Knot conormals, quantum curves, holomorphic disks, and quivers

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Overview

This talk reports on various joint works: with Shende, Kucharski, Longhi, Georgieva, Ng

Plan:

- Skein valued open GW-invariants
- Skein recursion for the toric brane and for knot conormals in the resolved conifold
- Some applications: quantum curves, basic holomorphic disks, and quivers.

  (Some of the material here is largely conjectural and uses results of many authors not appropriately referred to on these slides.)

Geometric setting:

- (X, ω) 3-dim symplectic Calabi-Yau, $c_1(X) = 0$.
  Main examples: $\mathbf{C}^3$, T^*S^3, and $\mathcal{O}(-1)^{\oplus 2} \to \mathbf{C}P^1$.
- $L \subset X$ Maslov zero Lagrangian.
  Main examples: toric brane, knot conormals, 0-section.

Holomorphic curves:

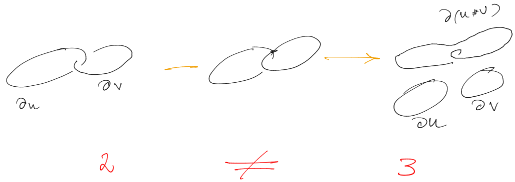
- J acs on X compatible with ω. (S, j) Riemann surface. *A holomorphic curve* is a map $u\colon (S, \partial S) \to (X, L)$ that solves the Cauchy-Riemann equation: $\bar{\partial}_J u = \frac{1}{2}(du + J \circ du \circ j) = 0$.
- The Cauchy-Riemann equation is Fredholm and the expected dimension of the moduli space of solutions is

$$(\dim_{\mathbf{C}} X - 3)\chi(S) + 2c_1^{\mathrm{rel}}(u^* TX) = 0 + 0.$$

- The A-model topological string partition function localizes on holomorphic curves.

Skeins on branes

The dimension count indicates after perturbation, moduli space of solutions is an oriented zero-manifold. Counting we compute the topological string partition function. The count should be invariant under deformations. For closed curves nodal solutions appear in codim 2, invariant under deformation. For open curves boundary nodes have codimension one, not invariant. There are invariant curve counts in this setting, in the skein.

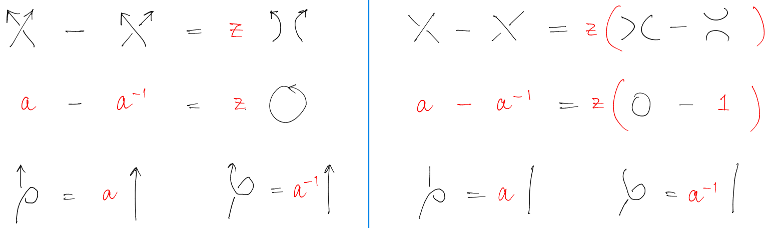


For general curves we use the HOMFLY skein. For curves invariant under an involution that fixes the Lagrangian we use the Kauffmann skein.

Skeins on branes

$\mathbb{Q}[a^{\pm}, z^{\pm}]$ - modules $\left(\text{or } \mathbb{Q}[a^{\pm}, q^{\pm 1/2}],\ z = q^{1/2} - q^{-1/2}\right)$

HOMFLY	Kauffman
oriented framed links	unoriented framed links



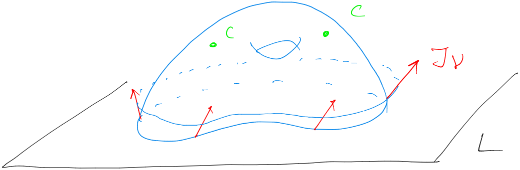
For example, $\mathrm{Sk}(S^3) = \mathbf{C}[q^{\pm 1}, a^{\pm 1}]$, $\mathrm{Sk}(S^1 \times \mathbf{R}^2)$ is a free commutative algebra on countably many generators A_m ($m - 1$ crossings, m times around).

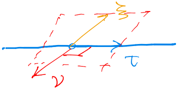
Skeins on branes

Bare curves: A stable map $u\colon S \to X$ is *bare* if all its irreducible components have positive symplectic area.

Skein valued curve counts are based on counting bare holomorphic curves by their boundary in the framed skein.

