

## Aspects of supergroup gauge theory

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### Keywords

- Instanton counting
- Brane dynamics, Seiberg-Witten geometry
- Topological string, Geometric engineering
- Gauge/Bethe correspondence
- Intersecting defects = Supergroup gauge theory

### References

- TK-Pestun: [1905.01513](#)
- Chen-TK-Lee: [1908.04928](#), [2003.13514](#)
- TK-Sugimoto: [2001.05735](#)
- TK-Nieri: [2105.02776](#)
- TK: [2012.11711](#) (Habilitation thesis)

### Overview

Supergroup YM theory e.g.  $G = U(n_+ | n_-)$

$$\begin{aligned} \mathcal{L}_{YM} &= -\frac{1}{2g^2} \text{Str } F_{\mu\nu} F^{\mu\nu} \\ &= -\frac{1}{2g^2} \text{tr}_+ F \cdot F + \frac{1}{2g^2} \text{tr}_- F \cdot F \end{aligned}$$

*wrong sign!*

spectrum unbounded, non-unitary ...

Quiver realization  $G \supset G_+ \times G_-$  ( $U(n_+ | n_-) \supset U(n_+) \times U(n_-)$ )  
*bifund.*

$$A = \begin{pmatrix} G_+ & G_+ \times G_- \\ G_+ \times G_- & G_- \end{pmatrix}$$

*bifund.*

( $\Omega$ -def.)



[DHJV]

( $\hat{A}_1$ -quiver)

(complex.)  $\tau$   
 ("T<sub>+</sub>")

$-\tau$   
 ("T<sub>-")</sub>

physical cond.  
 $\text{Im } \tau > 0$

ABJM :  $U(n)_k \times U(m)_{-k}$  theory

Brane realization

$U(n)$  gauge theory = stack of  $n$  branes

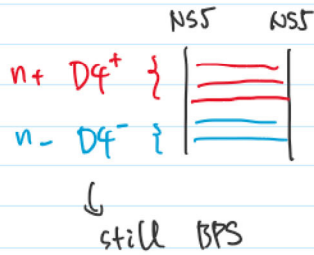
$U(n_+(n_-))$  theory =  $n_+$  (positive) branes

$\neq n_-$  (negative, ghost) branes

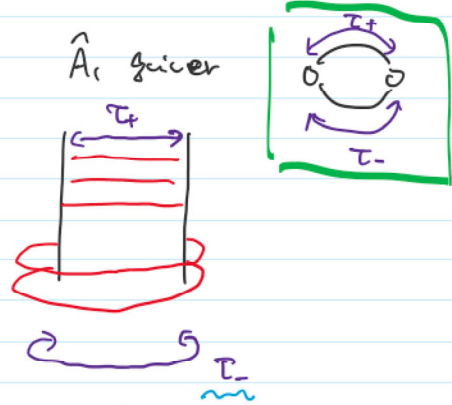
[DT]

Rem: Negative brane  $\neq$  anti-brane

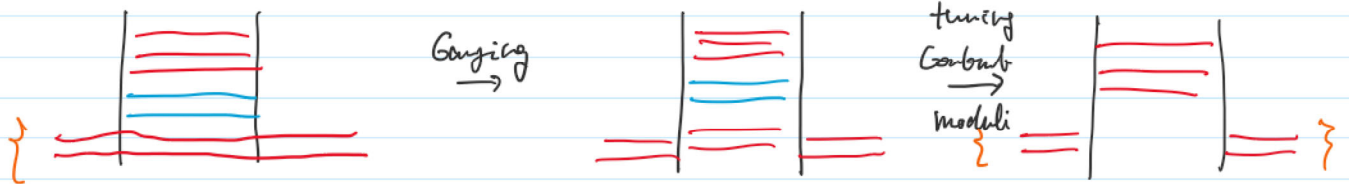
Hanany - Witten setup



$\tau_- \rightarrow -\tau_+$



decoupling  $\tau_- \rightarrow \infty$



$SU(n_+)$  SQCD  
w/  $\underline{2n_-}$  flavors

Rem: Ordering is not unique

$SU(2|2)$

Dynkin:  $\circ \text{---} \otimes$        $\otimes \text{---} \otimes$        $\otimes \text{---} \circ$

