# Geometry of Bethe equations and q-opers

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# Some history of quantum integrable systems and Bethe ansatz





R.P. Feynman: "I got really fascinated by these (1+1)-dimensional models that are solved by the Bethe ansatz and how mysteriously they jump out at you and work and you don't know why. I am trying to understand all this better."

Exactly solvable models of statistical physics: spin chains, vertex models

1930s: H. Bethe: Bethe ansatz solution of Heisenberg model

1960-70s: R.J. Baxter, C.N. Yang: Yang-Baxter equation, Baxter operator

1980s: Development of QISM by Leningrad school, leading to the discovery of quantum groups by Drinfeld and Jimbo

Since 1990s: textbook subject and an established area of mathematics and physics

# Geometric interpretation I: Quantum Cohomology, Quantum K-theory





Motivated by ideas of Dubrovin and Witten, Givental and collaborators pointed out the relations of quantum cohomology, quantum K-theory to integrability, particularly, to many-body systems.

In the past decade, enormous progress in this direction achieved by Okounkov and his school: in the case of quantum K-theory using a quasimap approach and quantum group/integrable structures.

D. Maulik, A. Okounkov, *Quantum Groups and Quantum Cohomology*, Astérisque, 408, 2019, arXiv:1211.1287

A. Okounkov, *Lectures on K-theoretic computations in enumerative geometry*, arXiv:1512.07363

M. Aganagic, A. Okounkov, *Quasimap counts and Bethe eigenfunctions*, Mosc. Math. J. 17 (2017) 565-600, arXiv:1704.08746

P.Pushkar, A. Smirnov, A.Z., *Baxter Q-operator from quantum K-theory*, Adv. Math. 360 (2020) 106919 arXiv:1612.08723

P. Koroteev, P.Pushkar, A. Smirnov, A.Z., *Quantum K-theory of Quiver Varieties and Many-Body Systems*, arXiv:1705.10419

In this talk we mainly will focus on:

q-deformed version of the classic example of geometric Langlands correspondence, studied in detail by E. Frenkel and his collaborators: correspondence between opers (certain connections with regular singularities) and Gaudin models.

P. Koroteev, D. Sage, A. Z., *(SL(N),q) -opers, the q-Langlands correspondence, and quantum/classical duality*, arXiv:1811.09937

E. Frenkel, P. Koroteev, D. Sage, A.Z., *q-opers, QQ-systems and Bethe ansatz*, arXiv:2002.07344

# Outline

Quantum groups and Bethe ansatz

Quantum equivariant K-theory and Bethe ansatz

QQ-systems and Bethe ansatz. Gaudin model and opers.

$(G, \hbar)$-opers and QQ-system

$(SL(r+1), \hbar)$-opers and QQ-systems

Quantum-classical duality via $(SL(r+1), \hbar)$-opers

$\hbar$-Opers for toroidal algebra

# Affine algebras and finite-dimensional modules





Let us consider Lie algebra $\mathfrak{g}$.

The associated *loop algebra* is $\hat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}]$ and $t$ is known as *spectral parameter*.

The following representations, known as *evaluation modules*, form a tensor category of $\hat{\mathfrak{g}}$:

$$V_1(a_1) \otimes V_2(a_2) \otimes \cdots \otimes V_n(a_n),$$

where

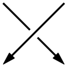
- $V_i$ are representations of $\mathfrak{g}$
- $a_i$ are values for $t$

Quantum group

$$U_\hbar(\hat{\mathfrak{g}})$$

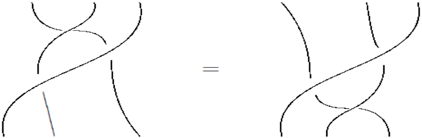
is a deformation of $U(\hat{\mathfrak{g}})$, with a nontrivial intertwiner $R_{V_1,V_2}(a_1/a_2)$:

$$V_1(a_1) \otimes V_2(a_2)$$



$$V_2(a_2) \otimes V_1(a_1)$$

which is a rational function of $a_1, a_2$, satisfying Yang-Baxter equation:



The generators of $U_\hbar(\hat{\mathfrak{g}})$ emerge as matrix elements of $R$-matrices (the so-called FRT construction).

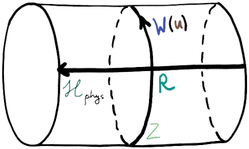
# Baxter algebra and Integrability

Source of integrability: commuting *transfer matrices*, generating *Baxter algebra* which are weighted traces of

$$\tilde{R}_{W(u),\mathcal{H}_{phys}} : W(u) \otimes \mathcal{H}_{\rm phys} \to W(u) \otimes \mathcal{H}_{\rm phys}$$

over auxiliary $W(u)$ space:

$$T_{W(u)} = \mathrm{Tr}_{W(u)}\Big(M(u)\Big) = \mathrm{Tr}_{W(u)}\Big((Z \otimes 1)\ \tilde{R}_{W(u),\mathcal{H}_{phys}}\Big)$$




Here $Z \in e^{\mathfrak{h}}$, where $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra.

