

Quantum geometry and physics of Looijenga pairs

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- Conference “BPS states, mirror symmetry, and exact WKB”, Sheffield 28 June–2 July 2021.
 - ▶ Speakers include: P. Bousseau, I. Coman, T. Ekholm (*), K. Iwaki, O. Lisovyi, P. Longhi, J. Manschot, S. Monavari, A. Neitzke, N. Nikolaev, J. Teschner (*), F. Yan, T.Y. Yu
 - ▶ Website: <http://alg-geom.group.shef.ac.uk/bps2021/>
 - ▶ tbc if in-person or online
- Associate/Full professor job (“SL/Chair”)
 - ▶ string-math applicants especially welcome!

Overview

The main character: log-Calabi–Yau surfaces with nef maximal boundary (X, D) (*nef Looijenga pairs*).

- X smooth complex projective surface
- $| -K_X | \ni D = D_1 + \cdots + D_l$, $l > 1$ s.n.c. divisor, D_i irreducible, smooth and nef

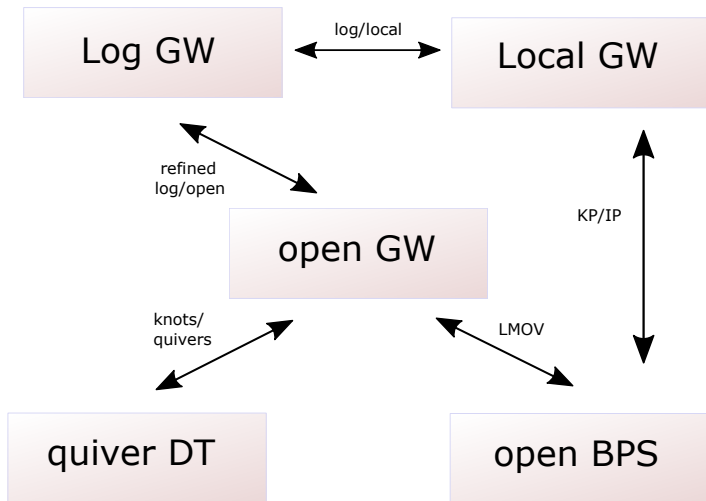
Overview

The two main messages:

- 1 five (different, but equivalent) string-theory motivated enumerative theories built from (X, D)
- 2 they are all closed-form solvable

Joint with P. Bousseau (ETH Zürich/Saclay) and M. van Garrel (Warwick/Birmingham).

Overview



Counting curves

- Fundamental subset of Q's in geometry: enumerative problems.
- Examples:
 - ▶ How many lines on a smooth cubic surface?
 - ▶ How many rational degree- d plane curves through $3d - 1$ points?
 - ▶ How many degree- d curves on a quintic threefold?
- relevance: both intrinsic and for other domains of maths (MathsPhys/Topology/Number Theory/...)

Counting curves

- Typical setup:
 - ▶ X : algebraic variety $_{\mathbb{C}}$ (e.g. $X = \mathbb{P}_{\mathbb{C}}^2$)
 - ▶ $\mathcal{M}(X)$ curves in X (e.g. plane conics)
 - ▶ $\int_{\mathcal{M}(X)}(\dots)$ numbers (“quantum invariants”)
 - ▶ (\dots) = “incidence cond’n” (e.g. conics thru 5 pts \rightarrow 1)
- No sense ($\mathcal{M}(X)$ non-compact)
- Different compactifications $\overline{\mathcal{M}}(X) \rightsquigarrow$ different invariants

Gromov–Witten theory

- Today (mostly):

$$\begin{aligned}\overline{\mathcal{M}}(X) &= \overline{\mathcal{M}}_{g,n}(X, d) \\ &= \overline{\{(C, p_1, \dots, p_n) \xrightarrow{\phi} X \mid h^1(C, \mathcal{O}_C) = g, \phi_*[C] = d\}} / \sim\end{aligned}$$

