# $T^* \operatorname{Gr}(k, n)$ as a Coulomb branch

Ben Webster
(based on joint work with Aiden Suter)

University of Waterloo
Perimeter Institute for Mathematical Physics

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UNIVERSITY OF WATERLOO
FACULTY OF MATHEMATICS
Department of Pure Mathematics

PERIMETER INSTITUT
INSTITUTE FOR THEORETICAL PHYSICS
PÉRIMÈTRE DE PHYSIQUE THÉORIQUE

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After that, things get a little blurry, but then I ended up here.

Hanany and Witten study configurations of D5, NS5 and D3 branes in type IIB string theory.

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| D3 | X | X | X | | | | X | | | |
| D5 | X | X | X | | | | | X | X | X |
| NS5 | X | X | X | X | X | X | | | | |

They consider how this theory localizes on the plane of the D3's and the result they obtain is a quiver gauge theory for a linear quiver (or a cyclic quiver if we make the $x^6$ direction a circle).

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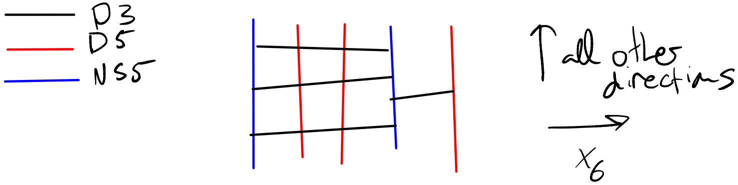


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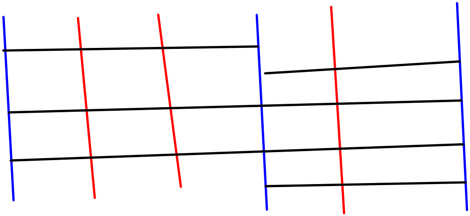
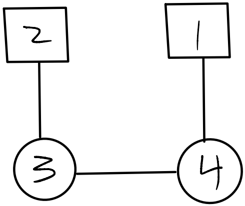
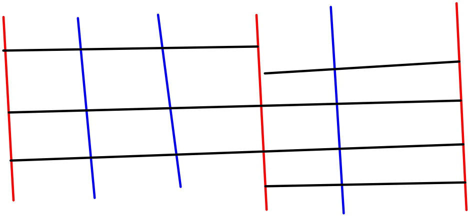
- If all D3 branes end on NS5 branes then we can describe 3d theory as a quiver gauge theory (this is called *cobalanced*), where:
  - nodes correspond to gaps between NS5 branes
  - rank of $U(v_i)$ is # of D3 branes joining consecutive pairs of NS5's.
  - matter is a bifundamental for each pair of consecutive branes, and a fundamental for each D5 between NS5's ($w_i$=# D5's).

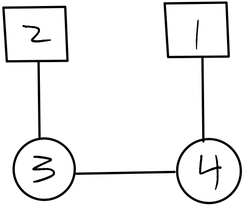




- If all D3 branes end on D5 branes then we can describe 3d theory as a quiver gauge theory (this is called *balanced*), where:
  - nodes correspond to gaps between D5 branes
  - rank of $U(v_i)$ is # of D3 branes joining consecutive pairs of D5's.
  - matter is a bifundamental for each pair of consecutive branes, and a fundamental for each NS5 between D5's ($w_i$=# NS5's).

but the supersymmetry acts differently!





In a 3d quantum field theory with $\mathcal{N} = 4$ supersymmetry, there are two topological twists, often called the A- and B-twists.

## Definition

The local operators in:

1. the A-twist are called the **Coulomb branch chiral ring** $A_{\text{Coulomb}}$
2. the B-twist are called the **Higgs branch chiral ring** $A_{\text{Higgs}}$

From the perspective of the whole theory, these are two natural classes of $\frac{1}{2}$BPS operators.

The spaces of the Coulomb and Higgs branch are $\mathfrak{M}_{\text{Coulomb}} = \operatorname{Spec}(A_{\text{Coulomb}})$ and $\mathfrak{M}_{\text{Higgs}} = \operatorname{Spec}(A_{\text{Higgs}})$

If you swap D5 and NS5 in the Hanany-Witten picture, you get the same theory, but with supersymmetry changed so that A- and B-twists switch.

This is an example of 3-dimensional mirror symmetry/S-duality.

In the balanced case, we have a nice description of the Coulomb branch: it is the Nakajima quiver variety of this quiver gauge theory.