Auxiliary framing data: Generic vector field ξ on L and 4-chain C with $\partial C = 2 \cdot L$ and $\pm J \cdot \xi$ along the boundary.





$$\mathrm{lk}(L, u) = u_{J\nu} \cdot C$$

The skein valued curve count is then a sum over all disconnected bare holomorphic curves where the contribution of $u\colon (S,\partial S) \to (X,L)$ is

$$w(u)\, z^{-\chi(S)}\, a^{\mathrm{lk}(L,u)}\, \langle u(\partial S)\rangle \;\in\; \mathrm{Sk}(L)$$

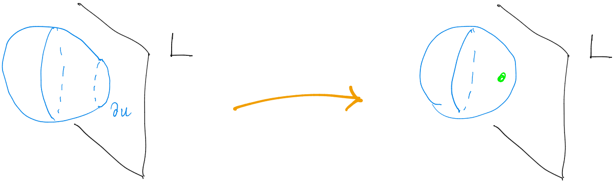
- $w(u)$ – rational weight of u as a weighted point in the moduli space
- $\chi(S)$ – Euler characteristic of S
- $\mathrm{lk}(L,u)$ – linking between u and L
- $\langle u(\partial S)\rangle$ – the boundary of u in the skein of L.

Skeins on branes – comments

The bare condition has to do with separating high and low energy contributions.

The 4-chain (or something similar) is needed for the deformation invariance of CS theory deformed by holomorphic curves:

$$\int \mathcal{D}A \; e^{\left(\frac{ki}{4\pi}\left(\mathrm{CS}+e^{-\mathrm{area}(u)} g_s^{-\chi(u)} \mathrm{tr}\left(P \exp \int_{\partial u} A\right)\right)+C\cdot u\right)}$$



With $z = (q - q^{-1})$ the bare curve count is like the first term (d=1) in the GV formula for the contribution to GW from a curve of Euler characteristic χ in homology class x:

$$\exp\left(\sum_{d=1}^{\infty} \frac{x^d}{d}(q^d - q^{-d})^{-\chi}\right)$$

Skeins on branes

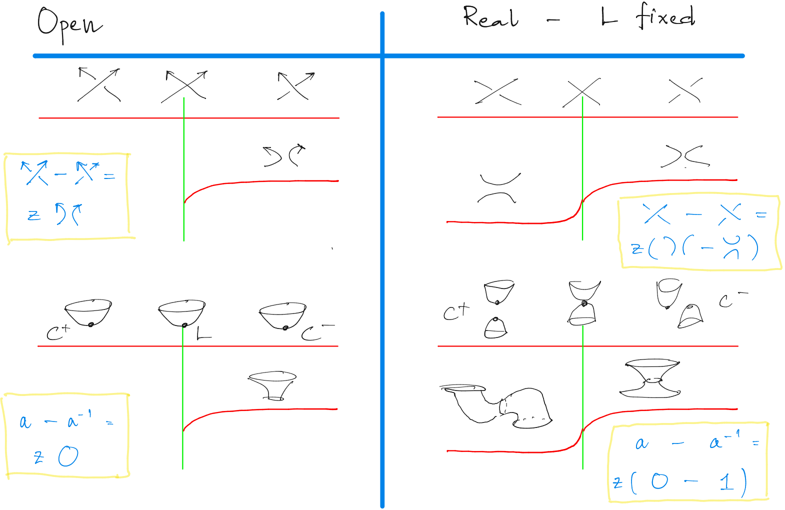
To show invariance one constructs a perturbation scheme that leaves constant maps unperturbed and shows.

1) Bare solutions transversely cut out, embeddings, tangent along boundary spans together with ξ a 2-plane everywhere.

2) Constant curves bubble off only in codimension $\geq$ two
$\Rightarrow$ for 1-parameter families, all solutions near boundary are bare with ghosts.

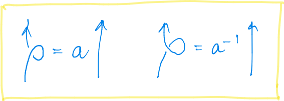
Skeins on branes

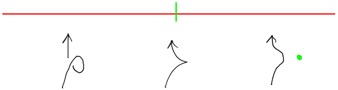
3) Degeneracies in 1-parameter families of solutions have standard form.