- Compactification: smooth $C \rightsquigarrow$ nodal + stability
- Proper, $\text{expdim} = (\dim X - 3)(1 - g) - K_X \cdot d + n$
- $[\overline{\mathcal{M}}_{g,n}(X, d)]^{\text{vir}} \in \mathbf{A}_{\text{expdim}}(\overline{\mathcal{M}}_{g,n}(X, d))$

$$\begin{aligned}n_{X,g,d}[B_1, \dots, B_n] &= \text{“}\# \text{ degree-}d, \text{ gen-}g \text{ curves in } X \text{ through } B_i\text{”} \\ &:= \int_{[\overline{\mathcal{M}}_{g,n}(X,d)]^{\text{vir}}} \prod_{i=1}^n \text{ev}_i^*[B_i]\end{aligned}$$

Example: counting rational plane curves

Q. $N_d =$ rational degree- d plane curves through $3d - 1$ points?

d	$3d - 1$	$N_d := n_{\mathbb{P}^2, 0, d}[\overbrace{\text{pt}, \dots, \text{pt}}^{3d-1}]$
1	2	1
2	5	1
3	8	12
4	11	620
5	14	87304
6	17	26312976
\vdots	\vdots	\vdots

- $N_d \sim (3d - 1)! x_0^d$

[Di Francesco–Itzykson, Tian–Wei, Zinger]

- \exists (non-linear) recursion, g.f. solution of Painlevé VI

[Kontsevich, Dubrovin]

Topological A-model

(Worldsheet) physics origin: topological A-twist of $\mathcal{N} = (2, 2)$ σ -model with target X (+ top. gravity)

- BRST cohomology $\leftrightarrow H^\bullet(X, \mathbb{C})$
- classical solutions: $\bar{\partial}\phi = 0$
- vevs of BRST-closed insertions \leftrightarrow integrals over on-shell trajectories

Topological A-model

(Macroscopic) physics origin: F-terms in $\mathcal{N} = 2$ IIA cpt'n $X = \text{CY3}$

- $\chi_I \in H^2(X, \mathbb{C}) \leftrightarrow$ vector multiplets $Y_I, I = 1, \dots, h^{1,1}(X)$
- genus zero Gromov–Witten g.f F_0 : prepotential $\int d^4x \int d^4\theta F_0(Y_I)$
- higher genus G–W g.f. F_g : graviphoton corrections
 $\int d^4x \int d^4\theta F_g(Y_I)(\mathcal{W}^2)^g$

Nef pairs

(X, D) smooth nef pair:

- X smooth complex projective, $\dim X = m$
- $| -K_X | \ni D = D_1 + \cdots + D_l$ s.n.c. divisor with D_i irreducible, smooth and nef $\forall i$

Examples:

- 1 $X = \mathbb{P}_{\mathbb{C}}^1, D = \{0\} + \{\infty\}$
- 2 $X = \mathbb{P}_{\mathbb{C}}^2, D =$
 - i smooth cubic ($l = 1$)
 - ii conic+line ($l = 2$)
 - iii $\sum_{i=1}^3 (\text{axes})_i$ ($l = 3$)

Nef pairs

Jargon: a smooth nef pair is

- *toric* if X is, and $X \setminus D \simeq (\mathbb{C}^*)^m$
- *Looijenga* (log-CY surface with maximal boundary) if $m = 2$, D is singular ($\Rightarrow l > 1$)
- a nef Looijenga pair is **tame** if either $l > 2$, or $D_i^2 > 0 \forall i$

Finite catalogue of 19 deformation families of nef Looijenga pairs; 16 are tame.

Local ($g = 0$) Gromov–Witten theory of (X, D)

- $E_{X,D} := \text{Tot}(\oplus_{i=1}^l \mathcal{O}_X(-D_i))$ (CY($2 + l$)-fold)
- $N_{(X,D),d}^{\text{loc}} = \text{“}\# \text{ degree-}d, g = 0 \text{ curves in } E_{X,D} \text{ through } l - 1\text{-points in } X\text{”}$

▶ $\text{Obs}_d^{(X,D)} \rightarrow \overline{\mathcal{M}}_{0,n}(X, d), \quad \text{Obs}_{[\mathcal{C}, \phi]}^{(X,D)} = H^1(\mathcal{C}, \oplus_{i=1}^l \phi^* \mathcal{O}_X(-D_i))$