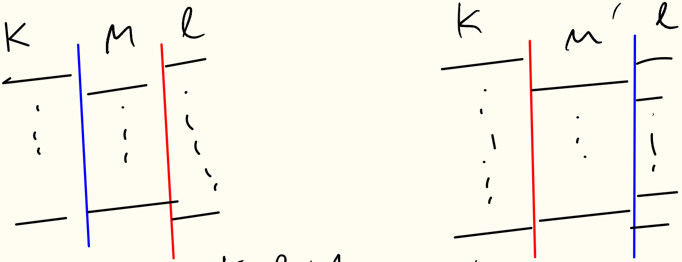
Dually, in the cobalanced case, the same Nakajima quiver variety is the *Higgs branch*.

More generally, for any good brane configuration, we can write both the Higgs and Coulomb branches as **bow varieties** which generalize quiver varieties.

This observation becomes much more powerful if we exploit the existence of Hanany-Witten transitions, which allow us to write a single theory in terms of both balanced and cobalanced brane configurations.

## Theorem

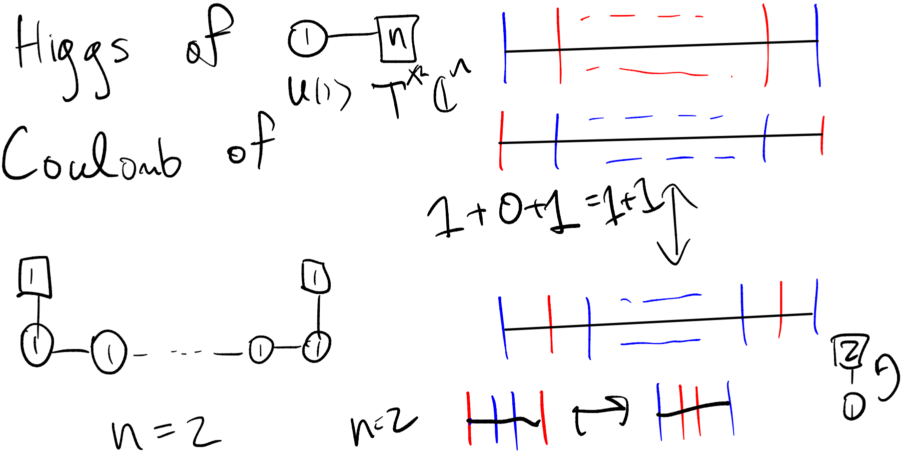
*When we swap the order of an NS5 and D5 brane, the number of D3 branes joining them in the new and old configurations are related by:*




$$k + l + 1 = m + m'$$

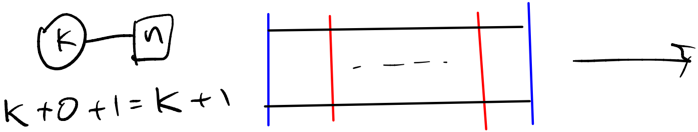
First example: $U(1)$ with $n$ hypers has Higgs branch

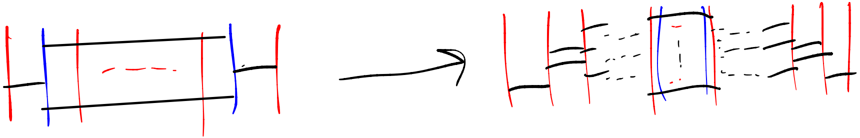
$$X_0^{(1,n)} = \{A \in M_{n\times n}(\mathbb{C}) \mid A^2 = 0, \mathrm{rk}(A) \leq 1\} \longleftarrow X^{(1,n)} = T^*\mathbb{CP}^{n-1}.$$

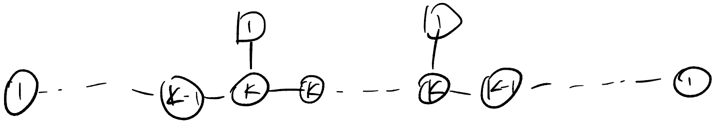


Main example: $U(k)$ with $n$ fundamentals ($k \leq n/2$) has Higgs branch

$$X_0^{(k,n)} = \{A \in M_{n\times n}(\mathbb{C}) \mid A^2 = 0, \mathrm{rk}(A) \leq k\} \longleftarrow X^{(k,n)} = T^* \mathrm{Gr}(k,n).$$

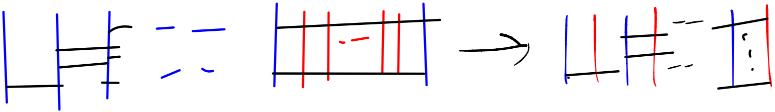


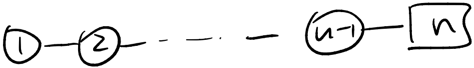





For funsies: we can also get

$$X_0 = \{A \in M_{n\times n}(\mathbb{C}) \mid A^n = 0\} \longleftarrow X = T^*\mathrm{Fl}_n!$$






Self-dual!