Integrability:

$$[T_{W'(u')}, T_{W(u)}] = 0$$

There are special transfer matrices called *Baxter Q-operators*. Such operators generate entire Baxter algebra.

Primary goal for physicists is to diagonalize $\{T_{W(u)}\}$ simultaneously.

# $\mathfrak{g} = \mathfrak{sl}(2)$: XXZ spin chain



Textbook example is XXZ Heisenberg spin chain:

$$\mathcal{H}_{XXZ} = \mathbb{C}^2(a_1) \otimes \mathbb{C}^2(a_2) \otimes \cdots \otimes \mathbb{C}^2(a_n)$$

States:

$$\uparrow\uparrow\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow\uparrow\uparrow\downarrow\downarrow\uparrow\uparrow\uparrow$$

Here $\mathbb{C}^2$ stands for 2-dimensional representation of $U_\hbar(\widehat{\mathfrak{sl}}_2)$.

Algebraic method to diagonalize transfer matrices:

Algebraic Bethe ansatz

as a part of Quantum Inverse Scattering Method developed in the 1980s.

# Bethe equations and Q-operator





The eigenvalues are generated by symmetric functions of Bethe roots $\{x_i\}$:

$$\prod_{j=1}^{n} \frac{x_i - a_j}{\hbar a_j - x_i} = z\, \hbar^{-n/2} \prod_{\substack{j=1 \\ j \neq i}}^{k} \frac{x_i \hbar - x_j}{x_i - x_j \hbar}, \quad i = 1, \ldots, k,$$

so that the eigenvalues $\mathcal{Q}(u)$ of the $Q$-operator are the generating functions for the elementary symmetric functions of Bethe roots:

$$\mathcal{Q}(u) = \prod_{i=1}^{k} (u - x_i)$$

A real challenge is to describe representation-theoretic meaning of $Q$-operator for general $\mathfrak{g}$ (possibly infinite-dimensional).

# qKZ-equations





Modern way of looking at Bethe ansatz: solving q-difference equations for

$$\Psi(z_1, \dots, z_k; a_1, \dots, a_n) \in V_1(a_1) \otimes \cdots \otimes V_n(a_n)[[z_1, \dots, z_k]]$$

known as quantum Knizhnik-Zamolodchikov (aka I. Frenkel-Reshetikhin) equations:

$$\Psi(qa_1, \dots, a_n, \{z_i\}) = (Z \otimes 1 \otimes \cdots \otimes 1) R_{V_1,V_n} \dots R_{V_1,V_2} \Psi$$
$$+$$
$$\text{commuting } q-\text{difference equations in } z-\text{variables}$$

Here $\{z_i\}$ are the components of twist variable $Z$.

The latter series of equations are known as dynamical equations, studied by Etingof, Felder, Tarasov, Varchenko, ...

In $q \to 1$ limit we arrive to an eigenvalue problem. Studying the asymptotics of the corresponding solutions we arrive to Bethe equations and eigenvectors.

## First geometric interpretation: enumerative geometry of Nakajima varieties





Conjecture of Nekrasov and Shatashvili '09 (through 3D gauge theory):

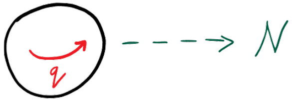
$$\text{Quantum K}-\text{theory ring of Nakajima variety} =$$

$$\text{symmetric polynomials in } x_{i_j} \ / \ \text{Bethe equations}$$

Okounkov'15, Okounkov-Smirnov'16:

$$\text{q}-\text{difference equations for vertex functions} =$$
$$\text{qKZ equations} + \ \text{dynamical equations}$$

through the study of quasimap moduli spaces for Nakajima varieties:



# Simplest example: $T^*Gr(k,n)$





$$N_{k,n}=T^*Gr(k,n)=T^*\mathcal{M}////GL(V), \quad \sqcup_k N_{k,n}=N(n).$$

Deformation of the product: $A \circledast B = A\otimes B+\sum_{d=1}^{\infty} A\otimes_d B\ z^d$.

