}_d)$ on the tangent fibres.

Case of Conifold: $d=1$, $\delta\sim D0$, $\beta\sim D2$,

$$\Gamma = \mathbb{Z}\cdot\delta\oplus\mathbb{Z}\cdot\beta\oplus\mathbb{Z}\cdot\beta^\vee\oplus\mathbb{Z}\cdot\delta^\vee, \qquad \begin{aligned} \delta &\equiv \gamma_0, & Z_\delta &\equiv Z^0,\\ \beta &\equiv \gamma_1, & Z_\beta &\equiv Z^1, \end{aligned}$$

with $t=Z^1/Z^0$, and $\check{Z}_0$ and $\check{Z}_1$ determined by special geometry (prepotential).

Relation to geometry of spaces of stability conditions II

Darboux coordinates on $\mathcal{M}$: Solutions to BPS Riemann-Hilbert problem

(Gaiotto-Moore-Neitzke; Bridgeland)

Define ζ-deformed complex structures by atlas of coordinates on $\mathcal{Z} \simeq \mathcal{M} \times \mathbb{C}^\times$ with charts $\{\mathcal{U}_\imath; \imath \in \mathbb{I}\}$, Darboux coordinates $(\mathsf{x}_\imath, \check{\mathsf{x}}^\imath)$,

$$x_\imath = (\mathsf{x}_\imath, \check{\mathsf{x}}^\imath), \qquad \Omega = \sum_{r=1}^{d} dx_\imath^r \wedge d\check{x}_r^\imath, \quad \text{such that}$$

- changes of coordinates across $\{\zeta \in \mathbb{C}^\times; a_\gamma/\zeta \in \mathrm{i}\mathbb{R}_-\}$ represented as

$$X_{\gamma'}^\jmath = X_{\gamma'}^\imath (1 - X_\gamma^\imath)^{\langle \gamma', \gamma \rangle \Omega(\gamma)}, \qquad \begin{array}{l} X_\gamma^\jmath = e^{2\pi \mathrm{i} \langle \gamma, x_\imath \rangle} = e^{2\pi \mathrm{i}(p_r^\imath x_\imath^r - q_\imath^r \check{x}_r^\imath)}, \\ \text{if } \gamma = (q_\imath^1, \dots, q_\imath^d; p_1^\imath, \dots, p_d^\imath), \end{array}$$

  determined by **BPS-indices** $\Omega(\gamma)$ satisfying Kontsevich-Soibelman-WCF.

- asymptotic behaviour

$$\mathsf{x}_\imath^r \sim \frac{1}{\zeta} Z_\imath^r + \Theta_\imath^r + \mathcal{O}(\zeta), \qquad \check{\mathsf{x}}_\imath^r \sim \frac{1}{\zeta} \check{Z}_r^\imath + \check{\Theta}_r^\imath + \mathcal{O}(\zeta).$$

Relation to geometry of spaces of stability conditions III

(i) Stokes jumps define a holomorphic **line bundle** $\mathcal{L}$ on $\mathcal{Z}$ with transition functions

$$\Phi_{Z_\gamma}(Z^0, Z^1; \zeta) = \exp\left(\phi_{Z_\gamma}\left(\tfrac{\zeta}{Z^0}, \tfrac{Z^1}{Z^0}\right)\right).$$

(ii) Holomorphic sections of $\mathcal{L}$: Partition functions

$$\mathcal{Z}(Z^0, Z^1; \zeta; \rho) = \exp\left(F\left(\tfrac{\zeta}{Z^0}, \tfrac{Z^1}{Z^0}; \rho\right)\right).$$

(iii) Jumps: Difference **generating functions** of changes of Darboux coordinates.

Remarks

- Observation (iii) generalizes a previous proposal of Coman-Longhi-J.T.
- The line bundle $\mathcal{L}$ is equivalent to the hyper-holomorphic line bundles previously considered by Alexandrov-Persson-Pioline and Neitzke, specialised to our case.

Strong coupling expansion I

There is an alternative integral representation:

$$F_{\mathrm{np}}(\lambda,t) = -\frac{\lambda}{(2\pi)^3}(2\mathrm{Li}_3(Q') + \alpha\,\mathrm{Li}_2(Q')) - \frac{\mathrm{i}}{4\pi}(\mathrm{Li}_2(Q') + \alpha\,\mathrm{Li}_1(Q'))$$

$$+ \int_{\mathbb{R}_+} \mathrm{d}v\, e^{-2\pi\lambda v} G_s(v,t'), \quad G_{\mathrm{s}}(v,t') = \frac{1}{(2\pi)^2} \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{n^3} \frac{v + 2\pi\mathrm{i}\, nt'}{1 - e^{-v/n - 2\pi\mathrm{i}t'}}\,,$$

where $t' = \frac{2\pi}{\lambda}t$, $Q' = e^{2\pi\mathrm{i}t}$.

Expanding $G_{\mathrm{s}}(v,t')$ in powers of v yields **strong coupling expansion** for $\lambda \to \infty$.

A family of functions $F'_{\rho'}(\lambda,t)$ having the same asymptotic expansion is obtained by replacing contour $\mathbb{R}_+$ of integration in formula above by rays ρ'.

