joint with D. Kalinov
and E. Rains 2005.13604

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1. Rational Cherednik algebras for S_n.

$\boxed{\text{work over } \mathbb{C}}$

$$H_{t,k}(n) = \left\langle S_n,\ \begin{matrix} x_1,\dots,x_n \\ y_1,\dots,y_n \end{matrix} \right\rangle \qquad \boxed{k\in\mathbb{C}}$$

1) S_n permutes x_i, y_i as usual

2) $[x_i,x_j]=[y_i,y_j]=0$

3) $[y_i,x_j]=ks_{ij}$

$$[y_i,x_i]=t-k\sum_{j\neq i}s_{ij}.$$

Realization via Dunkl operators

Action of $H_{t,k}(n)$ on $\mathbb{C}(x_1,\dots,x_n)$

scaling

$(t,k)\to(\lambda t,\lambda k)$

$t=\hbar$, $t\to 0$ classical limit

$x_i \longmapsto x_i\cdot$, $s_{ij}\longmapsto s_{ij}$

$$y_i\mapsto D_i=t\underbrace{\partial_i}_{p_i}+k\sum_{j\neq i}\frac{1}{x_i-x_j}s_{ij}.$$

Ex. $t = 1$.

$$H = \sum_i y_i^2 \Big|_{\mathbb{C}(x_1,\dots,x_n)^{S_n}} = \Delta - k(k+1) \sum_{\substack{i\neq j \\ 1\le i,j\le n}} \frac{1}{(x_i - x_j)^2}$$

$$\Delta = \sum \partial_i^2 .$$

Quantum Integrals of motion

$$H_m = \sum_i y_i^m \Big|_{\mathbb{C}(x_1,\dots,x_n)}^{S_n}$$

$H_m = \partial_1^m + \dots + \partial_n^m +$ lower terms.

$$[H_i, H_j] = 0 \qquad H_1 = \sum_i \partial_i$$

$H_2 = H.$ ⟵ quantum integrable.

Spherical subalgebra:

$\mathbb{C}S_n \ni e = \frac{1}{n!} \sum_{s\in S_n} s$ symmetrizer.

$$H_{1,k}(n) \; \| \; H_k(n)$$

$e^2 = e$

$eH_k^{(n)}e$ – spherical

subalgebra of $H_K(n)$
(with unit e).

For $k \neq -\frac{r}{m}$, $\begin{array}{l} 2 \le m \le n \\ 1 \le r \le m-1 \end{array}$

it is morita equivalent to $H_K(n)$ via $\underset{H_K(n)}{M} \longmapsto \underset{eH_K(n)e}{eM}$.

$eH_K(n)e$ is the algebra generated by $\mathbb{C}[x_1, \dots, x_n]$ and H, contains all H_m.

Want to take limit $N \to \infty$ of $H_K(N)$
(or rather, make $N \in \mathbb{C}$).

Consider $T_{m,\ell} = e\sum_{i=1}^{N} x_i^m y_i^\ell e$
$\cap$
$eH_K(n)e$

$T_{0,0} = N$, $T_{1,0} = \sum x_i$

$T_{0,1} = \sum y_i = \sum \partial_i$

$T_{0,2} = H, \dots$

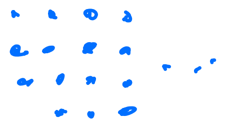
These are algebraically dependent for every fixed n, but asymp. $n \to \infty$ independent

finitely many $\neq 0$ values

$$Z_{\vec{n}} = \prod_{m,\ell} T_{m,\ell}^{n_{m,\ell}}$$

$\vec{n} : \mathbb{Z}_+ \times \mathbb{Z}_+ - (0,0) \to \mathbb{Z}_{\geq 0}$

$\vec{n} = (n_{m,\ell})$



(choose some ordering).

$$Z_{\vec{n}} \cdot Z_{\vec{p}} = \sum_{\vec{q}} C^{\vec{q}}_{\vec{n},\vec{p}}(N)\, Z_{\vec{q}}$$

poly in N.

Get an algebra

$\boxed{Y_{k,\nu}} = Y_{k,\nu}(\mathfrak{gl}_1)$ $\nu = N$ (viewed as elem of $\mathbb{C}$).