Quantum tautological classes –deformations of

$$\tau = T^*\mathcal{M}\times\tau(V)////GL(V), \quad \tau\in K_{GL(V)}(\cdot)=S(x_1^{\pm1},x_2^{\pm1},\dots x_k^{\pm1}):$$

$$\hat{\tau}(z)=\tau+\sum_{d=1}^{\infty}\tau_d z^d\in K_T(N(n))[[z]]$$

**Theorem.** [P. Pushkar, A. Smirnov, A.Z. '16]

1. The eigenvalues of operators of quantum multiplication by $\hat{\tau}(z)$ are given by the values of the corresponding Laurent polynomials $\tau(x_1,\dots,x_k)$ evaluated at the solutions of XXZ Bethe equations:

$$\prod_{j=1}^{n}\frac{x_i-a_j}{\hbar a_j-x_i}=z\,\hbar^{-n/2}\prod_{\substack{j=1\\ j\neq i}}^{k}\frac{x_i\hbar-x_j}{x_i-x_j\hbar}, \quad i=1\cdots k,$$

2. Baxter Q-operator: $Q(u)=\sum_{i=1}^{k}(-1)^i u^{k-i}\left[\Lambda^i V\right](z)\circledast$

Another modern view on Bethe ansatz one can find in the papers of D. Hernandez and E. Frenkel, following earlier papers by Bazhanov, Lukyanov and Zamolodchikov.

Extension of the category of representations of $U_\hbar(\hat{\mathfrak{g}})$ by representations of Borel subalgebra gives rise to the so-called QQ-systems, which serve as the relations in the Grothendieck ring.

In the case of $U_\hbar(\widehat{\mathfrak{sl}}(2))$ the QQ-system is:

$$z\widetilde{Q}(\hbar u)Q(u) - z^{-1}Q(\hbar u)\widetilde{Q}(u) = \prod_i (u - a_i)$$

Here $Q(u)$ can be viewed as an eigenvalue of the Q-operator.

# General form of QQ-system

For Lie algebra $\mathfrak{g}$ of rank $r$ we have:

$$\widetilde{\xi}_i Q^i_-(u)Q^i_+(\hbar u) - \xi_i Q^i_-(\hbar u)Q^i_+(u) = \Lambda_i(u)\prod_{j\neq i}\left[\prod_{k=1}^{-a_{ij}} Q^j_+(\hbar^{b^k_{ij}}u)\right]$$

$$i = 1,\dots,r, \quad b^k_{ij}\in\mathbb{Z}$$

Here polynomials $\Lambda_i(u)$ characterize the representation $U_\hbar(\hat{\mathfrak{g}})$ and $\xi_i$, $\widetilde{\xi}_i$ are related to $Z$.

Upon certain nondegeneracy conditions there is 1-to-1 correspondence between solutions of the $QQ$-system and Bethe ansatz equations.

# Classical limit: Gaudin model and opers





Gaudin model is a (semi)classical limit of our quantum group models (Sklyanin'89):

$$R(u) = 1 + \eta r(v) + O(\eta^2),$$
$$M(u) = 1 + \eta L(v) + O(\eta^2),$$
$$[L^1(v_1), L^2(v_2)] = \left[r^{12}(v_1 - v_2), L^1(v_1) + L^2(v_2)\right]$$

Gaudin Hamiltonians:

$$H_k = \sum_{j \neq k} \sum_{c} \frac{t_k^c \otimes t_j^c}{\mathfrak{a}_k - \mathfrak{a}_j} + \mathcal{Z}_k = Res_{\mathfrak{a}_k} \mathrm{tr}\left[(L(v))^2\right]$$

Geometric description of the spectrum via $G^L$-oper connections (special type of connections on a principal bundle over $\mathbb{P}^1$):

**Theorem** (E. Frenkel'03) There is 1-to-1 correspondence between the spectrum of Gaudin model for Lie algebra $\mathfrak{g}$ and nondegenerate Miura $G^L$-oper connections on $\mathbb{P}^1$ with regular singularities and trivial monodromy.

(case $\mathcal{Z} = 0$)

# Roots of this correspondence



Gaudin model eigenvalue problem is a critical level limit of Knizhnik-Zamolodchikov equations:

$$(k + h^{\vee})\partial_{\mathfrak{a}_i}\Phi(\mathfrak{a}_1, \mathfrak{a}_2, \dots, \mathfrak{a}_n) = H_i\Phi(\mathfrak{a}_1, \mathfrak{a}_2, \dots, \mathfrak{a}_n),$$

$$\Phi(\mathfrak{a}_1, \mathfrak{a}_2, \dots, \mathfrak{a}_n) \in V_1(\mathfrak{a}_1) \otimes \dots \otimes V_n(\mathfrak{a}_n)[[z]]$$

Feigin, E. Frenkel'92:

Completion of the center of $U(\hat{\mathfrak{g}})$ at the critical level is isomorphic to Gelfand-Dikii algebra associated to ${}^L\mathfrak{g}$, i.e. Poisson algebra of $Fun(Op_{{}^L\mathfrak{g}}(D^{\times}))$ (classical limit of W-algebra).

Feigin, E. Frenkel, Reshetikhin'94:

Explicit construction of eigenvectors of KZ equation using Wakimoto modules. Obtained Bethe equations via Miura transformations.