Seiberg - Witten geometry [DHJV]

Pure  $SU(n)$  SYM theory

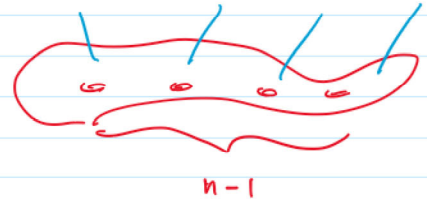
$\tau + 1/\tau = \prod_{i=1}^n (x - a_i) = \det(x - \phi)$

$\tau + 1/\tau = \frac{n_+ \prod (x - a_i^+)}{n_- \prod (x - a_i^-)} = \text{sdet}(x - \phi)$

$$\left| \eta + \frac{1}{\eta} = \frac{\prod_{i=1}^{n_+} (x - a_i^+)}{\prod_{i=1}^{n_-} (x - a_i^-)} \right| = \text{Sdet}(x - f)$$

SU(n) SQCD

$$\eta + \frac{1}{\eta} = \frac{\prod_{i=1}^n (x - a_i)}{\prod_{f=1}^{n_f} (x - m_f)}$$



$$(n_f = 2n_- : m = (m_1, m_1, m_2, m_2, \dots))$$

$$\rightarrow \frac{\prod_{i=1}^{n_+} (x - a_i)}{\prod_{f=1}^{n_-} (x - m_f)} = \text{Sdet}(x - f)$$

Super instanton counting [KP]

ADHM construction = T dual

$$(k_+(k_-) D_0 = k_+ D_0^+ + k_- D_0^-) \rightarrow n_+ D_0^+ + n_- D_0^- = n_+ D_0^+ + n_- D_0^-$$

$k$ -instanton in  $U(n)$  theory on  $T^f$  (radius  $R$ )  $\rightarrow \mathbb{R}^f$

$$\Downarrow$$

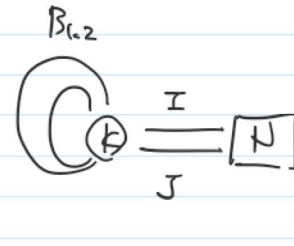
$(n_+(n_-))$   $n$ -instanton in  $U(k)$  theory on  $\check{T}^f$  ( $\check{R} = 1/R$ )  $\rightarrow \{pt\}$

Vector sp. :  $N = \mathbb{C}^n \rightarrow \mathbb{C}^{n_+(n_-)}$ ,  $K = \mathbb{C}^k \rightarrow \mathbb{C}^{k_+(k_-)}$   
 $\Rightarrow$  Chan - Paton for  $D_0^\pm$ ,  $D_0^\pm$

	instanton	anti-inst.	negative inst	anti-negative
SD or ASD	ASD	SD	ASD	SD
top #	+	-	-	+
D-brane	$D_0^+$	$\bar{D}_0^+$	$D_0^-$	$\bar{D}_0^-$

$$\left. \begin{array}{l} D_0^+ - \bar{D}_0^+ : \text{non-BPS} \\ D_0^+ - D_0^- : \text{BPS} \end{array} \right\} \begin{array}{l} (\text{ASD} + \text{SD}) \\ (\text{ASD}) \end{array}$$

▷ ADHM data

$$\left. \begin{array}{l} B_{1,2} : K \rightarrow K \\ I : N \rightarrow K \\ J : K \rightarrow N \end{array} \right\}$$


◦ ADHM moduli sp.

$$\mathcal{M}_{n,k} = \mu^{-1}(0) // U(K)$$

$$\Rightarrow \tilde{\mathcal{M}} = \mu_G^{-1}(0) // GL(K) \quad \text{w/ stability cond.}$$

$$\mu_{\mathbb{R}} = 0 \quad \rightarrow \quad \mu_{\mathbb{R}} = \zeta_{\neq 0}$$

Stab. cond.

