String Math, Berkeley

Monday, March 29, 2021 2:01 PM

Quantum Cohomology and Slices of the Affine Grassmannian

What structures?

$H^*(X, \mathbb{C})$ cohomology

$T \curvearrowright X$

$H^*_T(X)$ T-equivariant cohomology

algebra wrt $\cup$

deform $\cup$ to $*$

$QH^*_T(X)$

$$\langle a, b * c\rangle = \langle a, b \cup c\rangle + \sum_{d \in H_2^{\text{eff}}(X)} \underbrace{\langle a, b, c\rangle^X_{0,3,d}}_{\text{3-point GW invariants}} q^d$$

3-point GW invariants count curves of degree d

as a space $\underbrace{H^*_T(X)}_{\text{fibers}} \otimes \underbrace{\mathbb{C}[[q^d]]}_{\text{coordinates on a base}}$

Quantum Connection

$$\nabla^Q_\lambda = \frac{\partial}{\partial\lambda} + \lambda * .$$

$\lambda \in H^2_T(X)$ flat connection

$H^2_T(pt) \subset H^2_T(X)$ is not interesting

We'll focus on transversal directions.

We want to express ∇^Q in terms of more well-known flat connections. [For a special family of X]

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Let's start with the answer.

$[X = \widetilde{Gr}^{\lambda_1, \dots, \lambda_\ell}_{\mu}$ in ADE types$]$

Some Representation Theory

Let $G^\vee$ be a complex connected reductive group (SL_n, PSL_n, ...)

$V_1, \dots, V_\ell$ – $G^\vee$-reps.

- For $X \in \mathfrak{g}^\vee = \mathrm{Lie}\, G^\vee$ we'll write X^i for X acting on ith position in $V_1 \otimes \dots \otimes V_\ell$.
  $$X^i \cdot (v_1 \otimes \dots \otimes v_\ell) = v_1 \otimes \dots \otimes X v_i \otimes \dots \otimes v_\ell.$$
- Similarly, for $Y \in \mathfrak{g}^\vee \otimes \mathfrak{g}^\vee$ we'll write Y^{ij} for the operator acting by the first component of Y on ith place, the second component on jth place.

- $\mathfrak{g}^\vee \otimes \mathfrak{g}^\vee \to \mathbb{C}$ symmetric invariant pairing
  $\mathbb{C} \to \mathfrak{g}^\vee \otimes \mathfrak{g}^\vee$, the image of 1 is called the Casimir operator Ω

Explicitly,

$$\Omega = \sum_i x^i \otimes x_i + \sum_{\alpha\text{-root}} e_\alpha \otimes e_{-\alpha}$$

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x^i, x_i – dual bases in $\mathfrak{h}^\vee \subset \mathfrak{g}^\vee$ (Cartan)

e_α – elements in root subspaces with root α.

We'll need "half" Casimir operators:

$$\Omega_{\mathcal{C}} = \frac{1}{2}\sum_i x^i \otimes x_i + \sum_{\substack{\alpha\text{-root} \\ \alpha|_{\mathcal{C}} > 0}} e_\alpha \otimes e_{-\alpha}$$

Here $\mathcal{C}$ is a Weyl chamber

$\alpha|_{\mathcal{C}} > 0 \iff \alpha$ is positive w.r.t. $\mathcal{C}$.

Usually people take $\mathcal{C}$ to be dominant & antidominant.

The trigonometric Knizhnik-Zamolodchikov connection

$$\nabla_i^{kz} = z_i \frac{\partial}{\partial z_i} + H^i + \hbar \sum_{i \neq j} \frac{z_i \Omega^{ij}_{\mathcal{C}} + z_j \Omega^{ij}_{-\mathcal{C}}}{z_i - z_j}$$

$H^i \in \mathfrak{h}^\vee$.

We'll act on $V_{\lambda_1} \otimes \dots \otimes V_{\lambda_\ell}[\mu] \otimes \mathbb{C}(z_1, \dots, z_\ell)[\hbar]$

simple reps with h.w. λ_1

μ-weight space.