Let $X = T^*\mathrm{Fl}_n$ be the cotangent bundle of the flag variety $X_0 = \mathrm{Fl}_n$ over a field $\Bbbk$ of characteristic $p \geq 0$.

Let $\mathsf{Coh}_0(X)$ denote the abelian category of coherent sheaves on $X$ which are (set-theoretically) supported on $\mathrm{Fl}_n$.

Consider the algebra $A = U\mathfrak{gl}_n(\mathbb{k})$. Let $\mathfrak{U}\text{-mod}_0$ be the principal block of the category of finite dimensional modules with central character.

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**Theorem (Bezrukavnikov-Mirkovič)**

*If $p \gg 0$, there is an equivalence of derived categories*

$$D^b(\mathsf{Coh}_0(X)) \cong D^b(\mathfrak{U}).$$

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## Conjecture (Bezrukavnikov-Mirkovič)

*There is an equivalence of derived categories*

$$D^b(\mathsf{Coh}_0(X)) \cong D^b(\mathcal{U}).$$

Bezrukavnikov calls this a “non-commutative counterpart of the Springer resolution.”

This equivalence looks very strange, but that’s because you’ve been thinking about $X = T^*\mathrm{Fl}_n$ too Higgsily. It’s very natural when you use the Coulomb perspective.

| | |
|---|---|
| $\mathsf{Coh}_0(X)$ | $\mathfrak{U}\text{-mod}_0$ |
| geometry | representation theory |

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$\mathsf{Coh}_0(X)$
geometry

$\mathfrak{U}\text{-mod}_0$
representation theory

$\mathsf{Coh}_0(X) \longleftrightarrow \mathring{R}\text{-mod}_0 \longleftrightarrow \mathfrak{U}\text{-mod}_0$

$\mathring{R}\text{-mod}_0$
combinatorial algebra

The balanced Higgs branch/cobalanced Coulomb branch also has a mathematical description:

**Theorem (Braverman–Finkelberg–Nakajima)**

*The Coulomb branch chiral ring of the sigma model with matter N gauged by G is the Borel-Moore homology*

$$A_{\text{Coulomb}} = H_*^{BM}\left(\frac{N[\![t]\!]}{G[\![t]\!]} \times_{\frac{N((t))}{G((t))}} \frac{N[\![t]\!]}{G[\![t]\!]}\right) \qquad \mathfrak{M}_0 = \operatorname{Spec}(A_{\text{Coulomb}})$$

You can think of this computation in the category of lines in the A-twisted TQFT, which we can interpret mathematically as the D-modules on the loop space $N((t))/G((t))$.

$$ {}^{*1}\!/G[[t]] \times {}^{*}\!/G((t)) \; {}^{*}\!/G[[t]] = G(Q)\backslash G(K)/G(\mathcal{O}) $$

$$ G(\mathcal{O}) \hookrightarrow \qquad = \mathrm{Gr} $$

In our case, $G$ and $N$ define a quiver gauge theory for a linear quiver

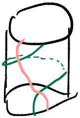
$$G = \prod_{i=1}^{r} GL(v_i) \qquad N = \bigoplus_{i=1}^{r-1} \mathrm{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{v_{i+1}}) \oplus \bigoplus_{i=1}^{r} \mathrm{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{w_i})$$

In this case, we can think of $N[\![t]\!]/G[\![t]\!]$ as the moduli space of quiver representations with $\mathbb{C}[\![t]\!]$ coefficients and $N(\!(t)\!)/G(\!(t)\!)$ with $\mathbb{C}(\!(t)\!)$ coefficients.

This algebra is generated by scalars in $(\mathrm{Sym}\,\mathfrak{t}^*)^W$ and monopole operators indexed by dominant coweights.

You can think of dominant coweights as paths in $T/W$ and scalars as coupons sitting on these paths. Remarkably, the relations of $A_{\text{Coulomb}}$ become simple and local if you write them this way.

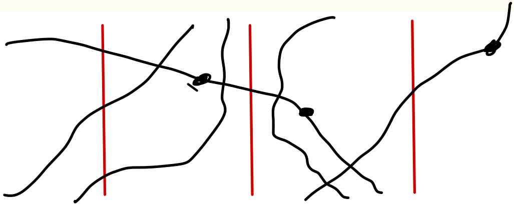
In the quiver case, we can think of $U(1)^n/S_n$ as an unordered $n$-tuple in $S^1$, and a path in this space as a diagram drawn on the cylinder.