$$K = \left. \begin{array}{l} \bullet \mathbb{C} [B_1, B_2] I(N) \quad (\zeta > 0) \\ \mathbb{C} [B_1^+, B_2^+] J^+(N) \quad (\zeta < 0) \end{array} \right\}$$

Super stab. cond.

$$\mu_{\mathbb{R}} = \begin{pmatrix} +\zeta \\ -\zeta \end{pmatrix}, \quad \mu_{\mathbb{C}} = 0$$

$$\left. \begin{array}{l} K_+ = \mathbb{C} [B_1, B_2] I(N_+) \\ K_- = \mathbb{C} [B_1^+, B_2^+] J^+(N_-) \end{array} \right\}$$

Equiv. action :  $U(1)^2 \subset SO(\mathbb{R})$

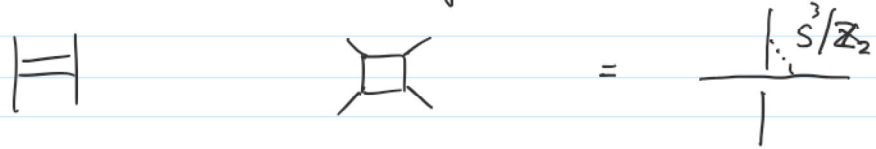
$$B_{1,2} \rightarrow g_{1,2} B_{1,2} \quad \text{w/} \quad g_{1,2} = e^{\epsilon_{1,2}}$$

$$\left. \begin{array}{l} \text{positive sector} : (\underline{\epsilon_1}, \underline{\epsilon_2}) \\ \text{negative} \quad \quad : (\underline{-\epsilon_1}, \underline{-\epsilon_2}) \end{array} \right\}$$

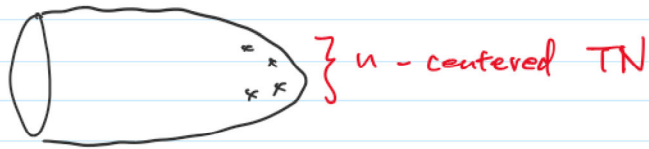
$$\left. \begin{array}{l} G - \text{SYM} \quad \quad \hat{G} - \text{ Toda} \quad \quad G = \text{supergroup.} \\ G - N=2^* \quad \quad G - \text{ Calogero} \\ G - \text{SQCD} \quad \quad sl(2) \times \mathbb{R} \quad \text{w/ positive \& negative magnetic flux} \end{array} \right\}$$

G - SQCD  $SL(2) \times XY$  w/ positive & negative monopoles  
 Topological string  $SL(n|m)$

Geometric engineering :  $A_n$ -singularity  $\Rightarrow$   $SU(n+1)$  theory  
 Brane web = toric diag



Taub-NUT :  $TN_n \sim S^1 \times \mathbb{R}^3 \xrightarrow{S^1 \rightarrow \infty} ALE : \mathbb{R}^4 / \mathbb{Z}_n$



$$V(\vec{x}) = 1 + \sum_{i=1}^n \frac{+1}{|\vec{x} - \vec{x}_i|}$$

positive

$$\rightarrow 1 + \sum_{i=1}^{n_+} \frac{+1}{|\vec{x} - \vec{x}_i^+|} + \sum_{i=1}^{n_-} \frac{-1}{|\vec{x} - \vec{x}_i^-|}$$

negative

ALE Lim  $\rightarrow \mathbb{R}^4 / \mathbb{Z}_{n_+ - n_-}$

$SU(n_+ | n_-)$  vs  $SU(n_+ - n_-)$

cf. signature change  
 [DHJV]

Resolution

$$y + y^{-1} = x^n \rightarrow x^n + \dots = \det(x - \phi) \quad \underline{SU(n)}$$

$$\rightarrow \frac{x^{n_+} + \dots}{x^{n_-} + \dots} = \text{sdet}(x - \phi) \quad SU(n_+ | n_-)$$

w/  $n_+ - n_- = n$

$\Rightarrow A_n$ -sing  $\simeq A_{n_+ | n_-}$ -sing. ( $n_+ - n_- = n$ ).