Deformed double current alg for $\mathfrak{gl}_1$

deformation of $Y_{0,\nu}$

$$Y_{0,\nu} = U(\underset{\substack{\nearrow \\ \text{basis}}}{}\mathrm{Diff}(\mathbb{C}))\big/(\mathbb{1}-\nu) .$$

$\uparrow$ Lie alg.

basis $\quad x^i\partial^j = z_{ij} \quad (i,j)\in\mathbb{Z}_+\times\mathbb{Z}_+$

$$\mathrm{gr}\, Y_{0,\nu} = \mathbb{C}[z_{ij}]_{i,j\geq 0\,,\ i+j>0.}$$

Also there is a version for $\mathfrak{gl}_n$:

$$\begin{tikzcd}
H_k(N) & \otimes & \mathrm{Mat}_n(\mathbb{C})^{\otimes N} \\
\mathbb{C}S_N \arrow[u, phantom, "\cup"] & \otimes & \mathbb{C}S_N \arrow[u, phantom, "\cup"] \\
& \mathbb{C}S_N \arrow[u, "\Delta"'] & \ni e
\end{tikzcd}$$

$$\Delta(e)\Big(H_k(N)\otimes\mathrm{Mat}_n(\mathbb{C})^{\otimes N}\Big)\Delta(e)$$

$$\shortparallel$$

$$Y_{k,\nu}(\mathfrak{gl}_n) \text{ — def.}$$

of $\mathcal{U}(\mathfrak{gl}_n \otimes \mathrm{Diff}(\mathbb{C}))/(\mathbb{I} = \nu)$.

$$\shortparallel$$

$Y_{0,\nu}(\mathfrak{gl}_n)$.

$Y_{k,\nu}(\mathfrak{gl}_N)$ – DDCA
rational degen.
of Yangian
$Y(\widehat{\mathfrak{gl}}_N)$

$\uparrow$
construct by
gen. and rel
(Guay) $N \geq 4$

$Y(\widehat{\mathfrak{gl}}_N)$ is a def of
$U(\widehat{\mathfrak{gl}}_N[y])$ $\qquad \mathrm{gr}\, Y(\mathfrak{g}) = U(\mathfrak{g}[y])$

$\shortparallel$ for zero central ext.

$U(\mathfrak{gl}_N[t, t^{-1}, y])$ $\qquad \mathbb{C}[t, t^{-1}, y]$
$= \mathbb{C}[\mathbb{C}^\times \times \mathbb{C}]$
$\wr$
$\downarrow$
degenerate
to $\mathbb{C}$.

$\mathfrak{sl}_2$-equivariant
presentation
of $Y_{k,\nu}(\mathfrak{gl}_1) = Y_{k,\nu}$

$$\mathrm{gr}\, Y_{k,\nu} = U(\mathbb{C}[x,p]_{\{,\}}).$$

$$\mathbb{C}[x,p]_{\{,\}} \;||\; \mathcal{P}$$

$\mathcal{P}$ – Lie alg of Hamilt in the plane. (classical).

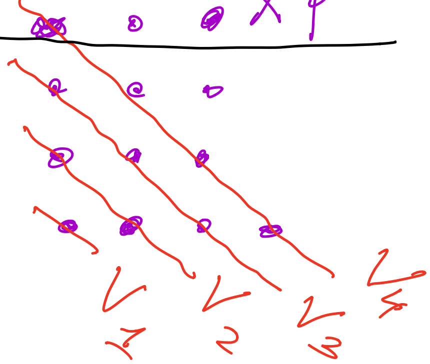
What's a presentation of this?

$$\mathbb{C}[x,p] \supset sl_2 = \langle p^2, xp, x^2 \rangle.$$

$$\mathbb{C}[x,p] = V_1 \oplus V_2 \oplus V_3 \oplus \boxed{V_4 \oplus V_5 \cdots \; x^2p^3}$$

V_1 ↑ 1.d. V_2 ↑ 2d V_3 ↑ 3d

$$sl_2 = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}$$



Generated by

V_1, V_2, V_3, V_4

What are relations?