## Definition

A (planar) KLRW diagram is a generic collection of curves in $\mathbb{R} \times [0,1]$ which are of the form $\{(\pi(t), t) \mid t \in [0,1]\}$ for $\pi\colon [0,1] \to \mathbb{R}$.

1. Each strand is labeled from $[1, r]$ and is colored red or black with $v_i$ black strands and $w_i$ red strands with label $i$.
2. Red strands must be vertical at fixed, distinct $x$-values (for example, $x = 1/n, 2/n, \ldots, 1$).
3. We place dots at a finite number of points on black strands, avoiding crossings.



## Definition

A cylindrical KLRW diagram is a generic collection of curves in $\mathbb{R}/\mathbb{Z} \times [0,1]$ which are of the form $\{(\pi(t), t) \mid t \in [0,1]\}$ for $\pi\colon [0,1] \to \mathbb{R}/\mathbb{Z}$.

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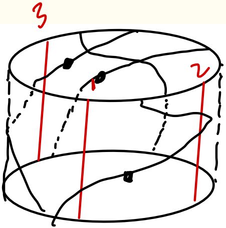
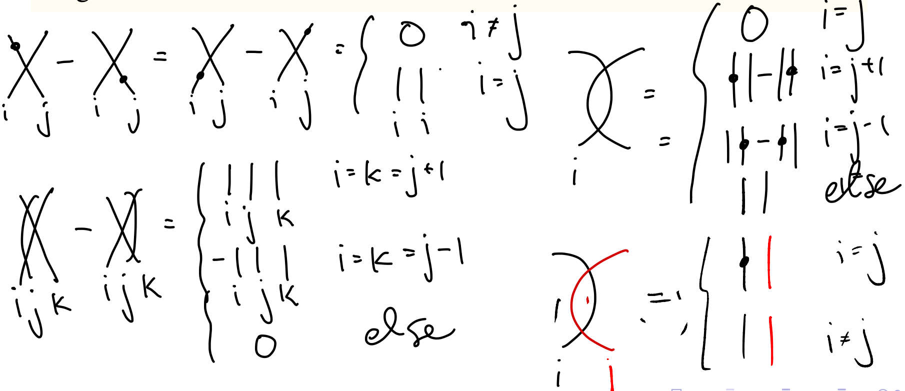


We can compose KLRW diagrams by stacking, if the labels on the bottom of one and top of the other match up to isotopy (never moving red strands).

## Definition

The (planar) KLRW algebra $R$ is the formal $\Bbbk$-span of planar KLRW diagrams modulo the local relations below.



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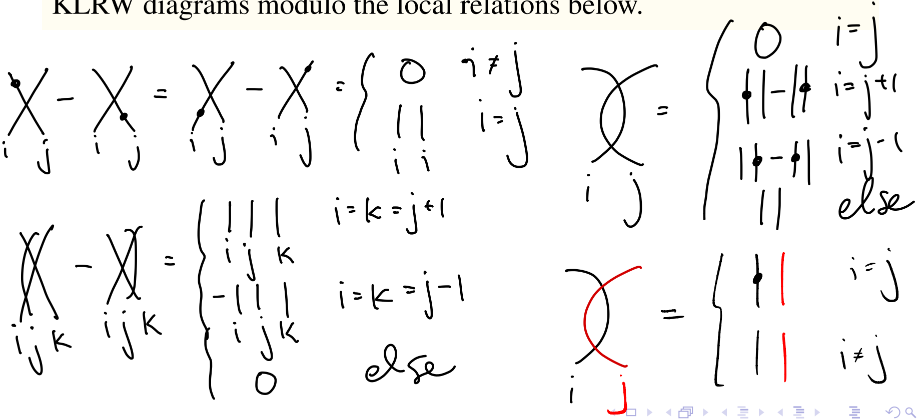
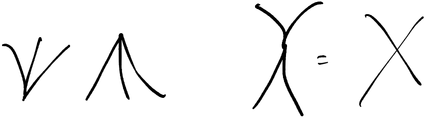
## Definition

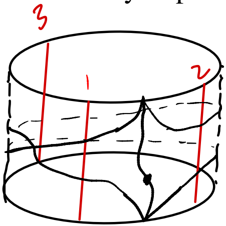
The cylindrical KLRW algebra $\mathring{R}$ is the formal $\Bbbk$-span of cylindrical KLRW diagrams modulo the local relations below.



## Coulomb from KLR

To get the algebra $A_{\text{Coulomb}}$, we have to include the ability to pinch together strands.