Skeins on branes

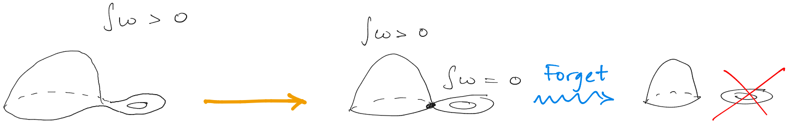
4) At tangencies with ξ a kink is traded for a 4-chain intersection.





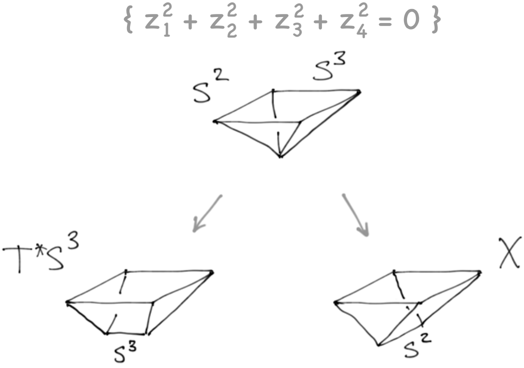
$\square$

Skein counts are inductive in Euler characteristic. Usual perturbative treatment is not.



Ooguri-Vafa large N duality

Geometric setup: $K \subset S^3$ – knot. $L_K \subset T^*S^3$ – Lagrangian conormal $\approx S^1 \times \mathbf{R}^2$. Shift L_K off of 0-section S^3 (non-exact). Transition to resolved conifold $X = \{\mathcal{O}(-1)^{\oplus 2} \to \mathbf{C}P^1\}$.



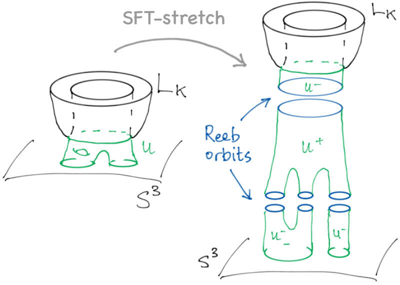
Ooguri-Vafa large N duality

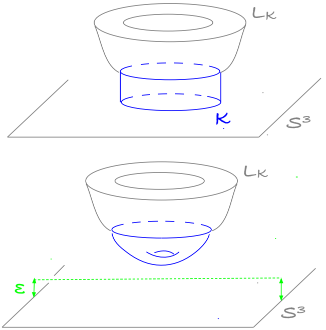
Theorem

The GW partition function equals the generating function for the colored HOMFLY:

$$\Psi_K(x, a, g_s) = \sum_{k \geq 0} H_{K,n}(a, e^{\frac{g_s}{2}})e^{nx},$$

Ooguri-Vafa large N duality

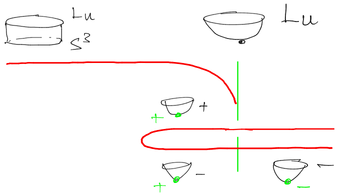




For a small shift of the conormal there is a unique holomorphic cylinder. SFT stretching removes all boundaries from the 0-section (outside curves asymptotic to Reeb orbits of index 2 gives negative dimension). Calculating the skein valued invariant gives the colored HOMFLY (obvious for once around, for many times we use info about the unknot). Curves in the stretched structure are the same as in the conifold for small area $\mathbf{C}P^1$. $\square$

Large N duality – comments

Moduli spaces for planar unknot.

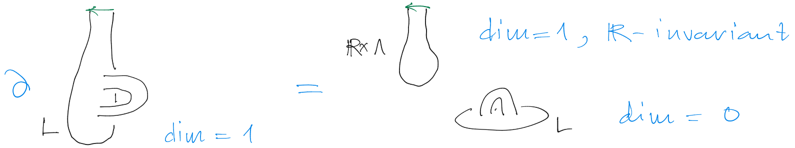


Real curves can be counted as in ordinary GW theory. For any knot K the count in the basic homology class in $H_2(T^*S^3; L_K \cup S^3)$ is one cylinder, i.e., 1. The skein count corresponds in the stretched picture to a count in $H_2(T^*S^3 \setminus S^3; L_K)$.