▶ $[\overline{\mathcal{M}}_{0,n}(E_{X,D}, d)]^{\text{vir}} = [\overline{\mathcal{M}}_{0,n}(X, d)] \cap c_{\text{top}}(\text{Obs}^{(X,D)})$

▶ $\text{expdim} = l + n - 1$

$$N_{(X,D),d}^{\text{loc}} := \int_{[\overline{\mathcal{M}}_{0,l-1}(E_{X,D}, d)]^{\text{vir}}} \prod_{i=1}^{l-1} \text{ev}_i^*[\text{pt}_X]$$

Log ($g = 0$) Gromov–Witten theory of (X, D)

- $N_{(X,D),d}^{\log}$ = “# degree- d , $g = 0$ curves in X through $l - 1$ points with maximal tangency at $\{D_i\}_i$ ”
- view X as log-scheme with D -log structure
- moduli of log-stable maps $\overline{\mathcal{M}}_{0,n}^{\log}((X, D), d)$
 - ▶ proper, $\text{expdim} = l - 1 + n$
 - ▶ $[\overline{\mathcal{M}}_{0,n}^{\log}((X, D), d)]^{\text{vir}} \in \mathbf{A}_{\text{expdim}}(\overline{\mathcal{M}}_{0,n}^{\log}((X, D), d), \mathbb{Q})$

[Chen, Abramovich–Chen, Gross–Siebert]

$$N_{(X,D),d}^{\log} := \int_{[\overline{\mathcal{M}}_{0,l-1}^{\log}((X,D);d)]^{\text{vir}}} \prod_{i=1}^{l-1} \text{ev}_i^* [\text{pt}_X]$$

Example II: $(\mathbb{P}^2, L + C)$

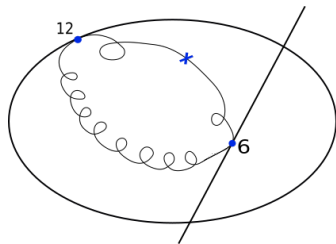


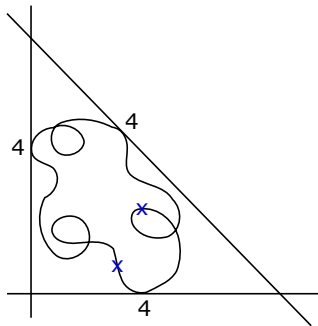
Figure: A degree 6 rational curve in \mathbb{P}^2 passing through 1 point and maximally tangent to line + conic.

d	N_d^{\log}	$N_d^{\log} / N_d^{\text{loc}}$	N_d
1	2	-2	1
2	6	8	1
3	20	-18	12
4	70	32	620
5	252	-50	87304
6	924	72	26312976
\vdots	\vdots	\vdots	

$$N_d^{\log} \sim 4^d \text{ (in fact } = \binom{2d}{d} \text{)}$$

$$N_d^{\log} / N_d^{\text{loc}} = (-1)^d 2d^2$$

Example III: $(\mathbb{P}^2, L_1 + L_2 + L_3)$



d	N_d^{\log}	$N_d^{\log} / N_d^{\text{loc}}$	N_d
1	1	1	1
2	4	-8	1
3	9	27	12
4	16	-64	620
5	25	125	87304
6	36	-216	26312976
\vdots	\vdots	\vdots	\vdots

Figure: A degree 4 rational curve in \mathbb{P}^2 passing through 2 points and maximally tangent to three lines.

$$N_d^{\log} \sim d^2 \text{ (in fact } = d^2)$$

$$N_d^{\log} / N_d^{\text{loc}} = (-1)^{d+1} d^3$$

The log-local correspondence

Conjecture (The log-local correspondence, vGGR '17)

For a smooth nef pair (X, D) ,

$$N_{(X,D),d}^{\log} = \prod_{i=1}^l (-1)^{d \cdot D_i + 1} (d \cdot D_i) N_{(X,D),d}^{\text{loc}}$$

Evidence:

- $l = 1$

[van Garrel–Graber–Ruddat]

- D_i very ample

[Nabijou–Ranganathan]

- toric pairs

[Bousseau–B–van Garrel]

The log-local correspondence for log-CY surfaces

Theorem (Bousseau–B–van Garrel '20)

The descendent log & local GW $g = 0$ GW theory of nef Looijenga pairs is closed-form solvable.