# Quantum Geometric Langlands correspondence



This lead to the proposed quantum Langlands correspondence between conformal blocks (correlation functions) of W-algebra $W({}^L\mathfrak{g})$ and WZW model associated to $\hat{\mathfrak{g}}$.

Correlation functions of $W({}^L\mathfrak{g})$ are subject to linear differential (ΨDO in general) equations with singularities.

In a particular case of $\mathfrak{sl}_2$ ($W(\mathfrak{sl}_2) = Vir$) it is a linear Sturm-Liouville problem with prescribed singularities of second order, known as BPZ (Belavin, Polyakov, Zamolodchikov'84) equation.

In $c \to \infty (k \to -h^\vee)$ limit these differential operators are:

$$\partial_v^2 - \sum_{i=1}^n \frac{\lambda_i(\lambda_i+2)/4}{(v-\mathfrak{a}_i)^2} - \sum_{i=1}^n \frac{c_i}{u-\mathfrak{a}_i}, \quad c_i = \lambda_i\Big(\sum_{j\neq i}\frac{\lambda_i}{\mathfrak{a}_i-\mathfrak{a}_j} - \sum_{j=1}^r \frac{1}{\mathfrak{a}_i - w_j}\Big)$$

naturally appear from Miura oper connections with regular singularities:

$$\partial_v - \begin{pmatrix} \sum_j \frac{1}{v-w_j} & \prod_{i=1}^n (v-\mathfrak{a}_i)^{\lambda_i} \\ 0 & -\sum_j \frac{1}{v-w_j} \end{pmatrix}$$

via the Drinfeld-Sokolov reduction.

# Quantum Geometric q-Langlands Correspsondence

In 2017 Aganagic, E. Frenkel and Okounkov introduced a q-deformed version of quantum Langlands correspondence and proved it in ADE case, explicitly identifying conformal blocks for $U_\hbar(\mathfrak{g})$ and $W_{q,t}({}^L\mathfrak{g})$.

Conformal blocks for $U_\hbar(\mathfrak{g})$ satisfy I. Frenkel-Reshetikhin (qKZ) equations.

Conformal blocks for $W_{q,t}({}^L\mathfrak{g})$ are satisfying some difference equations ($\hbar$-difference in $q \to 1$ limit: $t \to \hbar^{-1}$).

A natural question Igor could ask Edward:

What is the geometric meaning of such $\hbar$-difference equations when $q \to 1$ (critical level)?

# Path to answer: classical QQ-system and Miura G-opers

Bethe equations of Gaudin model can be related with what we call a polynomial solution of the classical $QQ$-system:

$$W(q_i^-, q_i^+)(v) + \langle \alpha_i, \mathcal{Z} \rangle q_i^+(v) q_i^-(v) = \Lambda_i(v) \prod_j q_j^+(v)^{-a_{ij}},$$

for $\mathfrak{g}^L$.

Relation of E. Frenkel '03 Miura opers with regular singularities to $q_i(v)$:

$$\partial_v + \sum_i \Lambda_i(v) e_i + \sum_i \partial_v \log(q_i^+(v)) \check{\alpha}_i + \mathcal{Z}$$

Here $\{e_i, \check{\alpha}_i\}_{i=1,\dots,r}$ are the generators of $\mathfrak{b}_+^L \subset \mathfrak{g}^L$.

Essentially we will be deforming this formula.

# $\hbar$-connections on $\mathbb{P}^1$





- Principal $G$-bundle $\mathcal{F}_G$ over $\mathbb{P}^1$
- $M_\hbar : \mathbb{P}^1 \to \mathbb{P}^1$, such that $u \mapsto \hbar u$.

  $\mathcal{F}_G^\hbar$ stands for the pullback under the map $M_\hbar$.

A meromorphic $(G, \hbar)$- connection on a principal $G$-bundle $\mathcal{F}_G$ on $\mathbb{P}^1$ is a section $A$ of $Hom_{\mathcal{O}_U}(\mathcal{F}_G, \mathcal{F}_G^\hbar)$, where $U$ is a Zariski open dense subset of $\mathbb{P}^1$.

Choose $U$ so that the restriction $\mathcal{F}_G|_U$ of $\mathcal{F}_G$ to $U$ is isomorphic to the trivial $G$-bundle.

The restriction of $A$ to the Zariski open dense subset $U \cap M_\hbar{}^{-1}(U)$ is an element $A(u)$ of $G(u) \equiv G(\mathbb{C}(u))$.