The toric brane in $\mathbf{C}^3$

The toric brane in $\mathbf{C}^3$ provides a universal model for 'crossing a basic disk' and illustrates how to calculate skein invariants 'from infinity'.

Strategy for curve counts from infinity: (X, L) has ideal contact boundary $(\partial X, \partial L)$. The boundary of 1-dimensional moduli spaces consists of $\mathbf{R}$-invariant curves at infinity and rigid curves in the bulk. The boundary is zero in the skein. The outside then determines the inside.



The toric brane in $\mathbf{C}^3$

$\mathbf{C}^3$ with coordinates $z = (z_1, z_2, z_3)$.

$\mathbf{C}^3 \to \mathbf{R}^3$, $z \mapsto (r_\alpha(z), r_\beta(z), r_\gamma(z))$,

$r_\alpha(z) = |z_1|^2 - |z_2|^2, \quad r_\beta(z) = |z_2|^2 - |z_3|^2, \quad r_\gamma(z) = \mathrm{Im}(z_1 z_2 z_3).$

Fiber at $(r_\alpha, r_\beta, r_\gamma)$:

$$
\begin{aligned}
&|z_1|^2 = r_\alpha + |z_3|^2, \qquad |z_2|^2 = r_\beta + |z_3|^2, \\
&\mathrm{Im}(|z_1||z_2||z_3| e^{i(\theta_1+\theta_2+\theta_3)}) = r_\gamma
\end{aligned}
$$

$\Rightarrow$ generic fiber $T^2 \times \mathbf{R}$.

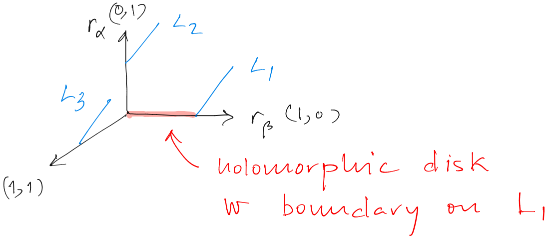
The toric brane in $\mathbf{C}^3$

Lagrangians L_1, L_2, $L_3 \approx S^1 \times \mathbf{R}^2$.

$$L_1: \quad r_\alpha = 0,\ r_\beta = r_1^*,\ r_\gamma \geq 0 \text{ and } \mathrm{Re}(z_1 z_2 z_3) = 0,$$
$$L_2: \quad r_\beta = 0,\ r_\alpha = r_2^*,\ r_\gamma \geq 0 \text{ and } \mathrm{Re}(z_1 z_2 z_3) = 0,$$
$$L_3: \quad r_\alpha - r_\beta = 0,\ r_\alpha = r_3^*,\ r_\gamma \geq 0 \text{ and } \mathrm{Re}(z_1 z_2 z_3) = 0,$$

We restrict attention to L_1 and parameterize it

$$\left(|z_3|e^{i\theta}, (|z_3| + r_1^*)e^{i\phi}, |z_3|e^{-i(\theta+\phi-\frac{\pi}{2})}\right)$$



The toric brane in $\mathbf{C}^3$

As $|z_3| \to \infty$, L_1 is asymptotic to the $\mathbf{R}$-invariant Lagrangian

$$\left(\rho e^{i\theta}, \rho e^{i\phi}, \rho e^{-i(\theta+\phi-\frac{\pi}{2})}\right).$$

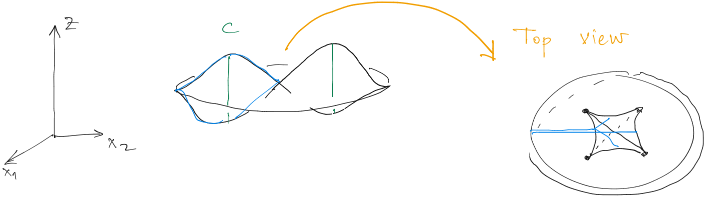
Consider the image under the Hopf map $S^5 \to \mathbf{C}P^2$:

$$[e^{i(2\theta+\phi)} : e^{i(2\phi+\theta)} : i].$$

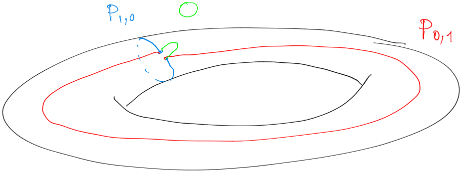
A Clifford torus and the Legendrian ∂L_1 is a 3-fold cover (Bohr-Sommerfeld). The Reeb chords of ∂L_1 are Bott degenerate and come in T^2-families, length $k\frac{2\pi}{3}$, index ≥ 1, with equality only for $k = 1$.