Obvious relations:

$$[V_1, ?] = 0$$

$$[V_2, V_4] \subseteq V_3$$

......

+ Interesting : $\mathfrak{n}$ is gen. by V_4 , what are relations?

Feigin, 1980s,

Hijligenberg & Post 1991

1) $\Lambda^2 V_4 = \underset{\phi_1}{\boxed{V_1}} \oplus \underset{\phi_2}{V_5}$

$\phi_1 = 0$

2) $V_4 \otimes V_5 = \underset{\psi_1}{\boxed{V_2}} \oplus \underset{\psi_2}{V_4} \oplus \underset{\psi_3}{V_6} \oplus \underset{\psi_4}{\boxed{V_8}}$

$\psi_1 = 0$, $\psi_4 = 0$

3) $\Lambda^2 V_5 = \underset{\chi_1}{\boxed{V_3}} \oplus \underset{\chi_2}{V_7}$

$\chi_1 = 0$

Thm. ([HP]) $\mathbb{C}[x,p]_{\geq 3}$ is
is gen by V_4 with
rel: $\phi_1=0$, $\psi_1=0$, $\psi_4=0$
$\chi_1=0$.

~~Thm. The most general~~

Deformation preserving filtrations:

$$V_0 = \langle K \rangle$$

$$\phi_1 = -\frac{s_1}{2} K \leftarrow \boxed{\text{third param.}}$$

$$\psi_1 = 15 s_1 V_2$$ (for appropr. normaliz).

$$\psi_4 = 0$$

$$\chi_1 = 90 s_1 V_3 + 42 s_2 \cdot V_1 V_3 + 7 s_2 \cdot S^2 V_2$$

have some relation between them.

∃ scaling symm: essential par:

$$s_1^* = s_1 K^2, \quad s_2^* = s_2 K^3$$

Thm. This is our $Y_{k,\nu}$ with

$$s_1^* = (k^2+k+1)\nu^2 - k(k+1)\nu^3$$

$$s_2^* = k(k+1)\nu^3$$

Set $\dfrac{(s_1^*+s_2^*)^2}{s_1^{*2}} = u$

$$z = k^{-1} + k + 1$$

$$\boxed{z^3 - uz + u = 0}$$

Galois symm. over $\mathbb{C}(u)$

S_3 (Galois gp of this eq).

$\mathbb{C}(z) \supset \mathbb{C}(u)$ cubic ext. (not Galois)

$\mathbb{C}(k)$ – splitting field.

$$S_{12}(k, \nu) = (-k-1, \nu)$$

$$S_{23}(k, \nu) = \left(\frac{1}{k}, k\nu\right)$$

<u>Triality</u>: (for Toroidal alg.

$$\underset{\overset{\curvearrowleft}{S_3}}{q_1 q_2} q_3 = 1$$

---

if $\nu = N$ integer

$$Y_{k,\nu}/\underset{\nwarrow \boxed{\text{ideal}}}{I} = eH_k(n)e$$

<u>Deligne categories.</u>

$\uparrow$

Machine which allows you to make N a continuous parameter, in any problem involving classical groups?

S_N, GL_N, $\breve{O}_N$, Sp_N.

G group, supergroup

Rep G category of f.d. reps.

- linear/$\mathbb{C}$, artinian
  additive // abelian
  \+ Hom(X,Y) f.dim
  objects fin length.

mult.

- monoidal $\otimes$
- rigid $X \to X^*$
- symmetric :

$$\begin{tikzcd} X\otimes Y \arrow[r] \arrow[rr, bend right, "id"'] & Y\otimes X \arrow[r] & X\otimes Y \end{tikzcd}$$

- $\otimes$ bilinear. $f,g \mapsto f\otimes g$
- $\mathrm{End}(\mathbb{1}) = \mathbb{C}$

Such categories are called symmetric tensor cat. (STC)

Deligne: Moderate growth:

STC $\mathcal{C}$ has moderate growth if for any $X \in \mathcal{C}$ $\exists\, C_X \geq 1$ s.t.

$$\underset{\substack{\uparrow \\ \text{of comp. series.}}}{\text{length}}\,(X^{\otimes N}) \leq C_X^N.$$

Thm. (Deligne, 2002)
A STC of moderate growth is Rep G.
G supergroup.