In particular, the log-local correspondence holds.

Strategy: the local side

Main idea: GW theory of $E_{X,D}$ reconstructed from GW theory of X

[Coates–Givental]

- local mirror theorems compute the J-function ($g = 0$ 1-point invariants) of $E_{X,D}$ from X

[Givental, Coates–Corti–Iritani–Tseng]

- J -function of X by degeneration to toric model+toric mirror symmetry
- tameness \leftrightarrow trivial mirror map
- multi-point invariants via big QH reconstruction theorem + WDVV
- TL;DR: there's an explicit formula.

Strategy: the log side

Two main ideas:

- $I = 2$: ACGS decomposition formula

[Abramovich–Chen–Gross–Siebert]

- tame:

① $\log GW$ counts in toric $(X, D) \leftrightarrow$ tropical counts in $(\Sigma(X), \Sigma(D))$

[Nishinou–Siebert, Mandel–Ruddat]

② construct toric model for (X, D) ; tameness \leftrightarrow finite scattering
(example later)

[Gross–Hacking–Keel, Gross–Pandharipande–Siebert, Abramovich–Wise, Mandel, Keel–Yu]

- TL;DR: there's an explicit formula, matches previous slide.

Higher genus logarithmic invariants

$$\begin{aligned} N_{(X,D),d}^{\log} &= \int_{[\overline{\mathcal{M}}_{0,l}^{\log}(X,D,d)]} \prod_{i=1}^l \text{ev}_i^*[\text{pt}] \\ &\rightsquigarrow \\ N_{(X,D),g,d}^{\log} &= \int_{[\overline{\mathcal{M}}_{g,l}^{\log}(X,D,d)]} \prod_{i=1}^l \text{ev}_i^*[\text{pt}] (-1)^g \lambda_g \\ &= [\log^{2g} q] \tilde{N}_{(X,D),d}^{\log}(q) \in \mathbb{Q}(q^{1/2}) \end{aligned}$$

Scattering calculation of $N_{(X,D),d}^{\log}$

\rightsquigarrow

(q -deformed) scattering calculation of $\tilde{N}_{(X,D),d}^{\log}(q)$

[Bousseau]

The $g > 0$ log-(local)-open correspondence

Higher genus local invariants?! (expdim < 0 for $g > 1$)

Proposal: GW invariants of local CY_n -folds

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$(n - 3)$ -holed open GW invariants of local CY_3 -folds

Toric A-branes

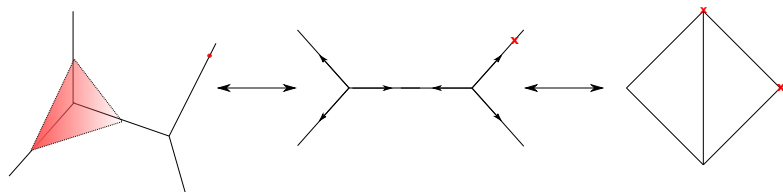
Harvey–Lawson fibration:

$$\begin{aligned}\mu : \mathbb{C}^3 &\longrightarrow \mathbb{R}^3 \\ (z_1, z_2, z_3) &\longrightarrow (|z_1|^2 - |z_2|^2, |z_1|^2 - |z_3|^2, \operatorname{Im}(z_1 z_2 z_3))\end{aligned}$$

- 1 generic fibre $\simeq \mathbb{T}^2 \times \mathbb{R}$
- 2 special fibres $(z_i = z_j = 0, i \neq j) \simeq \mathbb{R}^2 \times S^1$ (Aganagic–Vafa).

