Mysterious Triality

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Talk in Berkeley String-Math Seminar

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Setting up the problem

Once upon a time at Harvard...

in 2001, A. Iqbal, A. Neitzke, and C. Vafa discovered a "mysterious duality"

Math: Algebraic geometry of del Pezzo surfaces $\mathbb{B}_k$

$\downarrow$

Exceptional series E_k of root systems

$\uparrow$

Physics: M-theory, 11d supergravity

toroidal compactifications on $(S^1)^k$

(dimensional reduction)

Classical Algebraic Geometry: Cf. 27 lines on a cubic surface (A. Cayley and G. Salmon, 1849)

The del Pezzo surfaces

A *del Pezzo* (*dP*) *surface* is a complex compact smooth surface whose anticanonical class $-K$ is ample (sufficiently positive). Del Pezzo surfaces are classified topologically by belonging to one of the following types:

$$\mathbb{CP}^2 = \mathbb{B}_0, \mathbb{B}_1, \mathbb{B}_2, \ldots, \mathbb{B}_8$$

and

$$\mathbb{CP}^1 \times \mathbb{CP}^1.$$

Here

$$\mathbb{B}_k \quad = \quad \text{blowup of } \mathbb{P}^2 \text{ at } k \text{ generic points} \quad = \quad \mathbf{CP}^2 \# k\overline{\mathbf{CP}^2}$$

Blowup; picture credit: R. Hartshorne

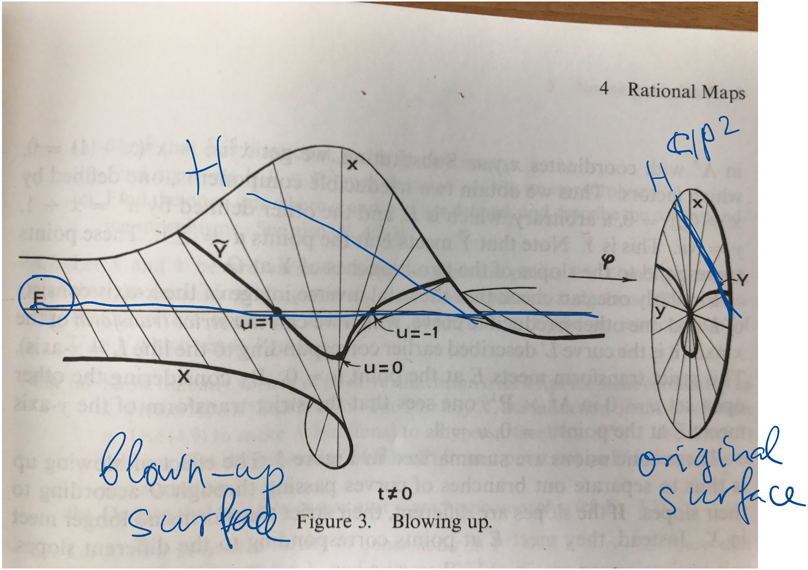
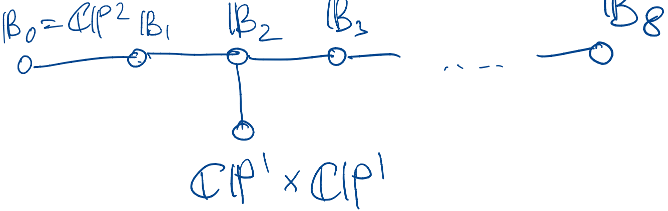


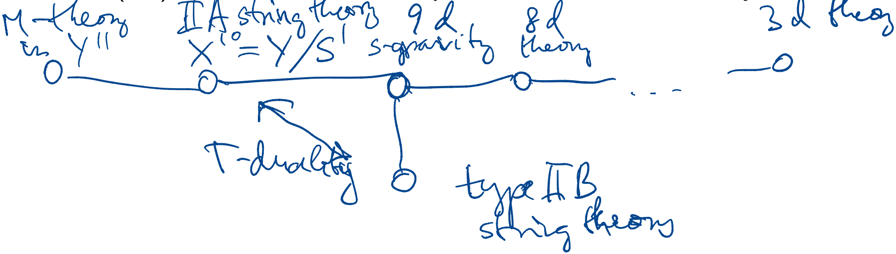
Figure 3. Blowing up.

