DT Invariants and Holomorphic Curves

Pierrick Bousseau

University of Georgia

Berkeley Informal String-Math Seminar
3 April 2023

Introduction

Relation between two topics:

- Donaldson-Thomas (DT) invariants of non-compact Calabi-Yau 3-folds: counts of stable coherent sheaves (or complexes of coherent sheaves) on X or special Lagrangian submanifolds of its mirror Y.
- Holomorphic curves in a hyperkähler manifold $\mathcal{M}$.

Basic relation between X and $\mathcal{M}$ through physics:

- IIA-IIB string theory on $X \times \mathbb{R}^4$: $\mathcal{N} = 2$ 4d field theory T
- $\mathcal{M}$: Coulomb branch of T on $S^1 \times \mathbb{R}^3$, Seiberg–Witten integrable system.

Plan

- General expected picture [Kontsevich-Soibelman 1303.3253]
- A concrete example [B 1909.02985-1909.02992, B-Descombes-Le Floch-Pioline 2210.10712]:
  - DT invariants for coherent sheaves on local $\mathbb{P}^2$: $X = K_{\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(-3)$, non-compact Calabi-Yau 3-fold.
  - holomorphic curves in $\mathcal{M}$, $(\mathcal{M}, I)$: elliptic fibration, $(\mathcal{M}, J) = \mathbb{P}^2 \setminus E$, $ALH^\star$ metric [Collins-Jacob-Lin 1904.08363].
- An heuristic/physics derivation of the general correspondence [B 2210.17001]
  - Holomorphic Floer theory for $\mathcal{M}$.

Geometry: DT invariants

- DT invariants:

$$\Omega_\gamma(u) \in \mathbb{Z}$$

counts of geometric objects on a Calabi-Yau 3-fold X, with given topology class $\gamma \in \mathbb{Z}^n$ and satisfying a (Bridgeland) stability condition u.

- Examples:
  - Stable holomorphic vector bundles of Chern character γ for a Kähler parameter u.
  - Special Lagrangian submanifolds of class γ for a complex parameter u.

- $\mathcal{N} = 2$ supersymmetric 4d field theories
  - B: Coulomb branch of vacua of the 4d theory, $B \simeq \mathbb{C}^r$.
  - In a generic vacuum $u \in B \setminus \Delta$, abelian gauge theory $U(1)^r$
  - Supersymmetry: charge γ, central charge $Z_\gamma(u) \in \mathbb{C}$, BPS bound

$$|M| \geq |Z_\gamma(u)|$$

  - Space of BPS states, saturating the BPS bound: $H_\gamma(u)$
  - BPS index

$$\Omega_\gamma(u) = \mathrm{Tr}_{H_\gamma(u)}(-1)^F$$

From geometry to physics

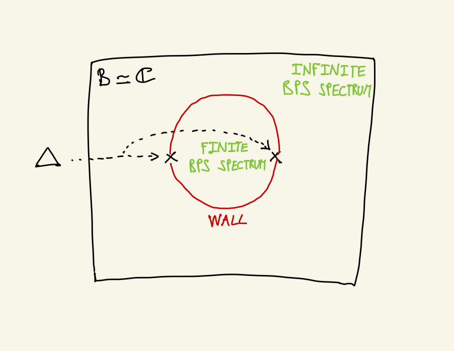
- Geometric constructions from string theory: IIA or IIB string on Calabi-Yau 3-fold X
- Expectation: the universal cover of $B \setminus \Delta$ naturally maps to the space of Bridgeland stability conditions.
- DT invariants = BPS indices: stability $u \in B \setminus \Delta$
- From now on: consider $\mathcal{N} = 2$ 4d field theories without gravity.
  - Geometrically: non-compact Calabi-Yau 3-folds.