The toric brane in $\mathbf{C}^3$

To find holomorphic curves one can either use curves on the Clifford torus or draw the front of ∂L_1 in $\mathbf{R}^5 \subset S^5$:



3 - disks, boundaries as follows



The toric brane in $\mathbf{C}^3$

We learn then that the skein valued curve count Ψ on the toric brane satsifies the operator equation:

$$(\bigcirc - P_{1,0} - P_{0,1})\,\Psi = 0.$$

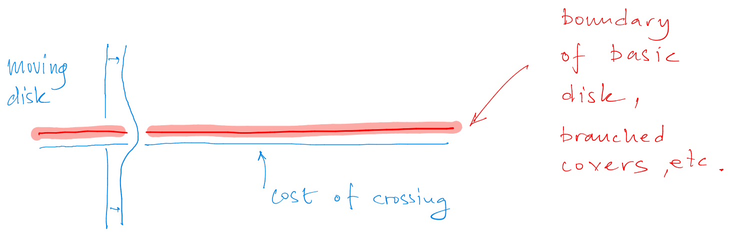
The operators $\bigcirc - P_{1,0}$ and $P_{0,1}$ have a common eigen-basis in the positive skein W_λ where λ runs over partitions of positive integers. The operator equation has a unique solution in Sk^+:

$$\Psi = \sum_{\lambda} W_\lambda \prod_{\square \in \lambda} \frac{q^{-c(\square)/2}}{q^{h(\square)/2} - q^{-h(\square)/2}},$$

where h is the hook length and c the content, here we use $z = q^{1/2} - q^{-1/2}$.

Interpretation of the equation

$$(\bigcirc - P_{1,0} - P_{0,1})\,\Psi = 0.$$

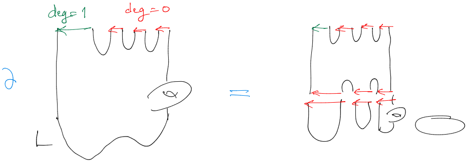


Conormals of knots

Consider the conormal $L_K \subset T^*S^3$ of a knot in the resolved conifold. If $K = U$ there is similarly an immediate recursion relation of the form

$$\left(\bigcirc - P_{1,0} - P_{0,1} + a^2 P_{1,1}\right)\Psi_U = 0.$$

For more general knots the recursion will appear from a skein valued Legendrian SFT. Schematically, the boundary of a 1-dimensional moduli space looks as follows:



The recursion relation will then appear after the degree 0 chords have been eliminated from the equation.

Generalized curves

The standard approach to open GW invariants with one copy of the Lagrangian correspond to $U(1)$ gauge theory and in the case of bare curves to $a = q = e^{g_s}$ after projection to 'homology + linking', we call them *generalized curves*.

$$w(u)\, z^{-\chi(S)}\, a^{\mathrm{lk}(L,u)}\, \langle u(\partial S)\rangle \;\in\; \mathrm{Sk}(L) \to$$

$$w(u)\, (q - q^{-1})^{-\chi(S)}\, q^{\mathrm{lk}(L,u)}\, [u(\partial S)] \;\in\; \mathbf{Q}[q^{\pm}][\widehat{H_2(X,L)}]$$

E.g., for the toric brane and the unknot conormal the recursion relations then read:

$$(1 - e^{\hat{x}} - e^{\hat{p}})\psi(x) = 0, \quad \psi(x) = \sum_k (q^2; q^2)_k^{-1} e^{kx},$$

$$(z; q^2)_k = (1 - z)(1 - zq^2) \dots (1 - zq^{2(k-1)}),$$

$$(1 - e^{\hat{p}} - e^{\hat{x}} + a^2 e^{\hat{x}} e^{\hat{p}})\Psi(x) = 0, \quad \Psi(x) = \sum_k C_k(a, q) e^{kx},$$

where x generates $H_1(L)$ and $\log a$ generates $H_2(X)$, $e^{\hat{x}}$ is multiplication by e^x and $e^{\hat{p}} = e^{g_s \partial_x}$.