For Y semi-projective toric CY3:

Critical value set \longleftrightarrow planar lattice graph \longleftrightarrow height-1 slice of the fan



Open Gromov–Witten theory

- 1 Several approaches to define open invariants of Aganagic–Vafa A-branes in class $(\beta, \vec{\nu}) \in H_2(Y; \mathbb{Z}) \oplus H_1(L; \mathbb{Z}) \simeq H_2(Y, L; \mathbb{Z})$

[Katz–Liu, Li–Song, Li–Liu–Liu–Zhou]

- 2 Upshot: $\overline{\mathcal{M}}_{g, |\vec{\nu}|}(Y, L; \beta, \vec{\nu})$, $\text{expdim} = 0 \forall g$.

$$\begin{aligned} N_{(Y, L); g, \beta, \vec{\nu}}^{\text{open}} &= \int_{[\overline{\mathcal{M}}_{g, |\vec{\nu}|}(Y, L; \beta, \vec{\nu})]^{\text{vir}}} 1 \\ &= [\log^{2g-2+|\nu|} q] \tilde{N}_{(Y, L); g, \beta, \vec{\nu}}^{\text{open}}(q) \end{aligned}$$

The $g > 0$ log-(local)-open correspondence

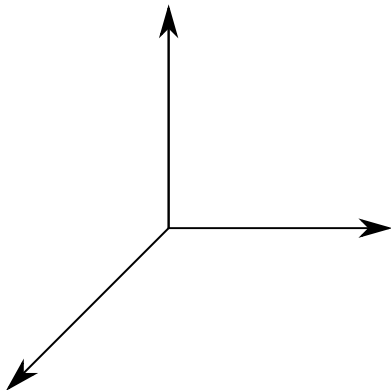
$(X, D = D_1 \cup \dots \cup D_l)$ tame Looijenga pair

\rightsquigarrow

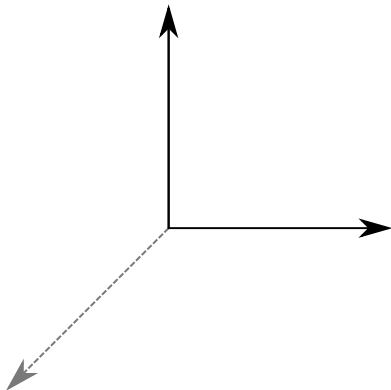
$(Y, L = L_1 \cup \dots \cup L_{l-1})$ semi-projective Aganagic–Vafa pair

- $Y := \text{Tot}(\mathcal{O}_{X \setminus \cup_{i=1}^{l-1} D_i}(-D_l))$
- L_i incident to non-compact torus 1-orbit intersecting D_i
- $d \in H_2(X) \leftrightarrow (\beta(d), \vec{\nu}(d)) \in H_2(Y, L)$ canonically

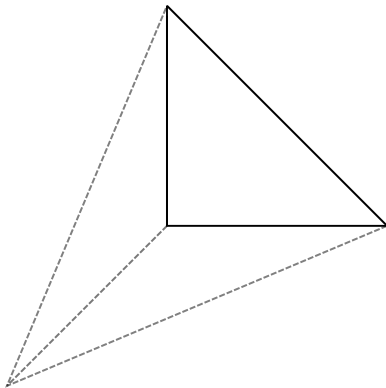
Example: $(X, D) = (\mathbb{P}^2, L + C)$



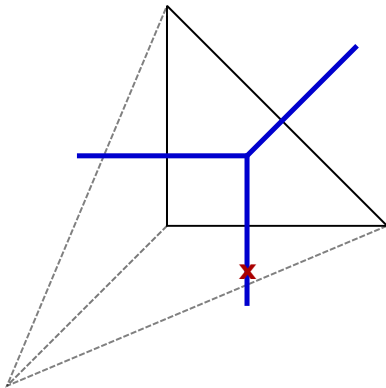
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Example: $(X, D) = (\mathbb{P}^2, L + C)$



The $g > 0$ log-(local)-open correspondence

Conjecture (The higher genus log-open correspondence, BBvG '20)

For a tame Looijenga pair,

$$\tilde{N}_{(X,D),d}^{\log}(q) = [1]_q^2 \prod_{i=1}^l (-1)^{d \cdot D_{i+1}} \frac{[d \cdot D_i]_q}{[1]_q} \prod_{i=1}^{l-1} \frac{d \cdot D_i}{[d \cdot D_i]_q} \tilde{N}_{(Y,L),\beta(d),\vec{\mu}(d)}^{\text{open}}(q)$$

where $[n]_q = q^{n/2} - q^{-n/2}$.