Changing the trivialization is given by $\hbar$-gauge transformation:

$$A(u) \mapsto g(\hbar u) A(u) g(u)^{-1}$$

A $(G, \hbar)$-oper on $\mathbb{P}^1$ is a triple $(\mathcal{F}_G, A, \mathcal{F}_{B_-})$:

- $\mathcal{F}_G$ is a $G$-bundle
- A is a meromorphic $(G, \hbar)$-connection on $\mathcal{F}_G$ over $\mathbb{P}^1$
- $\mathcal{F}_{B_-}$ is the reduction of $\mathcal{F}_{B_-}$ to $B_-$

Oper condition: there exists a Zariski open dense subset $U \subset \mathbb{P}^1$ together with a trivialization $\imath_{B_-}$ of $\mathcal{F}_{B_-}$, such that the restriction of the connection $A : \mathcal{F}_G \to \mathcal{F}_G^{\hbar}$ to $U \cap M_{\hbar}{}^{-1}(U)$, written as an element of $G(z)$ using the trivializations of $\mathcal{F}_G$ and $\mathcal{F}_G^{\hbar}$ on $U \cap M_{\hbar}{}^{-1}(U)$ induced by $\imath_{B_-}$ takes values in the Bruhat cell

$$B_-(\mathbb{C}[U \cap M_{\hbar}{}^{-1}(U)])cB_-(\mathbb{C}[U \cap M_{\hbar}{}^{-1}(U)]),$$

where $c$ is Coxeter element: $c = \prod_i s_i$.
Locally:

$$A(u) = n'(u)\prod_i(\phi_i(u)^{\check{\alpha}_i}s_i)n(u),\ \ \phi_i(u) \in \mathbb{C}(u),\ \ n(u), n'(u) \in N(u)$$

Here $N = [B, B]$, $H = B/[B, B]$.

$\hbar$-Drinfeld-Sokolov reduction: Semenov-Tian-Shansky, Sevostyanov'98

# $(G, \hbar)$-Miura opers

A Miura $(G, \hbar)$-oper on $\mathbb{P}^1$ is a quadruple $(\mathcal{F}_G, A, \mathcal{F}_{B_-}, \mathcal{F}_{B_+})$:

- $(\mathcal{F}_G, A, \mathcal{F}_{B_-})$ is a meromorphic $(G, \hbar)$-oper on $\mathbb{P}^1$.
- $\mathcal{F}_{B_+}$ is a reduction of the $G$-bundle $\mathcal{F}_G$ to $B_+$ that is preserved by the $\hbar$-connection $A$.

The fiber $\mathcal{F}_{G,x}$ of $\mathcal{F}_G$ at $x$ is a $G$-torsor with reductions $\mathcal{F}_{B_-,x}$ and $\mathcal{F}_{B_+,x}$ to $B_-$ and $B_+$, respectively. Choose any trivialization of $\mathcal{F}_{G,x}$, i.e. an isomorphism of $G$-torsors $\mathcal{F}_{G,x} \simeq G$. Under this isomorphism, $\mathcal{F}_{B_-,x}$ gets identified with $aB_- \subset G$ and $\mathcal{F}_{B_+,x}$ with $bB_+$.

Then $a^{-1}b$ is a well-defined element of the double quotient $B_- \backslash G / B_+$, which is in bijection with $W_G$.

We will say that $\mathcal{F}_{B_-}$ and $\mathcal{F}_{B_+}$ have a generic relative position at $x \in X$ if the element of $W_G$ assigned to them at $x$ is equal to 1 (this means that the corresponding element $a^{-1}b$ belongs to the open dense Bruhat cell $B_- \cdot B_+ \subset G$).

# Structural theorems



**Theorem.** For any Miura $(G, \hbar)$-oper on $\mathbb{P}^1$, there exists an open dense subset $V \subset \mathbb{P}^1$ such that the reductions $\mathcal{F}_{B_-}$ and $\mathcal{F}_{B_+}$ are in generic relative position for all $x \in V$.

What this means locally: if $g(\hbar u)A(u)g(u) = \widetilde{A}(u) \in B_+(u)$, then $g(u) \in B_+(u)N_-(u)$.

**Theorem.** i) For any Miura $(G, \hbar)$-oper on $\mathbb{P}^1$, there exists a trivialization of the underlying $G$-bundle $\mathcal{F}_G$ on an open dense subset of $\mathbb{P}^1$ for which the oper $\hbar$-connection has the form:

$$A(u) \in N_-(u) \prod_i (\phi_i(u)^{\check{\alpha}_i} s_i) N_-(u) \ \cap \ B_+(u).$$

ii) Any element from $N_-(u)\prod_i(\phi_i(u)^{\check{\alpha}_i} s_i)N_-(u) \ \cap \ B_+(z)$ can be written as:

$$\prod_i g_i^{\check{\alpha}_i}(u) e^{\frac{\phi_i(u) t_i(u)}{g_i(u)} e_i}$$

where each $t_i \in \mathbb{C}(u)$ is determined by the lifting of $s_i$.

In the following we set $t_i \equiv 1$.