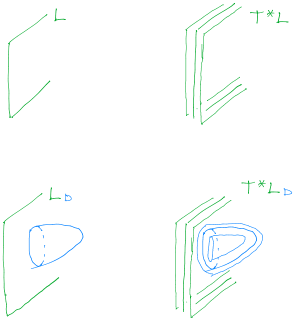
Generalized curves

For the trefoil T, $\widehat{A}_T \Psi_T(x) = 0$,

$$
\begin{aligned}
\widehat{A}_T(e^x, e^p, a, q) = {} & q a^6 e^{3p} (a^2 - q^{-3} e^{2p})(a^2 - q^{-1} e^p) \\
& + q^{-5/2} (a^2 - q^{-2} e^{2p}) \big((q^2 e^{2p} + q^3 e^{2p} - q^3 e^p + q^4) a^4 \\
& \qquad - (q e^{3p} + q^3 e^{2p} + q e^{2p}) a^2 + e^{4p} \big) \, e^x \\
& + (a^2 - q^{-1} e^{2p})(e^p - q) e^{2x}.
\end{aligned}
$$

Basic holomorphic disks and quivers

It was observed that the generating function for the colored HOMFLY can be written as a quiver partition function for a symmetric quiver. The geometry behind such expressions can be understood if we assume that there is a finite set of basic holomorphic disks (the quiver nodes) attached to L_K such that all holomorphic curves lie in a neighborhood of $L_K \cup \{\text{basic disks}\}$.



Basic holomorphic disks and quivers

As for generalized curves, we must keep track of the linking number between disks to count generalized curves. The result is an expression of the following form:

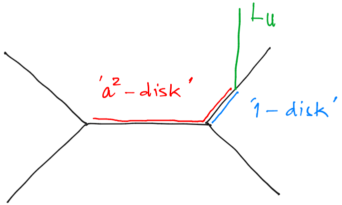
$$\begin{aligned}\Psi_K(e^x, a, q) &= \psi\left(e^{x_1} e^{\sum_{j=1}^n C_{1j}\, g_s \partial_{x_j}}\right) \cdots \psi\left(e^{x_m} e^{\sum_{j=1}^n C_{mj}\, g_s \partial_{x_j}}\right) \\ &= \sum_{(d_1,\ldots,d_m)\in \mathbf{Z}_+^m} (-q)^{\sum_{ij} C_{ij} d_i d_j} \prod_{j=1}^m \frac{e^{d_j x_j}}{(q^2, q^2)_{d_j}},\end{aligned}$$

$$\text{where} \qquad e^{x_i} = q^{n_i} a^{k_i} e^{l_i x}.$$

Geometric characters of nodes: C_{ij} is linking between disks i and j, C_{ii} self-linking or framing data for attaching the disk, n_i is 4-chain intersections (invariant self-linking minus framing), (k_i, l_i) homology class in $H_2(X, L_K)$.

Basic holomorphic disks and quivers

For the unknot the desired form can be obtained from toric geometry.

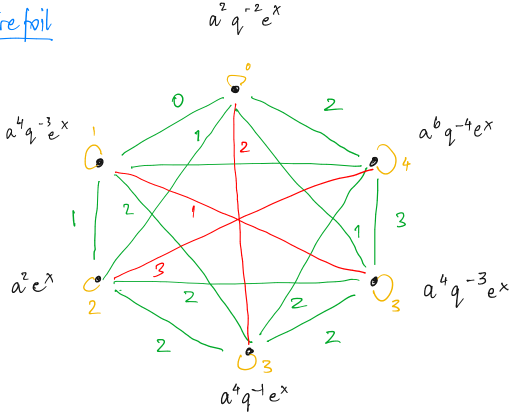


For conormals of other knots the quiver picture might come from viewing their conormals as a 'cover' or the unknot conormal.