Motivation: **QFT engineering**

- 1 GW potential of local CY3 \rightarrow Nekrasov instanton partition function on $\mathbb{R}^4 \times S^1$

[Katz–Klemm–Vafa, Lawrence–Nekrasov, Goettsche–Nakajima–Yoshioka]

- 2 GW potential of local CY4 \rightarrow superpotential terms in LEET on $\mathbb{R}^2 \times S^1 \dots$

[Greene–Morrison–Plesser, Gukov–Vafa–Witten, Mayr]

- 3 \leftarrow disk GW potential of local CY3

[Ooguri–Vafa]

- 4 $g = 0$ local invariants of toric CY4 *are* holomorphic disk counts on toric CY3 + certain AV branes

[Mayr, Aganagic–Beem]

Proposal: true in higher dimension, more general AV brane setups too, higher genera refine log-local.

Implications for log GW theory

To a tame Looijenga pair, we can assign:

- 1 large N dual Chern–Simons theory interpretation
[Gopakumar–Vafa, Ooguri–Vafa]
- 2 all-genus calculation scheme (localisation, topological vertex)
[Graber–Zaslow, Diaconescu–Florea–Grassi, Aganagic–Klemm–Mariño–Vafa, Li–Liu–Liu–Zhou]
- 3 random matrix/crystal/free fermion models
[Mariño, Okounkov–Reshetikhin–Vafa, Saulina–Vafa]
- 4 classical integrable hierarchy (2-Toda reduction)
[AB; AB–Carlet–Rossi–Romano]
- 5 higher genus mirror reconstruction theorem (remodelled B-model)
[Bouchard–Klemm–Mariño–Pasquetti; Eynard–Orantin; Fang–Liu–Zong]
- 6 integral structure via open BPS counts
[Ooguri–Vafa, Labastida–Mariño–Vafa]
- 7 symmetric quiver DT reformulation
[Kucharski–Reineke–Stošić–Sułkowski, Panfil–Sułkowski]
- 8 gauge theory interpretation
[Kozçaz–Pasquetti–Wyllard; Dimofte–Gukov–Hollands]

Implications for log GW theory

Example: $X = dP_1$, $D_1 = H$, $D_2 = 2H - E$.

$\tilde{N}_{(X,D),d}^{\log}(q)$:

- coloured HOMFLY unknot
- resolvent of SW matrix model
- τ -function of relativistic Toda (Ablowitz–Ladik)
- Eynard–Orantin/Hori–Vafa 1-point g.f.
- DT g.f. symmetric Q , $C_Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

Implications for log GW theory

To a tame Looijenga pair, we can assign:

- 1 large N dual Chern–Simons theory interpretation
- 2 all-genus calculation scheme (localisation, topological vertex)
- 3 random matrix/crystal/free fermion models
- 4 classical integrable hierarchy (2-Toda reduction)
- 5 higher genus mirror reconstruction theorem (remodelled B-model);
- 6 integral structure via open BPS counts
- 7 symmetric quiver DT reformulation
- 8 gauge theory interpretation (4d surface operator/vortex partition function)

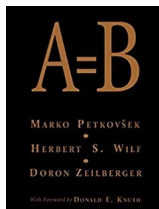
Today: 2) + 6) + 7)

The $g > 0$ log-(local)-open correspondence

Theorem (BBvG '20)

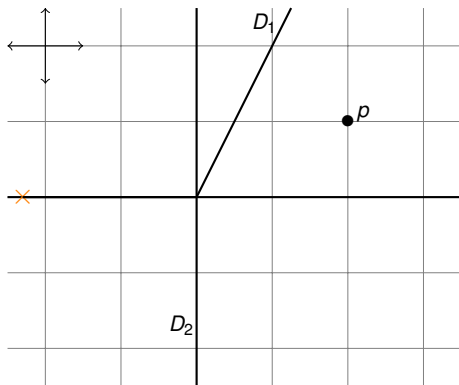
The higher genus log-GW theory of tame pairs (X, D) and open GW theory of the associated (Y, L) are closed-form solvable. In particular, the higher genus log-open correspondence holds for tame (X, D) .