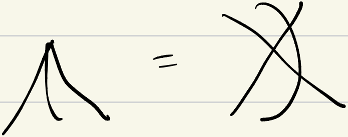


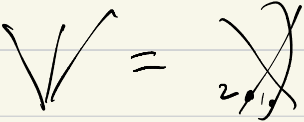



### Theorem

*The ring $A_{\text{Coulomb}} = e_C \mathring{R} e_C$ is the algebra of KLR diagrams where all strands with the same label are pinched together at the top and bottom (at the same x-value).*

$e_C$ = any idempotent w/ all black w/ same label together, take any idempotent pinching

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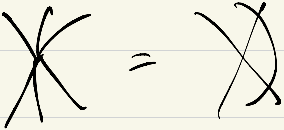
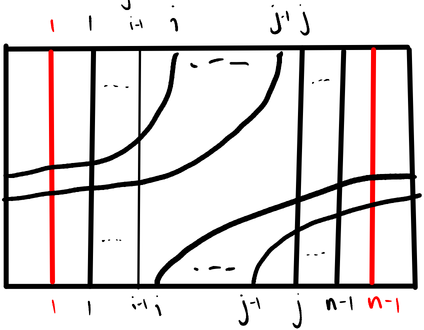


$$\{A \in M_{n\times n} \mid A^2 = 0,\ \mathrm{rk}(A) \leq 1\} \subset M_{n\times n}(\mathbb{C})$$


Example: $X_0^{(1,n)}$ as a Coulomb branch.

$e_{ij} \mapsto$

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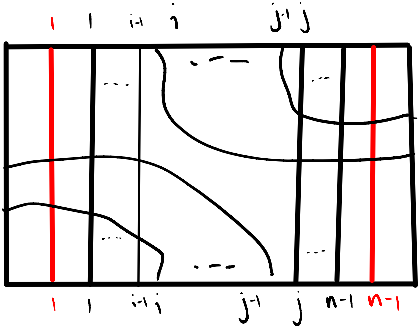
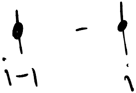


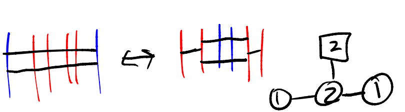

$e_{ii} \mapsto$



## Coulomb from KLR

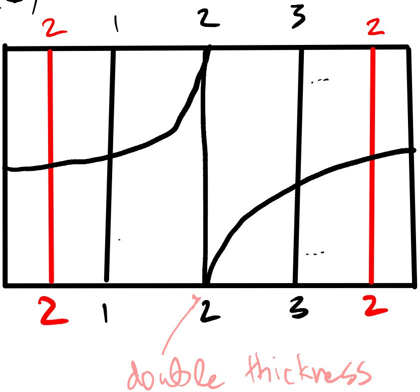
$\{A \mid A^2=0 \ \ rk(A)\le 2\}$

Example: $X_0^{(2,4)}$ as a Coulomb branch.




$M_{4\times 4}(\mathbb{C})$

$e_{23} \mapsto$




$i<j$

$e_{ij}$ = grab 1 strand w/ label $i, i+1, \ldots, j-1$

loop around cylinder once

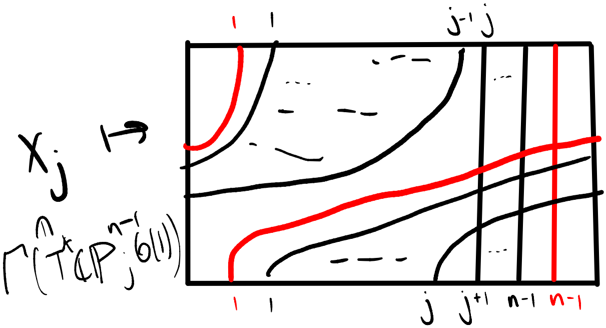
To get the resolution $T^*\mathbb{CP}^{n-1}$, need to find not just functions on $T^*\mathbb{CP}^{n-1}$ but sections $\Gamma(T^*\mathbb{CP}^{n-1}, \mathcal{O}(k))$.