More prec. have parity elt. $z \in G_0$
$z^2 = 1$.
Ad(z) – parity

Rep(G, z)
– reps of G so z acts

G, z uniquely determined.

by parity

Without moderate growth, not true, have also "interpolations" of Rep G.

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1) Rep GL_ν – interpol. of rep th of $GL_N(\mathbb{C})$ (Deligne–Milne, "Tannakian cat.", 1981)

Need to define Rep $GL_N(\mathbb{C})$ in a way that does not mention matrices.

$V = \mathbb{C}^N$ V^*, any irred repr is a direct summand in $V^{\otimes k}\otimes V^{*\otimes m}$ (Weyl)

$$[k,m] = V^{\otimes k}\otimes V^{*\otimes m}$$

$$\boxed{\mathcal{C} = \{[k,m]\}}$$

Rep $GL_N(\mathbb{C})$ = Karoubian closure of $\mathcal{C}$:

1) adjoin images of projectors
   if $p: X\to X$, $p^2=p$
   $\Rightarrow$ add object $\mathrm{Im}\, p$.
   (idempotent completion)
2) add finite direct sums.

$$\underset{X}{\underset{\shortparallel}{V^{\otimes k}\otimes V^{*\otimes m}}} = W\oplus W^\perp \underset{\oplus}{\supset} W$$

$$p:\underset{W\oplus W^\perp}{\underset{\shortparallel}{X}}\to X \qquad p|_W = 1 \qquad p|_{W^\perp}=0.$$

$$\mathrm{Hom}([k,m],[p,q])$$

$$= \mathrm{Hom}_{GL_N}(V^{\otimes k} \otimes V^{*\otimes m}, V^{\otimes p} \otimes V^{*\otimes q})$$

$$= \mathrm{Hom}_{GL_N}(V^{\otimes k+q}, V^{\otimes m+p})$$

$$= 0 \quad \text{if} \quad k+q \neq m+p$$

$$\iff k-m \neq p-q$$

otherwise:

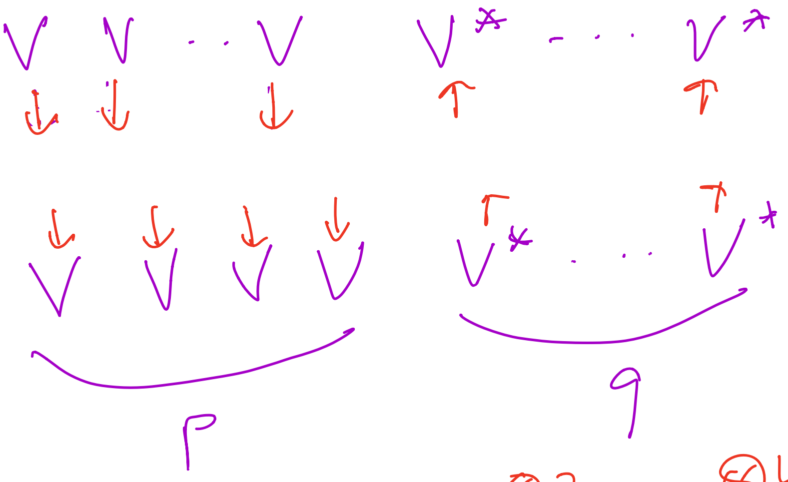
$\mathrm{Hom}_{GL_N}(V^{\otimes n}, V^{\otimes n})$

is generated by $\mathbb{C}S_n$.