Sequence of blowups and E_{10} Dynkin diagram

Del Pezzos:



Physics: Dimensional reduction of supergravity on a k-torus $T^k = (S^1)^k$ a.k.a. toroidal compactifications of M-theory

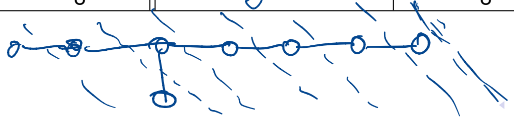


More on the E_k series

The E_k root system arises in the orthogonal complement to $-K$ (or $c_1(-K)$) in the cohomology group $H^2(\mathbb{B}_k;\mathbb{Z})$ with the intersection form $H^2(\mathbb{B}_k;\mathbb{Z}) \otimes H^2(\mathbb{B}_k;\mathbb{Z}) \to \mathbb{Z}$.

k	del Pezzo	Dynkin Diagram	Type of E_k	Lie Algebra
0	$\mathbb{P}^2$		A_{-1}	$\mathfrak{sl}_0 = \varnothing$
1	$\mathbb{B}_1$		A_0	$\mathfrak{sl}_1 = 0$
1	$\mathbb{P}^1 \times \mathbb{P}^1$		A_1	$\mathfrak{sl}_2$
2	$\mathbb{B}_2$		A_1	$\mathfrak{sl}_2$
3	$\mathbb{B}_3$		$A_2 \times A_1$	$\mathfrak{sl}_3 \oplus \mathfrak{sl}_2$
4	$\mathbb{B}_4$		A_4	$\mathfrak{sl}_5$
5	$\mathbb{B}_5$		D_5	$\mathfrak{so}_{10}$
6	$\mathbb{B}_6$	E_6	E_6	$\mathfrak{e}_6$
7	$\mathbb{B}_7$		E_7	$\mathfrak{e}_7$
8	$\mathbb{B}_8$		E_8	$\mathfrak{e}_8$

E_8



Branes in toroidal compactifications of M-theory

Same story but the E_k root system shows up in the yoga of branes, such as this table for type IIA string theory = M-theory$/S^1$:

$-K = 3H - E$

$(-K)\cdot(H-E) = 3 - 1 = 2 > 0$

$\mathbb{B}_1$ del Pezzo

homology class	brane tension	type IIA meaning
E	$R^{-1} = l_s^{-1} g_s^{-1}$	D0-brane
$H - E$	$(2\pi)^2 R l_p^{-3} = (2\pi)^{-1} l_s^{-2}$	F-string
H	$(2\pi)\, l_p^{-3} = (2\pi)^{-2}\, l_s^{-3} g_s^{-1}$	D2-brane
$2H - E$	$(2\pi)^2\, R l_p^{-6} = (2\pi)^{-4}\, l_s^{-5} g_s^{-1}$	D4-brane
$2H$	$(2\pi)\, l_p^{-6} = (2\pi)^{-5}\, l_s^{-6} g_s^{-2}$	NS5-brane
$3H - 2E$	$(2\pi)^3\, R^2 l_p^{-9} = (2\pi)^{-6}\, l_s^{-7} g_s^{-1}$	D6-brane
$4H - 3E$	$(2\pi)^4\, R^3 l_p^{-12} = (2\pi)^{-8}\, l_s^{-9} g_s^{-1}$	D8-brane
$5H - 2E$	$R^2 \ell_p^{-15} (2\pi)^{\cdots}$	?