Proof:



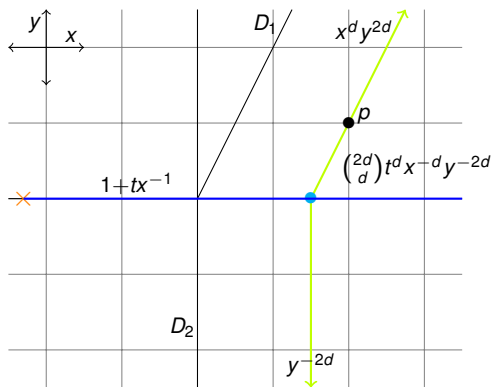
- 1 Log side: $\tilde{N}_{(X,D),d}^{\log}(q)$ from q -deformed scattering diagram
- 2 Open side: $\tilde{N}_{(Y,L),\beta,\bar{\mu}}^{\text{open}}(q)$ from the topological vertex

Example: $(X = \mathbb{P}^2, D = L + C)$, the log side



Scattering diagram for $(\mathbb{P}^2, L + C)$ is given by fan of \mathbb{F}_2 with a focus-focus singularity \times on the ray corresponding to a fibre class divisor.

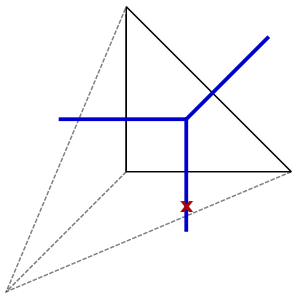
Example: $(X = \mathbb{P}^2, D = L + C)$, the log side



Multiplication of broken lines with asymptotic monomials $z^{d[D_1]} = x^d y^{2d}$, resp.
 $z^{2d[D_2]} = y^{-2d}$

$$\rightarrow N_{(X,D),d}^{\log} = \binom{2d}{d} \xrightarrow{q\text{-def}} \tilde{N}_{(X,D),d}^{\log}(q) = \left[\begin{matrix} 2d \\ d \end{matrix} \right]_q = \frac{\prod_{n=1}^{2d} [n]_q}{\prod_{m=1}^d [m]_q^2}$$

Example: $(X = \mathbb{P}^2, D = L + C)$, the open CY3 side



One-legged topological vertex:

$$\begin{aligned}
 \tilde{N}_{(Y,L),d}^{\text{open}}(q) &= \frac{1}{d} \sum_{R \vdash d} \chi_R((d)) q^{\kappa(R)/2} (-1)^{|R|} s_R(q^d) \\
 &= \frac{(-1)^d}{d[d]_q} \sum_{s=0}^{d-1} (-1)^s q^{\frac{3}{2} \binom{d}{2}} \begin{bmatrix} d-1 \\ s \end{bmatrix}_q (-q^d)^s q^{-ds/2} = \frac{(-1)^d}{d[2d]_q} \begin{bmatrix} 2d \\ d \end{bmatrix}_q
 \end{aligned}$$

Application I: log/local GW \leftrightarrow quiver DT

Gopakumar–Vafa invariants of CY($l + 2$)-folds:

$$\begin{aligned}\Omega_d(X, D) &= \sum_{k|d} \frac{\mu(k)}{k^{3-(l-1)}} N_{(X,D),d/k}^{\text{loc}} \\ &= \frac{1}{\prod_{i=1}^l d \cdot D_i} \sum_{k|d} \frac{\mu(k)(-1)^{d \cdot D_i/k+1}}{k^2} N_{(X,D),d/k}^{\text{log}}\end{aligned}$$

[Mayr, Klemm–Pandharipande, Ionel–Parker]

Conjecture (Klemm–Pandharipande)

$$\Omega_d(X, D) \in \mathbb{Z}$$

(Ionel–Parker: symplectic proof for compact CY n -folds)