- $(G,\hbar)$-oper with regular singularities at finitely many points on $\mathbb{P}^1$:

$$A(u)=n'(u)\prod_i(\Lambda_i^{\check{\alpha}_i}(u)s_i)n(u),\ \ \Lambda_i(u)\in\mathbb{C}[u].$$

For any Miura $(G,\hbar)$-oper with regular singularities:

$$A(u)=\prod_i g_i^{\check{\alpha}_i}(u)e^{\frac{\Lambda_i(u)}{g_i(u)}e_i}.$$

- $(G,\hbar)$-oper is $Z$-twisted if it is gauge equivalent to $Z\in H$, namely

$$A(u)=g(\hbar u)Zg^{-1}(u),\ \ \text{where } Z=\prod_i z_i^{\check{\alpha}_i},\ \ g(u)\in G(u).$$

We assume $Z$ is regular semisimple. In that case there are $W_G$ Miura opers for a given oper.

In the extreme case $Z=1$ we have $G/B$ Miura opers for a given oper.

# $(H,\hbar)$-connections and $(GL(2),\hbar)$-opers





- $(H,\hbar)$-connection associated to Miura $(G,\hbar)$-opers:

$$A^H(u)=\prod_i g_i(u)^{\check{\alpha}_i}.$$

In Z-twisted case: $A^H(u)=\prod_i y_i(\hbar u)^{\check{\alpha}_i}\ Z\ \prod_i y_i(u)^{-\check{\alpha}_i}$, $g_i(u)=z_i\frac{y_i(\hbar u)}{y_i(u)}$.

- Let $V_i$ be the fundamental representation for $\omega_i$, $W_i$ is a 2-dimensional subspace spanned by $\{v_i, f_i v_i\}$, where $v_i$ is the highest weight vector.

Associated $GL(2)$-oper:

$$A_i(u)=\begin{pmatrix} g_i(u) & \Lambda_i(u)\prod_{j>i} g_j(u)^{-a_{ji}} \\ 0 & g_i^{-1}(u)\prod_{j\neq i} g_j(u)^{-a_{ji}}\end{pmatrix},$$

# Z-twisted Miura-Plücker $(G, \hbar)$ -opers





A $Z$- twisted Miura-Plücker $(G, \hbar)$-oper is a meromorphic Miura $(G, \hbar)$-oper on $\mathbb{P}^1$ with the underlying $\hbar$-connection $A(u)$, such that there exists $v(u) \in B_+(z)$ such that for all $i = 1, \dots, r$, the Miura $(GL(2), \hbar)$-opers $A_i(u)$ associated to $A(u)$ can be written in the form:

$$A_i(u) = v(u\hbar) Z v(u)^{-1}|_{W_i} = v_i(u\hbar) Z_i v_i(u)^{-1}$$

where $v_i(u) = v(u)|_{W_i}$ and $Z_i = Z|_{W_i}$.

Nondegeneracy conditions (see detailed discussion in our paper):

$$A(u) = \prod_i g_i^{\check{\alpha}_i}(u) e^{\frac{\Lambda_i(u)}{g_i(u)} e_i}, \quad g_i(u) = z_i \frac{y_i(\hbar u)}{y_i(u)}$$

Each $y_i(u)$ is a polynomial, and for all $i, j, k$ with $i \neq j$ and $a_{ik} \neq 0, a_{jk} \neq 0$, the zeros of $y_i(u)$ and $y_j(u)$ are $\hbar$-distinct from each other and from the zeros of $\Lambda_k(u)$.

# Nondegenerate $(G, \hbar)$-opers and QQ-systems





Explicit formula for $v(u)$, such that

$$A_i(u) = v(u\hbar) Z v(u)^{-1}|_{W_i}$$

is:

$$v(u) = \prod_{i=1}^{r} y_i(u)^{\check{\alpha}_i} \prod_{i=1}^{r} e^{-\frac{Q^i_-(u)}{Q^i_+(u)} e_i} \dots,$$

where the dots stand for the exponentials of higher commutator terms in the Lie algebra $\mathfrak{n}_+$ of $N_+$, $\{Q^i_+(u), Q^i_-(u)\}$ are relatively prime polynomials and $Q^i_+(u)$ is a monic polynomial for each $i = 1, \dots, r$.