Table credit: Iqbal, Neitzke, and Vafa (2001)

Mystery: Physics and AG give rise to the E_k series, but no explicit connection between physics and del Pezzo surfaces.

Algebraic geometry $\longleftrightarrow$ Algebraic Topology

Iqbal, Neitzke, Vafa 2001

Physics

Sati – V 2021

Main results:

1. **Math Physics**: The RHT of iterated cyclic loop spaces $\mathcal{L}_c^k S^4$ is explicitly related to the M-theory story. (E.g., want equations of motion of M-theory wrapped on T^5, *i.e.*, 6d supergravity? Read them off the differential in the Sullivan minimal model of $\mathcal{L}_c^5 S^4$!);

   iterated cyclic loop spaces (cyclifications)

2. **Mathematics**: S^4, $\mathcal{L}_c S^4$, $\mathcal{L}_c^2 S^4, \dots$ is a new series of objects with hidden internal E_k symmetry. (Like, 27 "lines" in $\mathcal{L}_c^6 S^4 \dots$)

Cyclic loop spaces $\mathcal{L}_c^k S^4$

The *free loop space* of a topological space Z:

$$\mathcal{L}Z = \mathrm{Map}(S^1, Z).$$

It admits a natural action of the group S^1 by rotating loops, and we define the *cyclic loop space* $\mathcal{L}_c Z$ to be the *homotopy quotient*

$$\mathcal{L}_c Z := \mathcal{L}Z /\!\!/ S^1 = \mathcal{L}Z \times_{S^1} ES^1,$$

the *Borel construction*. For $k \geq 0$, the *iterated cyclic loop space* (*cyclification*) $\mathcal{L}_c^k Z$ is the k-fold iteration of the cyclic loop space construction:

$$\begin{aligned} \mathcal{L}_c^0 Z &:= Z, \\ \mathcal{L}_c^k Z &:= \mathcal{L}_c(\mathcal{L}_c^{k-1} Z) \qquad \text{for } k \geq 1. \end{aligned}$$

We will be interested mostly in the iterated cyclic loop spaces $\mathcal{L}_c^k S^4$ of the 4-sphere S^4 for $0 \leq k \leq 8$.

The Sullivan and Quillen models

Rational homotopy theory (RHT): $X \sim Y$ iff $X \to Y$ rational homotopy equivalence of path connected spaces, a conts. map inducing isomorphisms $H_\bullet(X; \mathbb{Q}) \xrightarrow{\sim} H_\bullet(Y; \mathbb{Q})$ on rational homology.

Rational homotopy category: topological spaces with inverses of rational h. equivalences formally added.
Fact (Quillen, Sullivan, '60–70s): the rational homotopy category (of good enough spaces) is equivalent to a category of DGCAs (or DGLAs, resp.) of a certain type:

$$X \mapsto M(X), \quad \text{the } \textit{Sullivan minimal model} \text{ of } X,$$
$$X \mapsto Q(X), \quad \text{the } \textit{Quillen minimal model} \text{ of } X.$$

We will use $\mathbb{R}$ in place of $\mathbb{Q}$ (*rational homotopy theory over the reals*)

Math physics part: M-theory

Sullivan minimal model

$$M(S^4) = (\mathbb{R}[g_4, g_7], d),$$
$$dg_4 = 0, \qquad dg_7 = -\tfrac{1}{2} g_4^2,$$
$$|g_4| = 4, \qquad |g_7| = 7.$$

$\underline{\Omega^\bullet(S^4)}$

$\uparrow$ q-is

$M(S^4)$

$$\varphi : Y^{11} \to S^4_{\mathbb{R}} \quad (\sim_{\mathbb{R}} S^4)$$
$$G_4 := \varphi^*(g_4) \quad \text{and} \quad G_7 := \varphi^*(g_7).$$