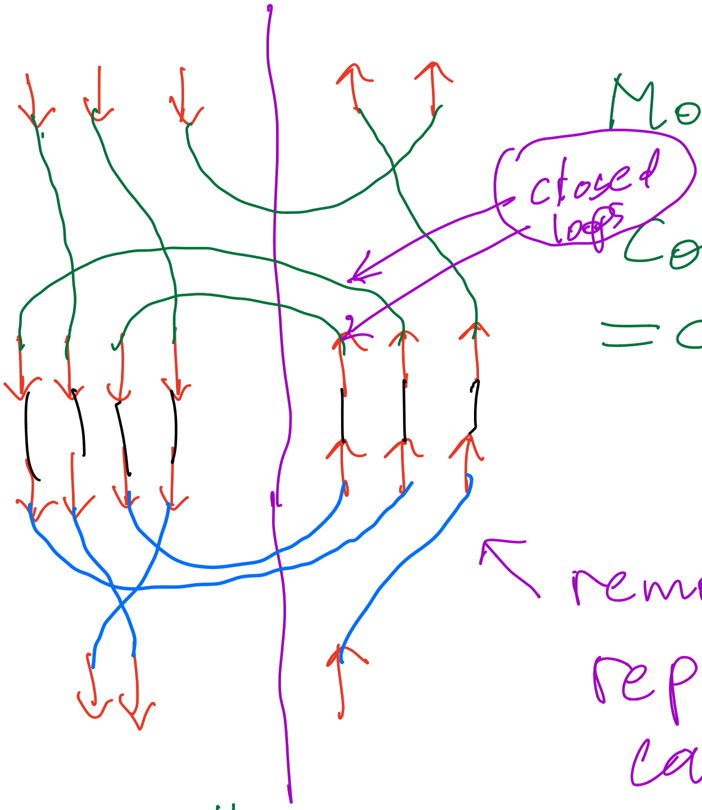
(coincides if $n \leq N$)

(Schur-Weyl duality)

We can draw this as pictures:



$\mathrm{Hom}(V^{\otimes 3} \otimes V^{*\otimes 2}, V^{\otimes 4} \otimes V^{*\otimes 3})$



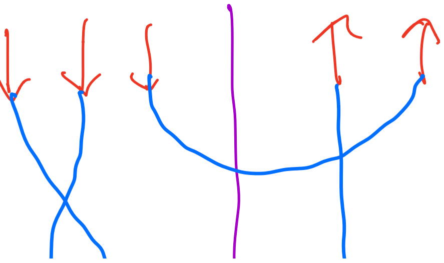
Morphisms

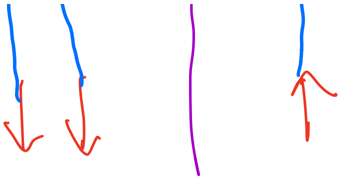
Comp. = concatenation.

remove loops replace each loop by N

$=$

$N^2 \cdot$



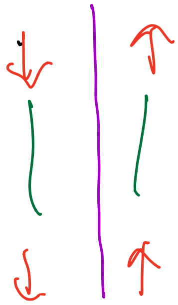


Now define $\widetilde{\mathrm{Rep}}\, GL_\nu$ with objects $[k,m]$ and morphisms as above (pictures), comp. as above with $N \mapsto \nu \in \mathbb{C}$.

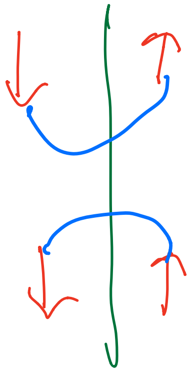
Def. $\mathrm{Rep}\, GL_\nu$ = Karoubian closure of $\widetilde{\mathrm{Rep}}\, GL_\nu$

$\mathrm{End}([k,m]) = \mathbb{C}S_{k+m}$ with a different product called Walled Brauer algebra $B_{k,m}(\delta)$

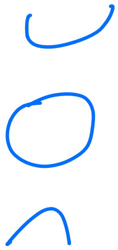
Ex. $B_{1,1}(\delta)$:



$= 1$



$= a$

$a^2 =$  $= \nu \cdot a$

$\Rightarrow$

$$B_{k,m}(\nu) = \mathbb{C}[a] \big/ a^2 = \nu \cdot a$$

Semisimple if $\nu \neq 0$

not if $\nu = 0$.

<u>Thm.</u> $B_{k,m}(\nu)$ are semisimple $\forall k,m \iff \nu \notin \mathbb{Z}$.