To compare with power series in QH^*, expand

$$|z_1| \ll |z_2| \ll \dots \ll |z_\ell|$$

$$\nabla_i^{kz} = z_i \frac{\partial}{\partial z_i} - H^i - \hbar\left[\sum \Omega^{ji}_{-\mathcal{C}} - \sum \Omega^{ij}_{-\mathcal{C}}\right]$$

$$\nabla_i^{KZ} = z_i \frac{\partial}{\partial z_i} \underbrace{- H^i - \hbar\Big[\sum_{j<i} \Omega_{-\varphi}^{ji} - \sum_{j>i} \Omega_{-\varphi}^{ij} \Big]}_{\text{classical } \cup}$$

$$\underbrace{-\hbar \Big[\sum_{j<i} \frac{(z_j/z_i)}{1 - z_j/z_i} \Omega^{ij} - \sum_{j>i} \frac{(z_i/z_j)}{1 - (z_i/z_j)} \Omega^{ij} \Big]}_{\substack{\text{corrections in } *, \\ \text{purely quantum part.}}}$$

Thm (D'20) Under identification:

$$V_{\lambda_1} \otimes \dots \otimes V_{\lambda_\ell}[\mu] \underset{\text{choice of basis}}{\longleftrightarrow} H_T^*(\widetilde{Gr}_\mu^{\lambda_1, \dots \lambda_\ell})$$

$$v_1 \otimes \dots \otimes v_\ell\text{-basis} \longleftarrow \mathrm{Stab}_\varphi \text{ - basis}$$

$$\varphi \text{ in } \Omega_{\varphi} \longleftrightarrow \mathfrak{C} \text{ is } \mathrm{Stab}_\varphi$$

$$z_i \longleftrightarrow q^{e_i} \longleftarrow \text{some homology classes}$$

$$|z_i| \ll |z_j| \longleftrightarrow e_i - e_j \text{ is effective}$$

$$\nabla_i \longleftrightarrow \nabla_{\lambda_i} \qquad \lambda_i = c_1^T(\mathcal{E}_i)$$

$\mathcal{E}_i$ - some T-equiv line bundle

$$\nabla^{KZ} = \nabla^Q .$$

The Affine Grassmannian

Let G be a complex connected reductive group

$$\mathcal{K} = \mathbb{C}((z)), \quad \mathcal{O} = \mathbb{C}[[z]]$$

$\operatorname{Spec} \mathcal{O}$ - formal disk

$\operatorname{Spec} \mathcal{K}$ - formal punctured disk.

Set-theoretically the Affine Grassmannian is

$$Gr_G = G(\mathcal{K}) / G(\mathcal{O})$$

There is a moduli space formulation

$$Gr_G = \left\{ (\mathcal{P}, \varphi) \;\middle|\; \begin{array}{l} \mathcal{P} \text{ - } G\text{-principle bundle on } \operatorname{Spec} \mathcal{O} \\ \varphi \text{ - isomorphism of } \mathcal{P}|_{\operatorname{Spec} \mathcal{K}} \text{ with} \\ \quad \text{the trivial } G\text{-principle bundle} \end{array} \right\}$$

Gr_G is an ind-proj scheme
and has an ample line bundle $\mathcal{O}(1)$
which generates $\operatorname{Pic} Gr_G$

We'll fix G, so we omit it from notation.

<u>Actions on Gr:</u>

1)
$$\begin{array}{l} G(\mathcal{K}) \curvearrowright Gr \\ \;\cup \\ G(\mathcal{O}) \\ \;\cup \\ G \\ \;\cup \text{ max torus} \\ A \end{array}$$

2) $\mathbb{C}^\times_\hbar \curvearrowright Gr$ by scaling z / disk

We'll need $T = A \times \mathbb{C}^\times_\hbar$ action on Gr.

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Fixed Points

Let $\lambda \in \mathrm{Hom}(\mathbb{C}^\times, A)$ be a cocharacter

Then we can make an element of $G(\mathcal{K})$:

$$\operatorname{Spec} \mathcal{K} \to \mathbb{C}^\times \xrightarrow{\lambda} A \subset G$$

We call it z^λ

This gives $[z^\lambda] \in Gr$.

$[z^\lambda]$ is A fixed

$\mathbb{C}^\times_\hbar$ - fixed

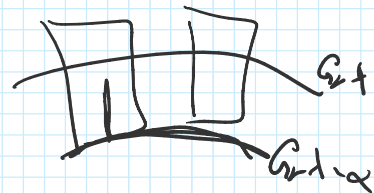
Prop $Gr^T = Gr^A = \displaystyle\bigsqcup_{\lambda\text{-cochar.}} \{[z^\lambda]\}$

Cell structure in Gr:

$Gr^\lambda = G(\mathcal{O}) \cdot [z^\lambda]$ ← as a space this is a vector bundle over G/P. $\mathbb{C}^\times_\hbar$ scales the fibers

$Gr = \displaystyle\bigsqcup_{\lambda\text{-dominant}} Gr^\lambda$ ← similar to Bruhat cells in G/B, G/P.