Application I: log/local GW \leftrightarrow quiver DT

l – 1-holed open Gopakumar–Vafa invariants of CY(3)-folds:

$$\Omega_d^{\text{open}}(Y, L) = \sum_{k|d} \frac{\mu(k)}{k^{3-(l-1)}} N_{(Y,L),d/k}^{\text{open}}$$

[Oguri–Vafa, Labastida–Mariño–Vafa]

Theorem (BBvG '20, after Panfil–Sułkowski '17)

Let (Y, L) be the AV pair corresponding to a tame Looijenga pair (X, D) . \exists symmetric quiver Q such that

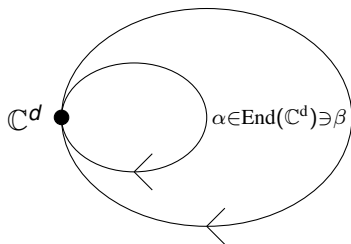
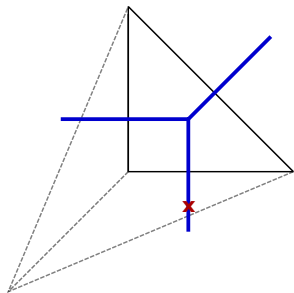
$$\Omega_d(X, D) = \Omega_d^{\text{open}}(Y, L) = \text{DT}_d(Q)$$

(cf Bousseau, “relative Gromov–Witten/Kronecker correspondence”)

Corollary (BBvG '20, after Efimov '11)

$$\Omega_d(X, D) \in \mathbb{Z}$$

Example: $(X, D) = (\mathbb{P}^2, L + C)$



Application II: log/local GW & DT/PT

- Cao–Leung, Borisov–Joyce, Oh–Thomas: invariants $DT_d^{(4)}$ from moduli of stable sheaves on CY4-folds
- Cao–Maulik–Toda: conjecturally $DT_d^{(4)} = \Omega_d$
- for $\mathcal{O}(-D_1) \oplus \mathcal{O}(-D_2) \rightarrow X$: checks by Cao–Kool–Monavari, based on BBvG

Application III: higher genus log GW & higher genus LMOV

$$\begin{aligned}\tilde{\Omega}_{(X,D),d}(q) &:= \sum_{k|d} \mu(k) (-1)^{\sum_i d \cdot D_i / k + 1} \frac{[1]_q^2}{[k]_q^2} \prod_{i=1}^l \frac{[k]_q}{[d \cdot D_i]_q} \tilde{N}_{(X,D),d/k}^{\log}(q^k) \\ &= [1]_q^2 \left(\prod_{i=1}^{l-1} \frac{d \cdot D_i}{[d \cdot D_i]_q} \right) \sum_{k|d} \mu(k) \frac{1}{k} \tilde{N}_{(Y,L),\beta(d)/k, \bar{\mu}(d)/k}^{\text{open}}(q^k)\end{aligned}$$

Theorem (The higher genus open BPS property, BBvG)

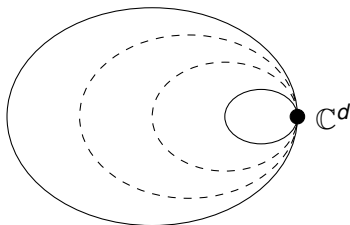
$$\tilde{\Omega}_{(X,D),d}(q) \in \mathbb{Z}[q, q^{-1}]$$

Strategy: direct arithmetic proof from the log/open calculation.

Orbifolds

Whole story generalises to orbifolds \Rightarrow infinite list.

Example: $(X = \mathbb{P}(1, 1, n), D_1 = L, D_2 = -K_X - L)$
 $\leadsto Q = (n + 1)$ -loop quiver



Conclusion

For nef/tame Looijenga pairs:

1. log GW are local GW are open GW are quiver DT are KP/IP/LMOV invariants

&

2. invariants are closed-form computed.

Food for thought

Some open questions (in very random order):

- $\dim X > 2$?
- D_i non nef? Non-tame cases?
- deduce higher genus log-(local)-open principle
 - ▶ geometrically?
 - ▶ algebraically?
- origin/meaning of quiver?