Consider the rings $\tilde{A}_{\text{Coulomb}}, \tilde{\mathring{R}}$ where instead of requiring red strands to be vertical, they wrap around the cylinder some number of times, specified by $\mathbf{m} \in \mathbb{Z}^\ell$. $\ell$ = # of red strand

This ring is graded by $\mathbb{Z}^\ell$, which induces a $U(1)^\ell$ action on $\mathrm{Spec}(\tilde{A}_{\text{Coulomb}})$, and we can define a partial resolution $\mathfrak{M}^{\mathbf{m}}$ as the GIT quotient for $\mathbf{m}$:

$$\mathfrak{M}^{\mathbf{m}} = \mathrm{Proj}\left(\bigoplus_{k\geq 0} \tilde{A}^{k\mathbf{m}}_{\text{Coulomb}}\right) \qquad \tilde{A}^{k\mathbf{m}}_{\text{Coulomb}} = \Gamma(\mathfrak{M}^{\mathbf{m}}, \mathcal{O}(k)).$$

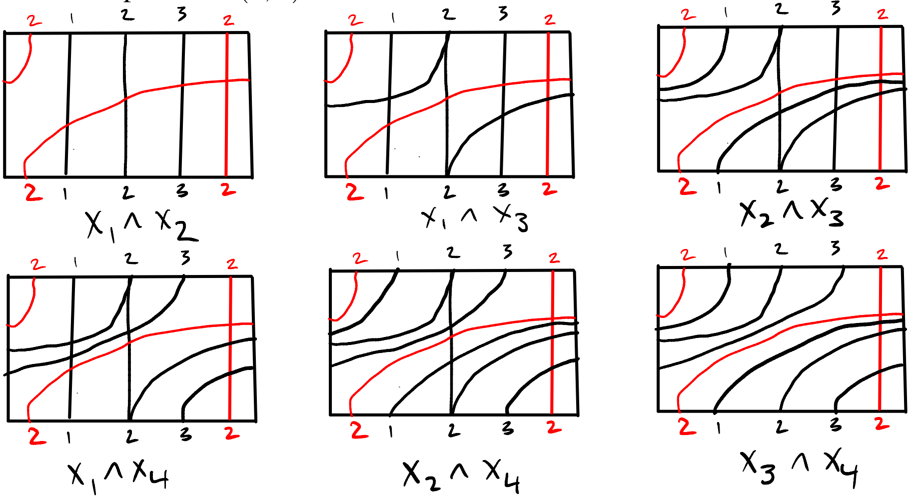
Example: $T^*\mathbb{CP}^{n-1}$ as a resolved Coulomb branch.



## Resolved Coulomb branches

$$\Gamma(\mathrm{Gr}, \mathcal{O}(1)) = \Lambda^2 \mathbb{C}^4$$


Example: $T^*\mathrm{Gr}(2,4)$ as a resolved Coulomb branch.



What new does this tell us about geometry?

Recall that we call a vector bundle $T$ on an algebraic variety $X$ a tilting generator if $\mathbb{R}\mathrm{Hom}(T, -)$ induces an equivalence of derived categories $D^b(\mathsf{Coh}(X)) \cong D^b(\mathrm{End}(T)^{\mathrm{op}}\text{-mod}.$

**Theorem (W.)**

*Over $\mathbb{C}$, there is a tilting generator $T$ on $\mathfrak{M}^{\mathbf{m}}$ such that* $\mathrm{End}(T)^{\mathrm{op}} = \mathring{R}$.

$$D^b(\mathsf{Coh}(\mathfrak{M}^{\mathbf{m}})) \cong D^b(\mathring{R}\text{-mod}).$$

*In particular, the ring $\mathring{R}$ is a non-commutative crepant resolution of singularities of $\mathfrak{M}$ which is D-equivalent to $\mathfrak{M}^{\mathbf{m}}$.*

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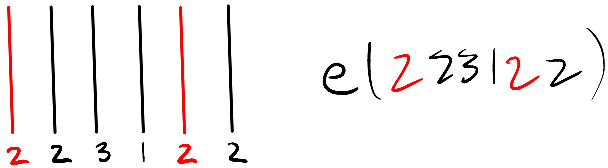
## Conjecture (W.)

*Over any field, there is a tilting generator $T$ on $\mathfrak{M}^{\mathbf{m}}$ such that* $\mathrm{End}(T)^{\mathrm{op}} = \mathring{R}$.

$$D^b(\mathsf{Coh}(X)) \cong D^b(\mathring{R}\text{-mod}).$$

*In particular, the ring $\mathring{R}$ is a non-commutative crepant resolution of singularities of $\mathfrak{M}$ which is D-equivalent to $\mathfrak{M}^{\mathbf{m}}$.*

To define $T$, need idempotents where all strands are vertical.




There's one of these for each possible order on strands. Can encode this in a word $\mathbf{i}$. Denote by $e(\mathbf{i})$.

This word is really cyclic, but can always start with red at $x = 0$.