$\overline{Gr^\lambda} = \displaystyle\bigsqcup_{\substack{\mu\text{-dominant} \\ \mu \le \lambda}} Gr^\mu$



$Gr^\lambda \subset \overline{Gr^\lambda}$ smooth part

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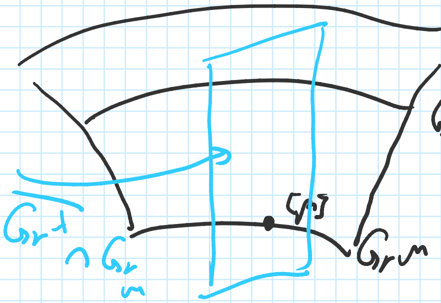
So $\overline{Gr^\lambda}$ is smooth $\Leftrightarrow$ $\overline{Gr^\lambda} = Gr^\lambda$ $\Leftrightarrow$ λ is minuscale.

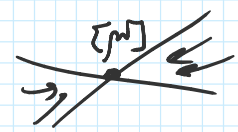


Prop 1) $Gr^\lambda_\mu \subset Gr$ is T-invariant

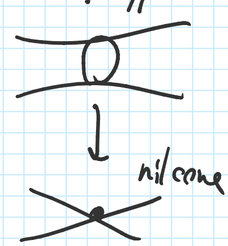
2) $(Gr^\lambda_\mu)^T = (Gr^\lambda_\mu)^A = [\mu]$

3) $\mathbb{C}^\times_\hbar$ contracts Gr^λ_μ to $[\mu]$.

4) Gr^λ_μ is normal affine Poisson variety $\langle 2\rho^\vee, \lambda-\mu\rangle$



It is known when Gr^λ_μ has a symplectic resolution [KWWY] if $\lambda = \sum \lambda_i$ of minuscale coweights



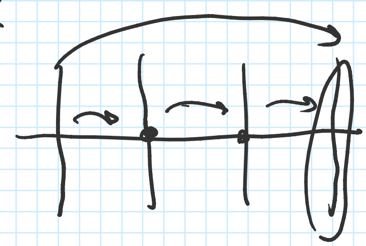
$$\widetilde{Gr}^{\overbrace{\lambda_1, \dots \lambda_\ell}^{\text{depends on order}}}_\mu \subset Gr^{\times \ell}$$

smooth symplectic form has weight $\hbar$

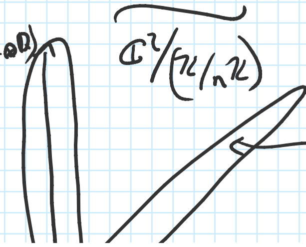
$\mathbb{C}^\times_\hbar$

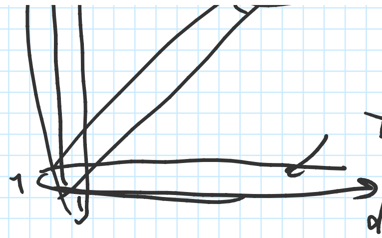
$\downarrow$ proper birational T-equivariant

Gr^λ_μ symplectic structure on the smooth part is the same.

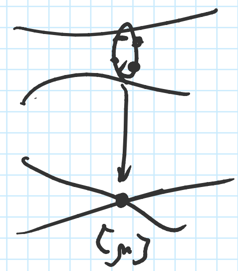


Examples: $\dim(\mathrm{Pic}\otimes\mathbb{R})$ $\overline{\mathbb{C}^2/(\mathbb{Z}/n\mathbb{Z})}$ T^*G/B. in type A.





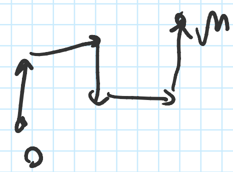
P - maximal parabolic corr. to minuscule coweight.



The T & A -fixed loci are the same for $\widetilde{Gr}_\mu^{\lambda_1,\dots,\lambda_\ell}$:

$$\left\{ ([z^{\Sigma_1}], \dots, [z^{\Sigma_\ell}]) \;\middle|\; \begin{array}{l} \Sigma_i \text{ are coweights} \\ \Sigma_i - \Sigma_{i-1} \text{ are in } W\lambda_i \iff \text{are weights in } V_{\lambda_i} \text{ - } G^\vee\text{-rep} \\ \Sigma_\ell = \mu \end{array} \right\}$$

They can be drawn as paths from 0 to μ



This is in bijection with a basis in $V_{\lambda_1} \otimes \dots \otimes V_{\lambda_\ell}[\mu]$.