Basic holomorphic disks and quivers

Unknot

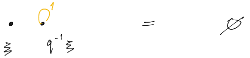
Trefoil

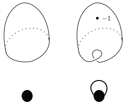


Non-uniqueness of quivers

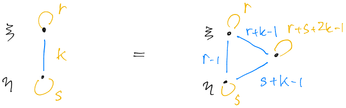
Different quivers can give rise to the same partition function.
There are two main sources.

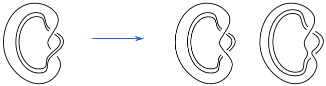
<u>Canceling pairs</u>



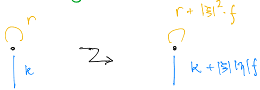


<u>Multi-cover skein</u>





<u>Framing change</u>



Basic holomorphic disks and quivers

In the right variables, the recursion relation for the quiver partition function takes a familiar form. Consider quiver variables x_i with duals y_i and symmetric quiver matrix C_{ij}.

1) Unlinking: $X_i' = (-q)^{C_{ii}} x_i \prod_{j=1}^{m} y_j^{C_{ij}} \Rightarrow X_i'X_j' = X_j'X_i'$.

2) Nesting: $Z_i = X_i' \prod_{j<i} y_j \Rightarrow Z_jZ_i = q^{2\mathrm{sgn}(j-i)} Z_iZ_j$.

3) Toric brane variable: $Z = Z_1 + \cdots + Z_m \Rightarrow$

$$Z^r = \sum_{|\mathbf{d}|=r} \begin{bmatrix} r \\ \mathbf{d} \end{bmatrix}_{q^2} Z_1^{d_1} \ldots Z_m^{d_m}, \qquad \begin{bmatrix} r \\ \mathbf{d} \end{bmatrix}_{q^2} = \frac{(q^2;q^2)_r}{(q^2;q^2)_{d_1} \ldots (q^2;q^2)_{d_m}}$$

Let

$$\mathbb{P} = \sum_{r=0}^{\infty} \frac{1}{(q^2;q^2)_r} Z^r \quad \Rightarrow \quad :\mathbb{P}: = \Psi(x_1, \ldots, x_m).$$

4) The meridian of L_K is $Y = \prod_{j=1}^{m} y_k$ and $YZ = q^2 ZY$. Since $Z^r = Z \cdot Z^{r-1}$ we have

$$(1 - Y - Z)\mathbb{P} = 0.$$

In general there are also basic disks with $x_i = x^{n_i}$, $n_i \neq 1$. The formula above generalizes with $Z^{(k)} = \sum_{n_i \geq k}$ and $Y = \prod_{j=1}^{m} y_j^{n_j}$ to total meridian.

$$Y\mathbb{P} = (1 - q^{2(n_{\max}-1)} Z^{(n_{\max})}) \dots (1 - Z^{(1)})\mathbb{P}$$

Basic holomorphic disks and quivers – observations and conjectures

Refined partition function:

$$\Psi_K(e^x, a, q, t) \;=\; \sum_{(d_1,\ldots,d_m)\in\mathbf{Z}_+^m} (-q)^{\sum_{ij} C_{ij} d_i d_j} \prod_{j=1}^{m} \frac{e^{d_j x_j}}{(q^2, q^2)_{d_j}},$$

$$\text{where} \qquad e^{x_i} = (-t)^{C_{ii}} q^{l_i} a^{k_i} e^{n_i x}.$$

In examples the n^{th} symmetrically colored HOMFLY homology can be computed from the refined partition function as the number of n-vortices. Concretely, the monomials of $(q^2; q^2)_n^{-1}$ which are the coefficients of e^{nx}.

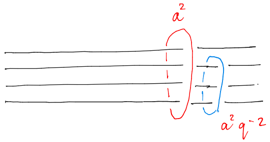
Basic holomorphic disks and quivers – observations and conjectures

To get to sl(N) homologies, $a = q^N$. There is a differential acting on HOMFLY homology coming from multiplication by closed BPS states in the conifold. The action can be viewed as unlinking of a small fiber circle with a disk with $a^2 q^{-2N} t$.

Basic holomorphic disks and quivers – observations and conjectures

Conjecture (E,Kucharski,Longhi)

The partition function of any knot conormal has the form of a generating function of a finite quiver. The quiver nodes come in unknot pairs and the pairs come in 'sl(1)-pairs' and such a quiver representation is unique.



Trefoil