Wall-crossing

- $\Omega_\gamma(u)$: constant function of u away from codimension one loci in B, called walls, across which $\Omega_\gamma(u)$ jumps discontinuously.
- Jumps controlled by a universal wall-crossing formula [Kontsevich-Soibelman]:

$$\{\Omega_\gamma(u^-)\}_\gamma \to \{\Omega_\gamma(u^+)\}_\gamma \,.$$

- Example: $\mathcal{N} = 2$ $SU(2)$ gauge theory



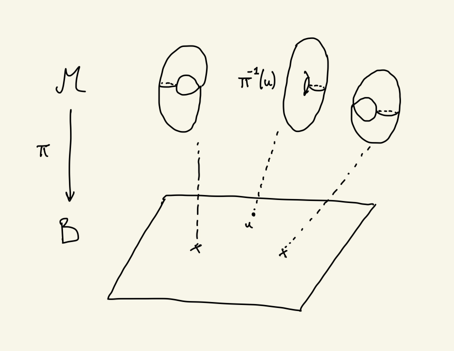
Seiberg-Witten integrable system

- $\mathcal{M}$: Coulomb branch of the theory on $\mathbb{R}^3 \times S^1$, hyperkähler manifold of complex dimension $2r$, complex integrable system:

$$\pi \colon \mathcal{M} \longrightarrow B$$

- Low energy: 3d $\mathcal{N} = 4$ sigma model with target $\mathcal{M}$
- Twistor sphere of complex structures I, J, K
    - π I-holomorphic: in complex structure I, generic fibers of π are abelian varieties of dimension r.
    - for every $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, generic fibers of π are special Lagrangians in complex structure $J_\theta = (\cos\theta)J + (\sin\theta)K$.
- $u \in B \setminus \Delta$, $\gamma \in \pi_2(\mathcal{M}, \pi^{-1}(u)) \to H_1(\pi^{-1}(u), \mathbb{Z}) = \mathbb{Z}^{2r}$,

$$Z_\gamma(u) = \int_\gamma \Omega_I$$



- Class S on C: $\pi : \mathcal{M} \to B$ is (essentially) the Hitchin integrable system for C.

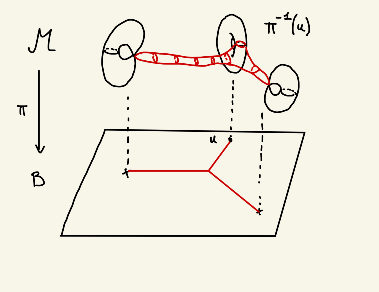
Expectation

- For every point $u \in B \setminus \Delta$, and class $\gamma \in \pi_2(\mathcal{M}, \pi^{-1}(u))$,

$$\Omega_\gamma(u) = N_\gamma(u) \, .$$

  - $\Omega_\gamma(u)$: DT/BPS invariants counting u-stable objects of class γ.
  - $N_\gamma(u)$: count of J_θ-holomorphic disks in $\mathcal{M}$ with boundary on the fiber $\pi^{-1}(u)$ and of class γ, where $\theta = \mathrm{Arg} Z_\gamma(u)$.
- Evidence:
  - BPS spectrum $\{\Omega_\gamma(u)\} \to$ hyperkähler geometry of $\mathcal{M}$ [Gaiotto-Moore-Neitzke]
  - J_θ-holomorphic disks: instantons/quantum corrections to construct the mirror of $(\mathcal{M}, \omega_\theta)$ [Fukaya, Kontsevich-Soibelman,...]
  - Same wall-crossing formula [Kontsevich-Soibelman]
  - Tropical curves in B from holomorphic disks and attractor trees from DT invariants [Kontsevich-Soibelman]

Expectation



Problems:

- The embedding of B in the space of Bridgeland stability conditions is not known in general.
- Defining counts of holomorphic disks is difficult in general (see Y-S. Lin for surfaces)

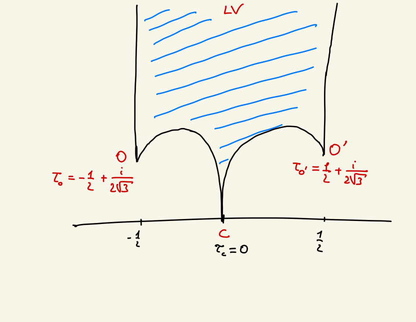
Examples:

- Log Gromov–Witten invariants: algebro-geometric version of holomorphic disks used by Gross-Siebert in their mirror symmetry construction.
- DT invariants of quivers with potential versus log Gromov–Witten invariants of toric and cluster varieties [Argüs-B, arXiv:2302.02068].
- This talk:
  - DT invariants counting coherent sheaves on local $\mathbb{P}^2$
  - One of the few examples where the embedding in the space of Bridgeland stability conditions is known.