That leads to the expression of Miura $(G, \hbar)$-oper connection:

$$A(u) = \prod_i g_i^{\check{\alpha}_i}(u) e^{\frac{\Lambda_i(u)}{g_i(u)} e_i}, \quad g_i(u) = z_i \frac{Q^i_+(\hbar u)}{Q^i_+(u)}.$$

# First of the main theorems



**Theorem.** There is a one-to-one correspondence between the set of nondegenerate $Z$-twisted Miura-Plücker $(G, \hbar)$-opers and the set of nondegenerate polynomial solutions of the $QQ$-system:

$$\widetilde{\xi}_i Q^i_-(u) Q^i_+(\hbar u) - \xi_i Q^i_-(\hbar u) Q^i_+(u) = \Lambda_i(u) \prod_{j>i} \left[Q^j_+(\hbar u)\right]^{-a_{ji}} \prod_{j<i} \left[Q^j_+(u)\right]^{-a_{ji}}, \qquad i = 1, \dots, r,$$

where $\widetilde{\xi}_i = z_i \prod_{j>i} z_j^{a_{ji}}$, $\xi_i = z_i^{-1} \prod_{j<i} z_j^{-a_{ji}}$.

In ADE case this QQ-system corresponds to the Bethe ansatz equations. Beyond simply-laced case: currently under investigation.

# Bethe equations

Let $\{w_i^k\}_{k=1,\dots,m_i}$ be the set of roots of the polynomial $Q_+^i(w)$. Then Bethe equations for the $QQ$-system are:

$$\frac{Q_+^i(\hbar w_i^k)}{Q_+^i(\hbar^{-1} w_i^k)} \prod_j z_j^{a_{ji}} =$$

$$-\frac{\Lambda_i(w_k^i)\prod_{j>i}\left[Q_+^j(\hbar w_k^i)\right]^{-a_{ji}}\prod_{j<i}\left[Q_+^j(w_k^i)\right]^{-a_{ji}}}{\Lambda_i(\hbar^{-1}w_k^i)\prod_{j>i}\left[Q_+^j(w_k^i)\right]^{-a_{ji}}\prod_{j<i}\left[Q_+^j(\hbar^{-1}w_k^i)\right]^{-a_{ji}}}$$

where $i = 1, \dots, r; k = 1, \dots, m_i$.

# Quantum Bäcklund transformations

Originally operators

$$A(u) = \prod_i g_i^{\check{\alpha}_i}(u) e^{\frac{\Lambda_i(u)}{g_i(u)} e_i}, \quad g_i(u) = z_i \frac{Q_+^i(\hbar u)}{Q_+^i(u)},$$

where $Q_\pm(u)$ are the solution of QQ-systems, were introduced by Mukhin, Varchenko'05 in the additive case with $Z = 1$.

They also introduced the following $\hbar$-gauge transformation of the $\hbar$-connection $A$:

$$A \mapsto A^{(i)} = e^{\mu_i(\hbar u) f_i} A(u) e^{-\mu_i(u) f_i}, \quad \text{where} \quad \mu_i(u) = \frac{\prod_{j \neq i} \left[Q_+^j(u)\right]^{-a_{ji}}}{Q_+^i(u) Q_-^i(u)}.$$

Then $A^{(i)}(u)$ can be obtained from $A(u)$ by substituting in formula for $A(u)$:

$$Q_+^j(u) \mapsto Q_+^j(u), \qquad j \neq i,$$
$$Q_+^i(u) \mapsto Q_-^i(u), \qquad Z \mapsto s_i(Z).$$

Suppose that the polynomial $Q^i_-(u)$ constructed as the solution of QQ-system is such that its roots are $\hbar$-distinct from the roots of $Q^j_+(u), j \neq i$, and $\Lambda_k(u)$ such that $a_{ik} \neq 0$ and $a_{jk} \neq 0$. Then the data

$$\{\widetilde{Q}^j_+\}_{j=1,\dots,r} = \{Q^1_+, \dots, Q^{i-1}_+, Q^i_-, Q^{i+1}_+ \dots, Q^r_+\}; \qquad (1)$$
$$\{\widetilde{z}_j\}_{j=1,\dots,r} = \{z_1, \dots, z_{i-1}, z_i^{-1} \prod_{j \neq i} z_j^{-a_{ji}}, \dots, z_r\}$$

give rise to a nondegenerate solution of the Bethe Ansatz equations, corresponding to $s_i(Z) \in H$.

Furthermore, there exist polynomials $\{\widetilde{Q}^j_-\}_{j=1,\dots,r}$ that together with $\{\widetilde{Q}^j_+\}_{j=1,\dots,r}$ give rise to a nondegenerate solution of the $QQ$-system corresponding to $s_i(Z)$.

Let $w = s_{i_1}\dots s_{i_k}$ be a reduced decomposition of an element $w$ of the Weyl group of $G$. A solution of the $QQ$-system is called $(i_1\dots i_k)$-generic if by consecutively applying the quantum Bäcklund transformations with $i=i_k,\dots,i=i_1$, we obtain a sequence of nondegenerate solutions of the $QQ$-systems corresponding to the elements $w_j(Z)\in H$, where $w_k = s_{i_{k-j+1}}\dots s_{i_k}$ with $j=1,\dots,k$.