Natural line bundles:

$$\widetilde{Gr}_\mu^{\lambda_1,\dots,\lambda_\ell} \subset Gr^{\times \ell}$$

This gives ℓ projection maps $p_1, \dots, p_\ell$.

$$\mathcal{L}_i = p_i^* \mathcal{O}(1)$$

$$\mathcal{E}_i = \mathcal{L}_i / \mathcal{L}_{i-1}$$ ← behave like e_i's in $A_{\ell-1}$ root system.

$$\sum_i c_i^T(\mathcal{E}_i) = 0 \iff \sum_i e_i = 0$$

$$\underline{c_1^T(\mathcal{E}_i)} \quad \text{span} \quad \underline{H^2(X)}$$

$$\updownarrow$$

$$\gamma_i$$

<u>Basis</u>

Localization: $\iota: X^T \to X$

$$\iota^*: H_T^*(X)_{loc} \xrightarrow{\sim} H_T^*(X^T)_{loc}$$ ← a vector space over $\mathrm{Frac}(H_T^*(pt))$ of dim $\# X^T$.

$$M_{loc} = M \otimes_{H_T^*(pt)} \mathrm{Frac}\, H_T^*(pt)$$

This gives two bases:

$(\iota^*)^{-1}(1_p)$ ← only localized classes. little control

$(\iota_*)\, 1_p$ ← non-localized, but $\in H_T^{2\dim X}(X)$

Maulik-Okounkov Stable Envelopes

- $\mathrm{Stab}_{\mathfrak{C}}(p) \in H_T^{\dim X}(X)$
  ↑ a refined version of P.D$[\mathrm{Attr}_{\mathfrak{C}}\, p]$
- $\mathrm{Stab}(p)|_q$ – triangular
- $\mathrm{Stab}(p)|_q = 0 \mod \hbar$ if $p \neq q$.

• $\mathrm{Stab}_{-\mathfrak{C}}(p)$ – the dual basis to $\mathrm{Stab}_{\mathfrak{C}}(p)$.

Classical Part

$D \cup \mathrm{Stab}_{\mathfrak{C}}(p) \qquad\qquad D \in H^2_T(X)$

$\langle \mathrm{Stab}_{-\mathfrak{C}}(q),\ D \cup \mathrm{Stab}_{\mathfrak{C}}(p) \rangle \in H^2_T(pt)$

If $p \neq q$ this is divisible by $\hbar$, so $\in \hbar\,\mathbb{Q}$.

If $p = q$ this is a straightforward computation.

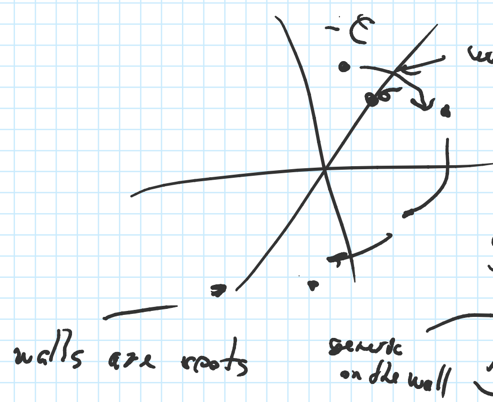
The only data you need is $\mathrm{Stab}(p)\big|_q \mod \hbar^2$.

1) Find this in A_1-type

There are explicit recursive relation.

2) For G of general type:

$p \neq q$; If $\mathrm{Stab}_{\mathfrak{C}}(p)\big|_q \neq 0 \Rightarrow \mathrm{Stab}_{-\mathfrak{C}}(p)\big|_q = 0$.



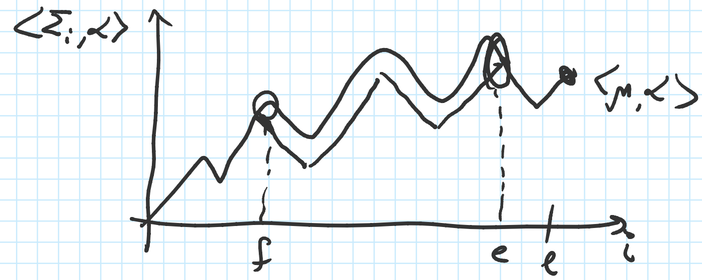
when you cross the wall you can control what happens to Stab

If you know $\underbrace{X^{\circ}}_{\text{slice of type } A_1} \subset X$

slice of type A_1.

In type A_1 the $\mathrm{Stab}(p)\big|_q \neq 0 \mod \hbar$

if p and q are related by "$f \otimes e$"


⟨Σ_i, α⟩
⟨μ, α⟩
f
e
ℓ
i