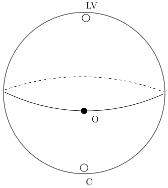
Local $\mathbb{P}^2$

- $X = K_{\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(-3)$ non-compact Calabi-Yau 3-fold
  - Zero section $\iota \colon \mathbb{P}^2 \hookrightarrow X$
- $D_{\mathbb{P}^2}(X)$: bounded derived category of sheaves on X set-theoretically supported on $\mathbb{P}^2$
  - $\iota_* : D^b Coh(\mathbb{P}^2) \to D_{\mathbb{P}^2}(X)$
  - $\mathcal{O}(n) := \iota_* \mathcal{O}_{\mathbb{P}^2}(n)$ (D4-branes with n units of D2-charges)
- IIA string theory on X: $\mathcal{N} = 2$ 4d theory.
  - Seiberg-Witten geometry $\pi : \mathcal{M} \to B$?
  - Mirror symmetry: $B \setminus \Delta = \mathbb{H}/\Gamma_1(3)$, modular curve. $\mathcal{M}$: universal family of elliptic curves.

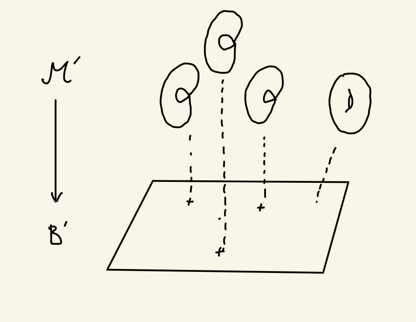
A fundamental domain F_C of $\Gamma_1(3)$ acting on $\mathbb{H}$:



The modular curve $B \setminus \Delta = \mathbb{H}/\Gamma_1(3)$:



- Work on the 3:1 cover B' of B resolving the orbifold point.
- $\mathcal{M}' \to B'$: elliptic fibration with 3 singular fibers.



Map to the space of stability conditions

- $Stab(D_{\mathbb{P}^2}(X))$: space of Bridgeland stability conditions on $D_{\mathbb{P}^2}(X)$, complex manifold of dimension 3
- Bayer-Macri (2009):

$$\widetilde{B \setminus \Delta} = \mathbb{H} \to Stab(D_{\mathbb{P}^2}(X))$$

$$\tau \mapsto (\mathcal{A}(\tau), Z(\tau))$$

- Central charge, additive map:

$$Z(\tau) : \Gamma = K_0(D_{\mathbb{P}^2}(X)) = \mathbb{Z}^3 \to \mathbb{C}$$

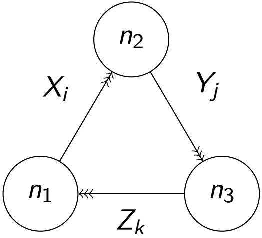
$$\gamma \mapsto Z_\gamma(\tau)$$

At the orbifold point

- At the orbifold point O.

$$\mathcal{A}(\tau_O) = Coh_0(\mathbb{C}^3/(\mathbb{Z}/3\mathbb{Z})) = Rep^{nilp}(Q, W)$$

induced by the exceptional collection $\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)$ on $\mathbb{P}^2$.



Potential $W = \sum_{i,j,k} \epsilon_{ijk} Z_k Y_j X_i$ with ϵ_{ijk} the totally antisymmetric tensor with $\epsilon_{123} = 1$.