The Borel summations of the strong-coupling expansion along different rays ρ' are found to have the following structure:

$$F'_{\rho'}(\lambda,t) = F_{\mathrm{BPS}}(\lambda,t;\rho') + F_{\mathrm{NS}}(\lambda,t).$$

Only F_{BPS} depends on ρ'. For $\rho' = \mathbb{R}_+$ we recover $F_{\mathrm{BPS}}(\lambda,t;\rho') = F_{\mathrm{GV}}(\lambda,t)$.

Strong coupling expansion II – Stokes phenomena

Compare $Z_{\mathrm{BPS}}(\lambda, t; \rho') := \exp\big(F_{\mathrm{BPS}}(\lambda, t; \rho')\big)$ to counting functions $\mathcal{Z}_{\mathrm{DT}}(u, v; \mathcal{C})$

$$\mathcal{Z}_{\mathrm{DT}}(u, v; \mathcal{C}) = \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} [\mathrm{DT}]^{\mathcal{C}}_{n\delta + k\beta + \delta^\vee} \, u^n v^k,$$

where BPS indices $[\mathrm{DT}]^{\mathcal{C}}_{n\delta + k\beta + \delta^\vee}$ are locally constant w.r.t. Kähler moduli, but jump across walls of marginal stability $\rightsquigarrow$ dependence on choice of chamber $\mathcal{C}$.[2]

In the large volume chamber $\mathcal{C}_\infty$ we may apply the MNOP-correspondence[3]

$$\mathcal{Z}_{\mathrm{DT}}(-q, Q; \mathcal{C}_\infty) = (M(q))^{\chi(X)} e^{F_{\mathrm{GV}}(\lambda, t)}, \qquad q = e^{\mathrm{i}\lambda}, \quad Q = e^{2\pi \mathrm{i} t}.$$

Dictionary $u = -e^{\mathrm{i}\lambda}$, $v = e^{2\pi\mathrm{i}\, t}$ $\rightsquigarrow$ map between chambers $\mathcal{C}$ and wedges in λ-plane.

Comparing Stokes jumps of $\mathcal{Z}_{\mathrm{BPS}}$ with wall-crossing of $\mathcal{Z}_{\mathrm{DT}}$ (Jafferis-Moore) yields:

$$\boxed{\mathcal{Z}_{\mathrm{BPS}}(\lambda, t; \rho') = \mathcal{Z}_{\mathrm{DT}}\big(-e^{\mathrm{i}\lambda}, e^{2\pi\mathrm{i}\, t}; \mathcal{C}\big), \qquad \forall\, \mathcal{C}, \ \ \rho' \in \mathcal{C}.}$$

[2] Jafferis-Moore 2008, Nagao-Nakajima 2008

[3] Maulik-Nekrasov-Okounkov-Pandharipande 2006

Strong coupling expansion II – Dual RH problem

Recall that $Z_{\mathrm{BPS}}(\lambda, t; \rho')$ is the part which jumps at strong coupling.

Observation: Jumps of $Z_{\mathrm{BPS}}(\lambda, t; \rho')$ also related to the BPS-RH problem.

More precisely: There is a **dual** BPS RH problem for coordinates X'_γ having changes of coordinates across **dual** walls $\{\zeta \in \mathbb{C}^\times; \zeta a'_\gamma \in \mathrm{i}\mathbb{R}_-\}$ obtained by replacing X_γ by X'_γ. In our case, in particular

$$X^0 = e^{2\pi\mathrm{i}\frac{Z^0}{\zeta}}, \qquad X'^0 = e^{2\pi\mathrm{i}\frac{\zeta}{Z^0}},$$

$$X^1 = e^{2\pi\mathrm{i}\frac{Z^1}{\zeta}}, \qquad X'^1 = e^{2\pi\mathrm{i}\frac{Z^1}{Z^0}}.$$

And furthermore:

$$\text{Jumps of } Z_{\mathrm{BPS}}(\lambda, t; \rho') \quad = \quad \text{Jumps of } \check{X}'_0.$$

Corresponding change of homogenous coordinates $\lambda' = -\frac{4\pi^2}{\lambda}$, $t' = \frac{2\pi}{\lambda} t$ related to realisation of **S-duality** on hypermultiplet moduli space.[4]

[4] Alexandrov-Pioline-Saueressig-Vandoren 2009, Alexandrov-Persson-Pioline 2011

Conclusions

Non-perturbative effects encoded in the perturbative expansion of the topological string partition function (resurgence).

Mathematical coding of non-perturbative effects:

Weak coupling Stokes jumps $\rightsquigarrow$ canonical line bundles on hypermultiplet moduli space ($\hookleftarrow$ Bridgeland's RH problems, generalises proposal of Coman-Longhi-J.T.).

Strong coupling Stokes jumps $\rightsquigarrow$ **dual** RH problem of Bridgeland type ($\hookleftarrow$ S-duality on hypermultiplet moduli space).

Different origin of weak/strong framed wall-crossing: GW versus DT theory

All this was fully worked out in the case of the resolved conifold, but there is hope that the essential ingredients of the resulting picture can hold in larger generality.