<u>Cor.</u> $\mathrm{Rep}\, GL_\nu$, $\nu \notin \mathbb{Z}$ is a semisimple STC.

No moderate growth:

$$X = \sum n_i \underset{\uparrow \text{simple}}{X_i} \qquad \ell(X) = \sum n_i$$

$$\mathrm{Hom}(X,X) = \bigoplus \mathrm{Mat}_{n_i}(\mathbb{C})$$

$$\dim \mathrm{Hom}(X,X) = \sum n_i^2$$

$$\ell(X) \geq \sqrt{\dim \mathrm{Hom}(X,X)}$$

$$\left(\sum n_i\right)^2 \geq \sum n_i^2$$

$$X = V^{\otimes n}$$

$$\mathrm{Hom}(X,X) = \mathbb{C}S_n$$

$$\ell(V^{\otimes n}) \geq \sqrt{n!} > C^n$$

$$\forall C.$$

$$n \gg 1.$$

$\nu \in \mathbb{Z}_{\geq 0}$:

$\mathrm{Rep}\, GL_\nu$ is not semis,
not abelian

Has a tensor ideal
of negligible morphisms
$\mathcal{N}eg$

$$\mathrm{Rep}\, GL_\nu / \mathcal{N}eg = \mathrm{Rep}_{cl}\, GL_n$$
$\uparrow$
classical.

Ex.

$eH_k(n)e$ = Hamiltonian
= quantum reduction
from diff. operators
on $\mathfrak{gl}_n$
(deformed HC homom.)
E-Ginzburg.

$eH_k(n)e$ is a quantiz.

of Calogero–Moser space

$$\{(X,Y) \mid [X,Y]+1 \text{ has rk } 1\} / GL_n$$

$n\times n$

$V = \mathbb{C}^n$

ADHM

$\wr\, C^\infty$

$W = \mathbb{C}$

$Hilb_n(\mathbb{C}^2)$

Nakajima variety

= reduction from space of rep of doubled quiver

Rep DQ

$$D(\operatorname{Rep} Q) /\!/ GL_N$$
$$\shortparallel$$
$$\mathfrak{gl}_N \times \mathbb{C}^N$$

$\rightsquigarrow$ We can interpolate this in Rep GL_ν

$\rightsquigarrow$ get $Y_{K,\nu}$.

(Costello's paper).

$\boxed{\text{skein categ.}}$ Kalinov

Can define

$\mathrm{Rep}_q\, GL_\nu$, parameters q and $a = q^\nu$

$\uparrow$

Braided $\otimes$ cat (semisimple if ν, q generic).

RT inv. of links

= Homfly pol of q, a.

$\mathrm{Rep}\, S_\nu$:

write $\mathrm{Rep}\, S_N$ without mentioning S_n.

$P = \mathbb{C}^N$ − perm. repr. as an S_n-mod.

$P^* = P$ algebra

∀ irr. repr is a dir. summand in $p^{\otimes n}$

$$\mathcal{C} = \langle \underset{\underset{[n]}{\parallel}}{p^{\otimes n}} \rangle$$

$\mathrm{Hom}([k],[p])$ for large N

$$= \mathrm{Hom}_{S_N}(p^{\otimes k}, p^{\otimes p})$$

$$= \mathrm{Hom}_{S_N}(\mathbb{C}, \underset{\underset{(p^{\otimes k+p})^{S_N}}{\parallel}}{p^{\otimes k+p}})$$

$r = k+p.$

$$= \mathrm{Fun}([1,N]^r)^{S_N}$$

basis corr. to S_N-orbits $\mathcal{O}$

on $(x_1, \dots, x_r)$

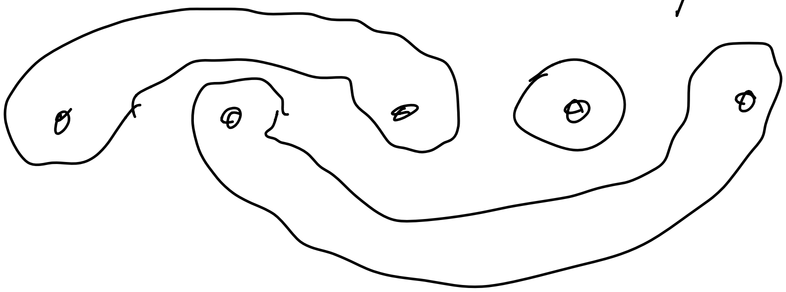
$x_i \in [1,N]$

$\mathcal{O} \longleftrightarrow$ equality patterns.