DT/BPS invariants

- To summarize:

$$\widetilde{B \setminus \Delta} = \mathbb{H} \to Stab(D_{\mathbb{P}^2}(X))$$

$$\tau \mapsto (\mathcal{A}(\tau), Z(\tau))$$

- We can then do DT theory.
  - Moduli spaces

$$M(\gamma, \tau) = \{\tau\text{-semistable objects in } \mathcal{A}(\tau) \text{ of class } \gamma\}$$

  - DT/BPS invariants:

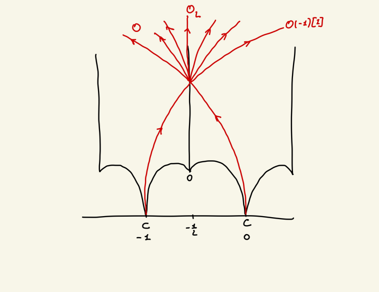
$$\Omega(\gamma, \tau) \in \mathbb{Z}$$

  - Wall-crossing as a function of $\tau \in \mathbb{H}$.
- Goal: study of the DT/BPS invariants using flow trees organized in "scattering diagrams" in $\widetilde{B \setminus \Delta} = \mathbb{H}$
  - supergravity attractor picture
  - Kontsevich-Soibelman wall-structure on base of complex integrable systems.

- Pick a phase $\theta \in \mathbb{R}/2\pi\mathbb{Z}$
  - For every $\gamma \in \Gamma$, consider the 1-dimensional locus, "rays":

$$\mathcal{R}_\gamma^+(\theta) := \{\tau \in \mathbb{H} \,|\, \mathrm{Arg}(Z_\gamma(\tau)) = \theta \,, \Omega(\gamma, \tau) \neq 0\} \subset \mathbb{H}$$

  - Orient rays such that $|Z_\gamma(\tau)|$ increases.
  - Decorate the rays by generating functions of DT invariants, get a scattering diagram $\mathfrak{D}_\theta$



Main result

Theorem (B., Descombes, Le Floch, Pioline, 2022)

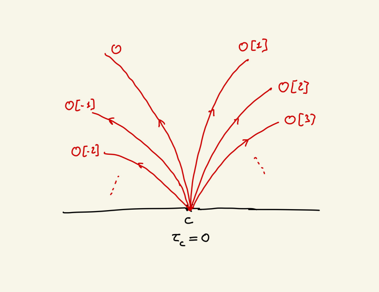
For every $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, the scattering diagram $\mathcal{D}_\theta$ can be uniquely reconstructed from:

- *Explicit initial rays coming from the conifold points.*
- *Scatterings imposed by the consistency condition.*

- Algorithmic reconstruction of the full BPS spectrum (except pure D0) at any point of the physical space of stability conditions.

Initial rays

At the conifold point $\tau_O = 0$, $Z_{\mathcal{O}}(\tau_O) = 0$. Infinitly many initial rays corresponding to the objects $\mathcal{O}[k]$, $k \in \mathbb{Z}$.



General conifold point: apply $\Gamma_1(3)$, spherical object E becoming massless, infinitly many initial rays corresponding to the objects $E[k]$, $k \in \mathbb{Z}$.

Reconstruction from initial rays

- Rays of $\mathfrak{D}_\theta$ are gradient flow lines of $\mathrm{Re}(e^{-i\theta} Z_\gamma(\tau))$.
- Key point: for every $\gamma \in \Gamma$, the holomorphic function

$$\mathbb{H} \to \mathbb{C}$$

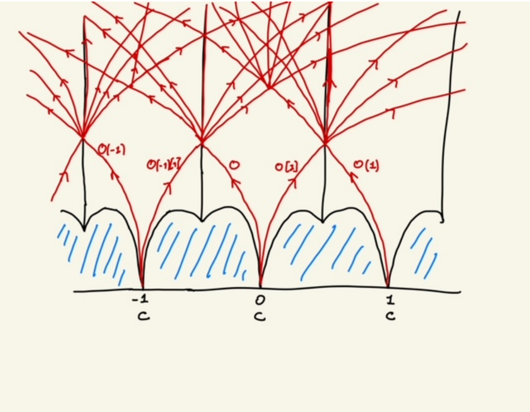
$$\tau \mapsto Z_\gamma(\tau)$$

has no critical point on $\mathbb{H}$:

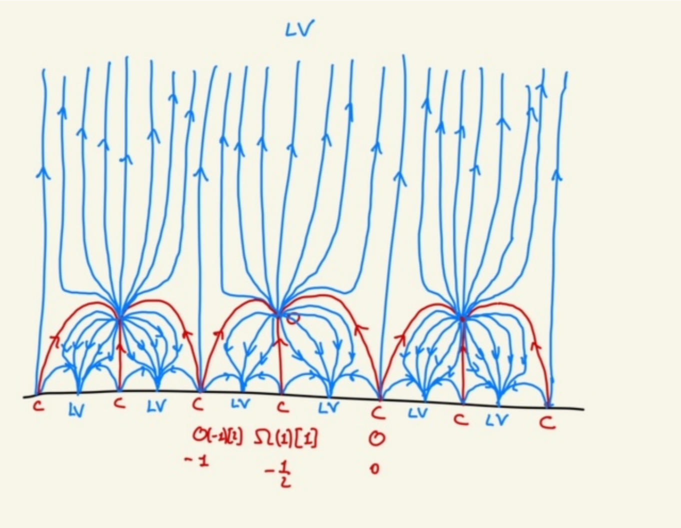
$$\frac{d}{d\tau} Z_\gamma(\tau) = (-r\tau + d) C(\tau) \neq 0$$

- Study of the boundary behavior: $C(\tau) \to 0$ when τ goes to a conifold point, not otherwise.

The scattering diagram $\mathfrak{D}_{\frac{\pi}{2}}$



The scattering diagram $\mathfrak{D}_0$



The scattering diagram $\mathfrak{D}_0$

- The global picture of $\mathfrak{D}_0$ also give a clear description of the correspondence between normalized $(-1 < \mu \leq 0)$ torsion free Gieseker semi-stable sheaves on $\mathbb{P}^2$ and representations of the Beilinson quiver.
- For these objects $\mathfrak{D}_0$ gives a path from the large volume point to the orbifold point avoiding the walls of marginal stability.

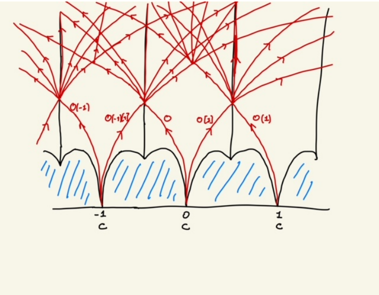
Holomorphic disks?

- Expectation: for every $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, the scattering diagram $\mathfrak{D}_\theta$ should describe J_θ-holomorphic disks in $\mathcal{M}'$.
- Problem: how to describe $\mathcal{M}'$ as a complex manifold for the complex structure J_θ?
  - We only know that $(\mathcal{M}', I)$ is an elliptic fibration over B.
- [Collins-Jacob-Lin]:
  - $(\mathcal{M}', J_{\frac{\pi}{2}}) = \mathbb{P}^2 \setminus E$, where $E \subset \mathbb{P}^2$ is a smooth cubic. Affine algebraic variety.
  - $(\mathcal{M}', J_0) \simeq (\mathcal{M}', I)$, elliptic fibration. Twin torus fibrations.
- In both cases, use algebro-geometric definition of counts of holomorphic disks as log Gromov–Witten invariants.

Holomorphic disks?

Theorem (Gräfnitz, B.)

The scattering diagram $\mathfrak{D}_{\frac{\pi}{2}}$ describes log curves in $(\mathcal{M}', J_{\frac{\pi}{2}}) = \mathbb{P}^2 \setminus E$.



Corollary (B.)

Correspondence between DT invariants of $K_{\mathbb{P}^2}$ of phase $\frac{\pi}{2}$ and counts of log curves in $\mathbb{P}^2 \setminus E$

Holomorphic disks?