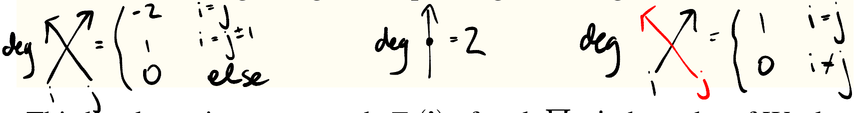
We can think of $\tilde{\mathring{R}}e_C$ as a right module over $\tilde{A}_{\text{Coulomb}} = e_C\tilde{\mathring{R}}e_C$. This has a left $\mathring{R}$-module structure by left multiplication.

## Theorem

*$T$ is the coherent sheaf obtained on $\mathfrak{M}^{\mathbf{m}}$ by GIT quotient. That is, for $k \gg 0$, we have $\Gamma(\mathfrak{M}^{\mathbf{m}}, T \otimes \mathcal{O}(k)) \cong \tilde{\mathring{R}}^{k\mathbf{m}} e_C$*

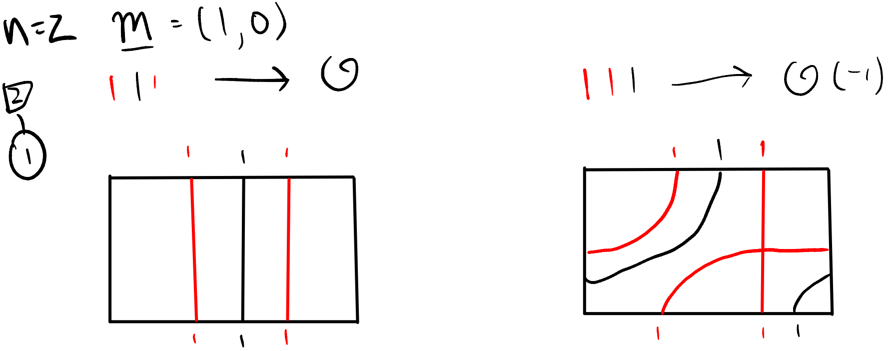
In physical terms, these come from vortex line operators, that is, natural D-modules on loop space.

There is a natural grading, corresponding to scaling $\mathbb{C}^\times$ action.




This breaks up into summands $Te(\mathbf{i})$ of rank $\prod v_i!$, the order of Weyl group of $G$.

What tilting generator do we get on $T^*\mathbb{CP}^{n-1}$? The only possibility is $\mathscr{O}(a) \oplus \cdots \oplus \mathscr{O}(a-n+1)$ for some $a$.




Can get $a = 0, 1, \ldots, n-1$ depending on conventions.

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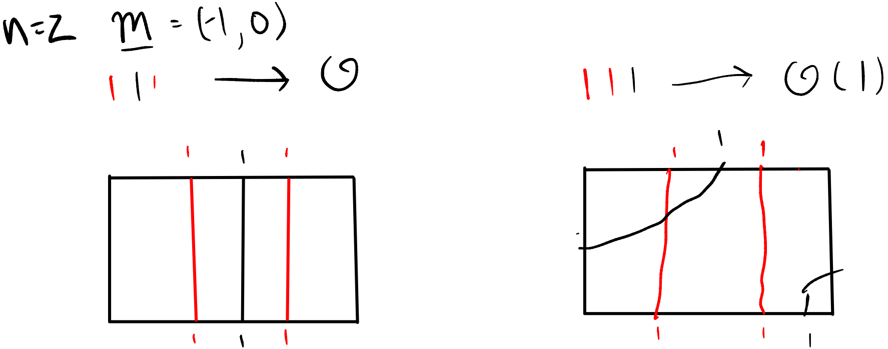
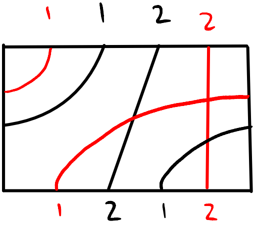


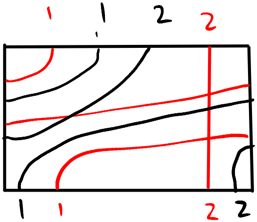

Can get $a = 0, 1, \ldots, n-1$ depending on conventions.

What tilting generator do we get on $T^*\mathbb{CP}^{n-1}$? The only possibility is $\mathscr{O}(a) \oplus \cdots \oplus \mathscr{O}(a-n+1)$ for some $a$.

n=3 1122 → $\mathscr{O}$ 1221 → $\mathscr{O}(-2)$

1212 1212 → $\mathscr{O}(-1)$
1221 1122



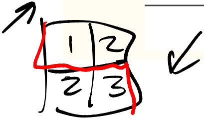

Can get $a = 0, 1, \ldots, n-1$ depending on conventions.

What about $T^*G(2,4)$? First non-projective space, and summands of $T$ have ranks 1 and 2.