$S^4_{\mathbb{R}} \sim S^4$

Equations of motion of 11d supergravity:

$$dG_4 = 0, \qquad dG_7 = -\tfrac{1}{2} G_4 \wedge G_4, \qquad *G_4 = G_7.$$

Duality-Symmetric formulation (metric-free background):

$$dG_4 = 0, \qquad dG_7 = -\tfrac{1}{2} G_4 \wedge G_4.$$

$S^1 \to \mathcal{L}S^4 \to \underbrace{\mathcal{L}_c S^4} \to BS^1 = \mathbb{C}P^\infty$

$$\varphi_1 : X^{10} = Y^{11} /\!\!/ S^1 \to \mathcal{L}_c S^4_{\mathbb{R}} \quad (\sim_{\mathbb{R}} \mathcal{L}_c S^4)$$

$|w| = 2$

$S^1 \to Y^{11} \to Y^{11}/\!\!/S^1 \to BS^1$

$|sg_4| = 3$

$$M(\mathcal{L}_c S^4) = (\mathbb{R}[g_4, g_7, sg_4, sg_7, w], d),$$

$|sg_7| = 6$

$$dg_4 = (sg_4) \cdot w, \qquad dg_7 = -\tfrac{1}{2} g_4^2 + (sg_7) \cdot w,$$
$$d(sg_4) = 0, \quad d(sg_7) = (sg_4) \cdot g_4, \quad dw = 0.$$

$$F_2 := \varphi_1^*(w), \quad H_3 := \varphi_1^*(sg_4), \quad F_4 := \varphi_1^*(g_4), \quad H_7 := \varphi_1^*(g_7).$$

Equations of motion (EOMs) of 10d type-IIA supergravity:

$$dF_4 = H_3 \wedge F_2, \qquad dH_7 = -\tfrac{1}{2} F_4 \wedge F_4 + F_6 \wedge F_2,$$
$$dH_3 = 0, \qquad dF_6 = H_3 \wedge F_4, \qquad dF_2 = 0.$$

Recipe for looping/cyclifying/wrapping

This pattern continues for all $k \geq 0$**:** $\varphi_k : Y^{11} /\!\!/ (S^1)^k \to \mathcal{L}_c^k S^4_{\mathbb{R}}$ **with** $\mathcal{L}_c^k S^4$ **serving as the universal** $(11-k)$**-dim spacetime!!!**

$$\text{If} \quad M(\mathcal{L}_c^k S^4) = (S(V), d), \quad \text{then}$$
$$M(\mathcal{L}_c^{k+1} S^4) = (S(V \oplus V[1] \oplus \mathbb{R}w), d_c) \quad \text{with}$$

$$\begin{aligned} d_c v &:= dv + sv \cdot w, \\ d_c sv &:= -sdv, \\ d_c w &:= 0. \end{aligned}$$

Vigué-Poirrier, Burghelea

For example,

$$M(S^4) = (S(V), d) = \mathbb{R}[g_4, g_7 \mid dg_7 = -\tfrac{1}{2} g_4^2],$$
$$V = \mathbb{R}g_4 \oplus \mathbb{R}g_7,$$

whence

$$M(\mathcal{L}_c S^4) = \mathbb{R}[g_4, g_7, sg_4, sg_7, w \mid dg_4 = (sg_4)w \dots]$$

Math part: E_k from $\mathcal{L}_c^k S^4$

Toroidal symmetries of $M(\mathcal{L}_c^k S^4)$ (and of the rational homotopy type of $\mathcal{L}_c^k S^4$):

Theorem (Sati-V)

For each k, $0 \le k \le 8$, the automorphism group $\operatorname{Aut} M$ of the Sullivan minimal model $M = M(\mathcal{L}_c^k S^4) \otimes_{\mathbb{Q}} \mathbb{R}$ is a real algebraic group which contains a canonically defined maximal $\mathbb{R}$-split torus

$$T \cong (\mathbb{R}^\times)^{k+1} \subseteq \operatorname{Aut} M,$$

where $\mathbb{R}^\times = \mathbb{R} \setminus \{0\} = \mathbb{G}_m(\mathbb{R})$.