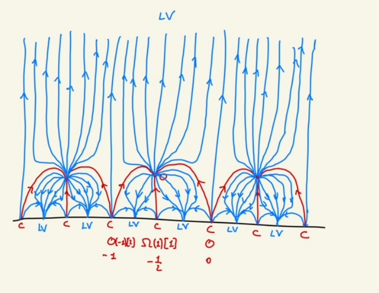
Applications:

- Proof of Takahashi's conjecture on Gromov–Witten invariants of $(\mathbb{P}^2, E)$ [B.].
- Proof of quasimodularity of generating series of DT invariants [B-Fan-Guo-Wu].

The scattering diagram $\mathfrak{D}_0$

Theorem (Gross-Hacking-Keel)

The scattering diagram $\mathfrak{D}_0$ describes log curves in $(\mathcal{M}', J_0)$.



Corollary (B.)

Correspondence between DT invariants of $K_{\mathbb{P}^2}$ of phase 0, DT invariants of the quiver (Q, W), and counts of log curves in the elliptic fibration $(\mathcal{M}', J_0)$.

General question

Why are the counts of BPS states of a $\mathcal{N} = 2$ 4d theory given by counts of holomorphic curves in the Seiberg–Witten geometry $\pi : \mathcal{M} \to B$?

- Mirror symmetry and hyperkähler rotation for $X = K_{\mathbb{P}^2}$.
- In general?
  - Stronger conjecture formulated using holomorphic Floer theory.
  - Physics derivation.

Mirror symmetry and hyperkähler rotation

- How to go from coherent sheaves on $X = K_{\mathbb{P}^2}$ to J_θ-holomorphic curves in the Coulomb branch $\pi\colon \mathcal{M} \to B$?
- Claim: the mirror Y of X is the non-compact Calabi-Yau 3-fold $Y : uv = \pi - t$.
  - Hyperkähler rotation: J_θ-holomorphic disks in $\mathcal{M} \to$ special Lagrangian disks in $(\mathcal{M}, I)$ of phase θ.
  - Suspension $\to$ closed special Lagrangians in Y.
  - Mirror symmetry $\to$ stable coherent sheaves on X.
- Physics: IIA on $X \leftrightarrow$ IIB on $Y \leftrightarrow$ IIA on $\mathcal{M}$ and NS5 on $\pi^{-1}(u) \leftrightarrow$ M on $\mathcal{M}$ and M5 on $\pi^{-1}(u) \leftrightarrow$ IIB on B, D3 on u (string junctions on D3-brane probe)

Holomorphic Floer theory

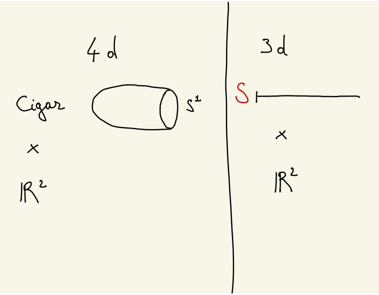
[Kontsevich-Soibelman] [Doan-Rezchikov], [B.]

- $(\mathcal{M}, I, \Omega_I)$: holomorphic symplectic manifold.
  - Hyperkähler structure I, J, K, $J_\theta := (\cos\theta)J + (\sin\theta)K$.
  - $L_1, L_2 \subset \mathcal{M}$: I-holomorphic Lagrangian, $\Omega_I|_{L_1} = \Omega_I|_{L_2} = 0$.
- P: space of paths between L_1 and L_2, $W := \int_{\mathfrak{p}} d^{-1}\Omega_I$ (multivalued!)
  - Critical points: intersection points $L_1 \cap L_2$.
  - Gradient flow lines: J_θ holomorphic curves, $u : \mathbb{R}^2 \to \mathcal{M}$.
  - ζ-instantons, $u : \mathbb{R}^3 \to \mathcal{M}$, solutions to Fueter equation

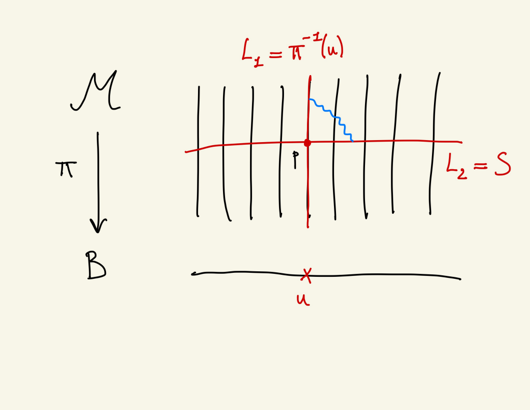
$$\partial_\tau u + I\partial_s u + J_\theta \partial_t u = 0 \,.$$