## Theorem (Suter-W.)

*Let $\mathcal{T}$ be the tautological bundle on $T^*G(2,4)$ and $\mathcal{O}(1) = \bigwedge^2 \mathcal{T}^*$. Every summand $Te(\mathbf{i})$ is isomorphic to one of:*

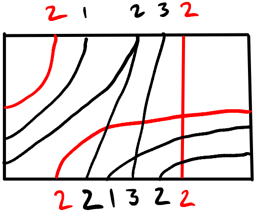
| KLRW idempotent | Grassmanian sheaf |
|---|---|
| 222231 | $\mathcal{O} \oplus \mathcal{O}(-1)$ |
| 223122 | $\mathcal{O}(-1) \oplus \mathcal{O}$ |
| 223221 | $\mathcal{T}$ |
| 221223 | $\mathcal{T}^\perp \subset \mathcal{O}^{\oplus 4}$ |
| 223122 | $0 \to \mathcal{O}(-2) \to \mathcal{V} \to \mathcal{O} \to 0$ |
| 222312 | $0 \to \mathcal{O}(-1) \to \mathcal{W} \to \mathcal{O}(1) \to 0$ |



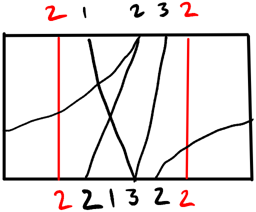
How do we check something like this?

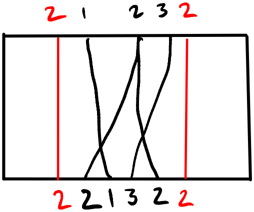
- Elements of $\bigwedge^2\mathbb{C}^4$ give divisors on $T^*G(2,4)$. Vector bundle is trivial when complement is $\mathbb{A}^n$.
- Only need to check that vector bundle is right on open subset of codim $\geq 2$, so enough to find transition function between patches.
- Calculate!
- Sneaky trick: Bezrukavnikov says $T$ is $GL_4$-equivariant, so $T$ must be induced by a representation of $P = \left[\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right] \subset GL_4$.

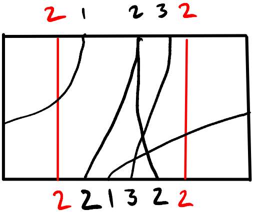
## Grassmannians

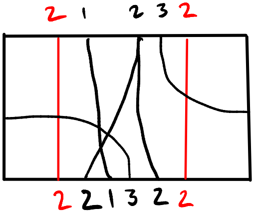


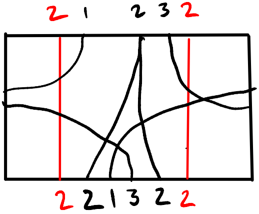

$$\mathcal{O}(-1) \rightarrow Te_{221322} \rightarrow \mathcal{O}(1)$$

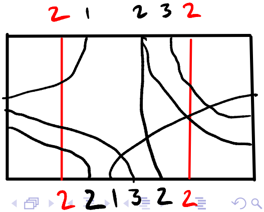


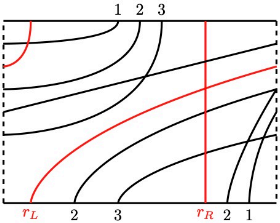



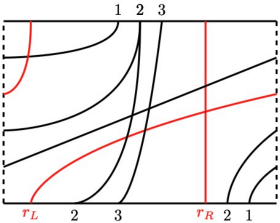
## Grassmannians

$(x_1 \wedge x_3)^{-1}$

$\mathcal{Z}^1_{13} =$

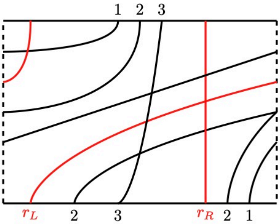



$\mathcal{Z}^2_{13} =$

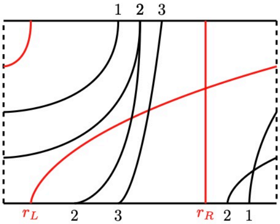



$(x_2 \wedge x_4)^{-1}$

$\mathcal{Z}^1_{24} =$




$\mathcal{Z}^2_{24} =$




$$\mathcal{Z}^1_{13} = \mathcal{D}^{-1}_{24}(\mathcal{D}_{24}\mathcal{Z}^1_{24} + \mathcal{D}_{12}\mathcal{Z}^2_{24}), \qquad \mathcal{Z}^2_{13} = \mathcal{D}^{-1}_{24}(\mathcal{D}_{14}\mathcal{Z}^2_{24} - \mathcal{D}_{34}\mathcal{Z}^1_{24}).$$

# Thanks!