θ : $(x_1, x_2, x_3, x_4, x_5)$

$x_1 = x_3$, $x_2 = x_5$.

no other equalities.



Set partitions of $\{1, \dots, r\}$ into $\leq N$ subsets.

N large $\Rightarrow$ all set partit. occur.

$\mathrm{Hom}([K], [p]) = \mathbb{C}\mathcal{P}_{K+p}$

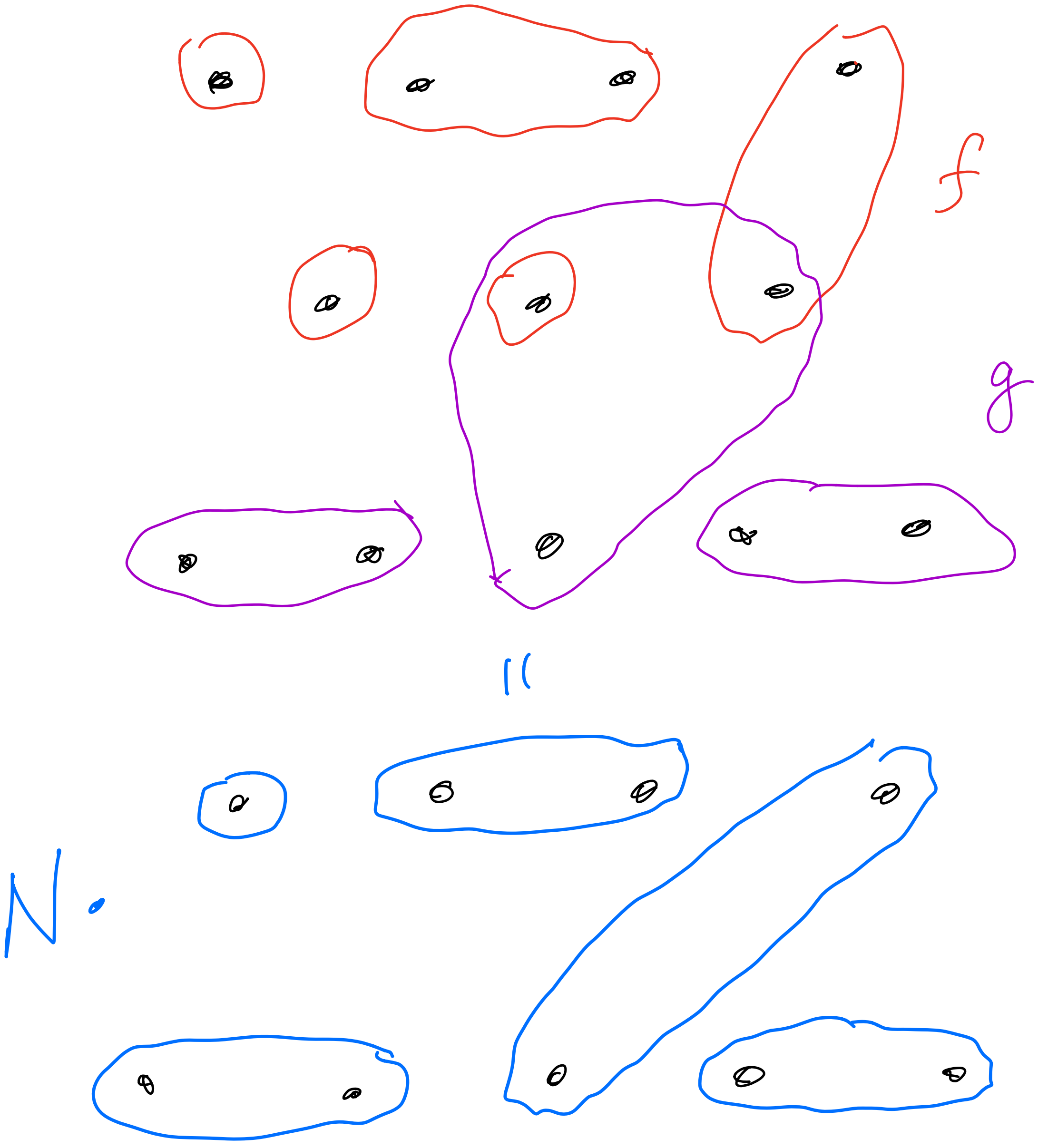
– set partitions of $\{1, \dots, K+p\}$.