- LG model for (P, W):
  - $p, q \in L_1 \cap L_2 \to$ vector space H_{pq} of 2d BPS states of (P, W)
  - $L_1, L_2 \to$ category $\mathrm{Brane}(P, W)$
  - $\mathcal{M} \to$ 2-category of I-holomorphic Lagrangians (A-model versus Rozansky-Witten B-model).

- Back to a $\mathcal{N} = 2$ 4d field theory.
- How to recover the BPS spectrum $\{\Omega_\gamma(u)\}$ from holomorphic Floer theory? Correct holomorphic symplectic manifolds $\mathcal{M}$ and holomorphic Lagrangians L_1, L_2 ?
  - $\mathcal{M}$: Seiberg-Witten integrable system
  - $L_1 = \pi^{-1}(u)$: fiber of $\pi : \mathcal{M} \to B$ over $u \in B$.
  - $L_2 = S$: natural section of π. Physical definition: boundary condition for the 3d sigma model of target $\mathcal{M}$ defined by the cigar geometry [Nekrasov-Witten]. Hitchin system example: Hitchin section.

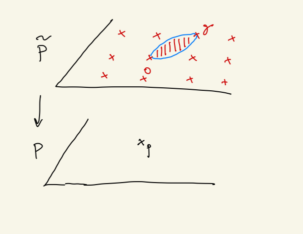


Holomorphic Floer theory and DT invariants



- $L_1 \cap L_2 = \pi^{-1}(u) \cap S = \{p\}$
- But $\pi_1(P) \neq 0$ and W is multivalued.
- $\pi_1(P) = \pi_2(\mathcal{M}, \pi^{-1}(u))$: on $\widetilde{P}$, critical points of W indexed by

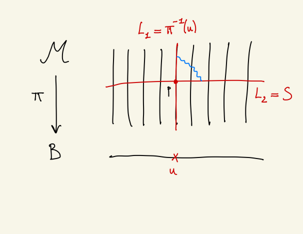
$$\gamma \in \pi_2(\mathcal{M}, \pi^{-1}(u))$$



Conjecture (B)

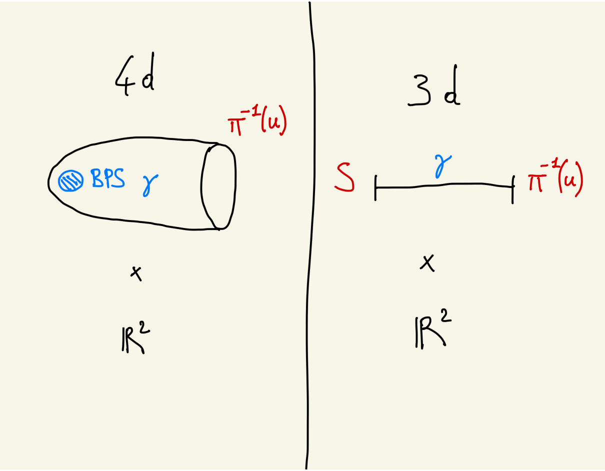
Given a $\mathcal{N} = 2$ 4d field theory, the space of BPS states $H_\gamma(u)$ of class γ in the vacuum u is isomorphic to the vector space $H_{0\gamma}$ associated by holomorphic Floer theory for the Seiberg-Witten integrable system $\mathcal{M}$ to the lifts 0 and γ of the intersection point between the fiber $\pi^{-1}(u)$ and the section S:

$$H_\gamma(u) \simeq H_{0\gamma}$$



Gradient flow lines are naturally J_θ-holomorphic disks with boundary on $\pi^{-1}(u)$ and so one recovers the previous expectation in the numerical limit.

Physics derivation



Thank you for your attention !