Math part: E_k from $\mathcal{L}_c^k S^4$

Theorem (Sati-V)

The abelian Lie algebra $\mathfrak{h}_k = \mathrm{Lie}(T)$ *of* $T \subseteq \mathrm{Aut}\, M(\mathcal{L}_c^k S^4)$ *has a natural basis, giving a lattice* $\mathfrak{h}_k^{\mathbb{Z}} \subseteq \mathfrak{h}_k$*, an integral inner product, and a distinguished element* $K_k \in \mathfrak{h}_k^{\mathbb{Z}}$*.*
The triple $(\mathfrak{h}_k^{\mathbb{Z}}, (-,-), K_k)$ *associated to the cyclic loop spaces* $\mathcal{L}_c^k S^4$ *and their Sullivan minimal models* $M(\mathcal{L}_c^k S^4)$ *consists of*

- *a free abelian group* $\mathfrak{h}_k^{\mathbb{Z}}$ *with a basis* $h_0, h_1, \ldots, h_k$*;*
- *a symmetric bilinear form* $\mathfrak{h}_k^{\mathbb{Z}} \otimes \mathfrak{h}_k^{\mathbb{Z}} \to \mathbb{Z}$ *given by*

$$(h_0, h_0) = 1, \qquad (h_i, h_j) = -\delta_{ij}, \qquad i > 0, j \geq 0;$$

- *an element* $K_k = -3h_0 + h_1 + \cdots + h_k$*, with* $(-K_k$ *being the unique element of* $\mathfrak{h}_k$ *which acts on the Quillen model* $Q(\mathcal{L}_c S^4)$ *by degree).*

This algebraic structure produces the root system E_k *and the Weyl group* $W(E_k)$*, now in the context of cyclifications* $\mathcal{L}_c^k S^4$*.*

$K_k \in \mathfrak{h}_k^{\mathbb{Z}}$ $\qquad (-,-)$ on $\mathfrak{h}_k^{\mathbb{Z}}$

$$K_k^\perp = \{x \in \mathfrak{h}_k \mid (x, K_k) = 0\}$$

$\dim \mathfrak{h}_k = k+1$

$\dim K_k^\perp = k$ $\qquad (-,-)\big|_{K_k^\perp}$

is pos. def.

$\Longleftrightarrow \quad k \leq 8$

$$\boxed{(\text{roots } E_k) \subseteq K_k^\perp}$$

a finite set

$$M(\mathcal{L}_c^3 S^4) = \bigoplus_{\alpha \in (\mathfrak{h}_k^{\mathbb{Z}})^*} M(\mathcal{L}_c^3 S^4)_\alpha$$

weight space of ~~the~~ ~~torus~~ $T \subseteq \mathrm{Aut}(\mathcal{L}_c^3 S^4)$

$$\alpha_i = \varepsilon_i - \varepsilon_j$$

$$\alpha_0 = \varepsilon_0 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3$$

s_{α_i} reflection in α_i

$$s_{\alpha_i} : \mathfrak{h}_k \to \mathfrak{h}_k$$

Conjecture: duality between del Pezzo surfaces and loop spaces of S^4

$$\text{Algebraic geometry} \overset{?}{\longleftrightarrow} \text{Algebraic Topology}$$

Conjecture

There must be an explicit relation between the series of del Pezzo surfaces $\mathbb{B}_k$, $0 \leq k \leq 8$, and the series of iterated cyclic loop spaces $\mathcal{L}_c^k S^4$, $0 \leq k \leq 8$. This relation should match the E_k symmetry patterns occurring in both series, as well as relate other geometric data, such as the volumes of curves on del Pezzo surfaces, with some geometric data, such as the radii of S^4 and S^1s, for the iterated cylic loop spaces $\mathcal{L}_c^k S^4$.