How to compose pictures:

$$\begin{tikzcd} \theta \arrow[r] & \delta_\theta \arrow[d, squiggly] \\ & \varepsilon_\theta = \sum_{\theta' < \theta} \delta_{\theta'} \end{tikzcd}$$

nonstrict

$[4] \xrightarrow[f]{} [3] \xrightarrow[g]{} [5]$ eg, part.



$\widetilde{\mathrm{Rep}}\, S_\nu$ – the cat defined so with $N \mapsto \nu$

Rep S_ν — Karoubian completion

$\mathrm{End}([K]) = \mathrm{Par}_k(\nu)$

k-th partition algebra

Thm $\mathrm{Par}_k(\nu)$ semisimple

$\Longleftrightarrow \nu \neq 0, 1, 2, \dots$

$\Rightarrow$ If $\nu \neq 0, 1, 2, \dots$

$\Rightarrow \mathrm{Rep}\, S_\nu$ is a semisimple STC,

$\nu = 0, 1, 2, \dots \quad \nu = N$

$\mathrm{Rep}\, S_\nu \supset I$

$\mathrm{Rep}\, S_\nu / I = \mathrm{Rep}_{cl}(S_N)$.

Rem for $\mathrm{Rep}\, GL_\nu$, simple objects $V_{\lambda,\mu}$

λ, μ partitions

interpolates

$$V_{\underbrace{(\lambda_1,\dots,\lambda_k,0,\dots,0,-\mu_\ell,\dots,-\mu_1)}_{N}}$$

For Rep S_γ:

simples are

X_λ λ partition.

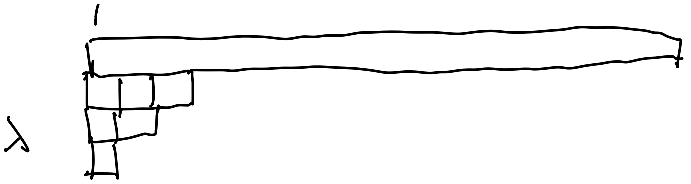
interpolates irrep

$$\pi(N-|\lambda|,\lambda_1,\lambda_2,\dots,\lambda_k)=\tilde{\lambda}(N)$$

of S_N

ex: $\lambda=(3,2,1)$

$$\underbrace{\hphantom{xxxxxxxxxxxxxxxxxxxx}}^{\,N-|\lambda|}$$



$\nwarrow$ N-padding of λ

$$\tilde{\lambda}(N)$$

Can define

$$H_\kappa(\nu)\text{-mod} =$$

$$\{ M \in \operatorname{Rep} S_\nu \mid$$

$$x : P \otimes M \longrightarrow M$$

$$y : P \otimes M \longrightarrow M$$

$$[x,x] = 0$$

$$[y,y] = 0$$

$$[y,x] = \cdots \sim$$

Classically! $eH_\kappa(\nu)e = \operatorname{End}_{H_\kappa(\nu)}(H_\kappa(\nu)e)$

$$eH_\kappa(\nu)e = \operatorname{End}_{H_\kappa(\nu)}(H_\kappa(\nu)e)$$

$$\parallel \qquad\qquad\qquad\qquad \uparrow$$

$Y_{K,\gamma}$ interpolate

<u>Cor.</u> If M is a $H_K(\gamma)$-module in cat $\mathcal{O}$ then $eM = \mathrm{Hom}_{\mathrm{Rep}\,S_\gamma}(\mathbb{I}, M)$ is a $Y_{K,\gamma}$-module.