$$\Rightarrow A_n\text{-sing} \approx A_{n_+|n_-}\text{-sing} \quad (n_+ - n_- = n)$$

o Topological vertex for supergroup theory [KS]

where  $\begin{matrix} \uparrow \\ \text{Y} \\ \downarrow \\ \lambda \end{matrix} = C_{\mu \nu \lambda}(\eta_1, \eta_2^{-1})$

cf. standard convention :  $(\eta, t) = (\eta_1, \eta_2^{-1})$

o Negative node :  $\eta_{1,2} \rightarrow \eta_{1,2}^{-1}$

$\Rightarrow$  Negative vertex (anti) :  $\bar{C}_{\mu \nu \lambda}(\eta_1^{-1}, \eta_2) =$

e.g.  $U(1|1)$  theory

o Unrefined slice :  $\eta_1, \eta_2 = 1$  ( $\eta = t$ )

$$\begin{matrix} \text{Y} \\ C_{\dots}(\eta) \end{matrix} = \begin{matrix} \text{Y} \\ (\bar{C}(\eta) =) C(\eta^{-1}) \end{matrix} \quad (\neq \text{branching})$$

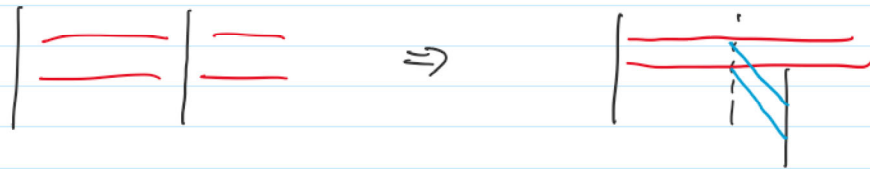
classical VOA

$\mathfrak{gl}_n\text{-alg} \quad \mathfrak{gl}_{n_+|n_-}\text{-alg}$

Gaiotto - Witten / Nekrasov - Witten

$$\begin{array}{c} \text{NSS} \\ \left| \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \right| \begin{array}{c} n D3 \\ \text{---} \\ \text{---} \end{array} \end{array} \Rightarrow \begin{array}{c} n_+ \\ \left| \begin{array}{c} \text{---} \\ \text{---} \end{array} \right| \begin{array}{c} n_+ \\ \text{---} \\ \text{---} \end{array} \end{array} = \begin{array}{c} \left| \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \right| \begin{array}{c} n_+ \\ \text{---} \\ \text{---} \end{array} \\ \text{---} \\ \text{---} \end{array}$$

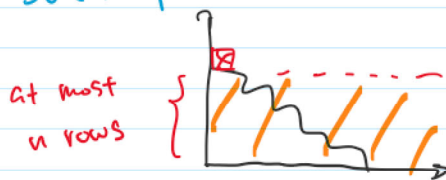
Intersecting defects & super group str. [KN]



$\Omega$  - deformation:

$$a = m + n \cdot \epsilon_1 + m \cdot \epsilon_2$$

$SU(n)$  rep.



trapped inst.  
= vortex

$SU(n|m)$  rep.



intersecting defects.  
 $\mathbb{C}_{\epsilon_1} \vee \mathbb{C}_{\epsilon_2}$

5d / 3d (4d / 2d) correspondence of triality

(non-supergroup)

$$5d \text{ theory} : Z = \sum_{\lambda} Z_{\lambda}$$

$\Downarrow$

$$3d \text{ theory (Higgs br.)} : Z = \sum_{\lambda \in \Lambda} Z_{\lambda}$$

$\Downarrow$

$$3d \text{ theory (Coulomb br.)} : Z = \int \dots$$

Coulomb br. formula

$$Z_{\text{Coulomb}} = \int dz dw \frac{\Delta(z; q, t) \Delta(w; t, q)}{\prod_{i,j} (1 - z_i w_j)} \times \dots$$

super Macdonald measure

J

$\prod_{i,j} (1 - q^{w_j})$   
super Macdonald measure

$$\Delta(z; q, t) = \prod_{i \neq j}^n \left( \frac{z_i}{z_j} : q \right) / \left( t \frac{z_i}{z_j} : q \right)$$

$\downarrow$   
q
 $\downarrow$   
adj mass