Let $w_0 = s_{i_1}\dots s_{i_\ell}$ be a reduced decomposition of the maximal element of the Weyl group of $G$. In what follows, we refer to a $(i_1,\dots,i_\ell)$-generic object as $w_0$-generic.

**Theorem.** Every $w_0$-generic $Z$-twisted Miura-Plücker $(G,\hbar)$-oper is a nondegenerate $Z$-twisted Miura $(G,\hbar)$-oper.

Proof involves playing with double Bruhat cells and implies only existence of the diagonalizing element $v(u)\in B_+(u)$ in this case. However, there is no explicit formula (so far).

# $SL(r+1)$ opers: explicit formula



QQ-system:

$$\xi_{i+1} Q_i^+(\hbar u) Q_i^-(u) - \xi_i Q_i^+(u) Q_i^-(\hbar u) = \Lambda_i(u) Q_{i-1}^+(u) Q_{i+1}^+(\hbar u)\,,\; i=1,\dots,r$$

$$\xi_1 = \frac{1}{z_1}, \quad \xi_2 = \frac{z_1}{z_2}, \quad \dots \quad \xi_r = \frac{z_{r-1}}{z_r}, \quad \xi_{r+1} = \frac{1}{z_r}\,,$$

Introducing notation:

$$\phi_i(u) = \frac{Q_i^-(u)}{Q_i^+(u)}\,, \qquad \rho_i(u) = \Lambda_i(u) \frac{Q_{i-1}^+(u) Q_{i+1}^+(\hbar u)}{Q_i^+(u) Q_i^+(\hbar u)}\,.$$

We have a sequence of quantum Bäcklund transformations:

$$\xi_i\, \phi_i(u) - \xi_{i+1}\, \phi_i(\hbar u) = \rho_i(u)\,,\;\; i = 1, \dots, r,$$

$$\xi_i\, \phi_{i,i+1}(u) - \xi_{i+2}\, \phi_{i,i+1}(\hbar u) = \rho_{i+1}(u)\phi_i(u)\,,\;\; i = 1, \dots, r-1,$$

$$\dots\dots$$

$$\xi_i\, \phi_{i,\dots,r-2+i}(u) - \xi_{r-2+i}\, \phi_{i,\dots,r-1+i}(\hbar u) = \rho_{r-1}(u)\phi_{i,\dots,r-3+i}(u)\,, i = 1, 2$$

$$\xi_1 \phi_{1,\dots,r}(u) - \xi_{r+1}\phi_{1,\dots,r}(\hbar u) = \rho_r(u)\phi_{1,\dots,r-1}(u)\,,$$

where

$$\phi_{i,\dots,j}(u) = \frac{Q^-_{i,\dots,j}(u)}{Q_j^+(u)}.$$

For Z-twisted oper:

$$A(u) = v^{-1}(\hbar u) Z v(u)$$

$$v(u) = \begin{pmatrix} \frac{1}{Q_1^+(u)} & \frac{Q_1^-(u)}{Q_2^+(u)} & \frac{Q_{12}^-(u)}{Q_3^+(u)} & \dots & \frac{Q_{1,\dots,r-1}^-(u)}{Q_r^+(u)} & Q_{1,\dots,r}^-(u) \\ 0 & \frac{Q_1^+(u)}{Q_2^+(u)} & \frac{Q_2^-(u)}{Q_3^+(u)} & \dots & \frac{Q_{2,\dots,r-1}^-(u)}{Q_r^+(u)} & Q_{2,\dots,r}^-(u) \\ 0 & 0 & \frac{Q_2^+(u)}{Q_3^+(u)} & \dots & \frac{Q_{3,\dots,r-1}^-(u)}{Q_r^+(u)} & Q_{3,\dots,r}^-(u) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \dots & \dots & \frac{Q_{r-1}^+(u)}{Q_r^+(u)} & Q_r^-(u) \\ 0 & \dots & \dots & \dots & 0 & Q_r^+(u) \end{pmatrix}.$$

Moreover, $w_0$-genericity is not needed in this case!

# $SL(r+1)$ opers: alternative definition

A meromorphic $(GL(r+1), \hbar)$-*oper* on $\mathbb{P}^1$ is a triple $(A, E, \mathcal{L}_\bullet)$, where $E$ is a vector bundle of rank $r+1$ and $\mathcal{L}_\bullet$ is the corresponding complete flag of the vector bundles,

$$\mathcal{L}_{r+1} \subset \ldots \subset \mathcal{L}_{i+1} \subset \mathcal{L}_i \subset \mathcal{L}_{i-1} \subset \ldots \subset E = \mathcal{L}_1,$$

where $\mathcal{L}_{r+1}$ is a line bundle, so that $A \in Hom_{\mathcal{O}_U}(E, E^\hbar)$ satisfies the following conditions:

- $A \cdot \mathcal{L}_i \subset \mathcal{L}_{i-1}$,
- There exists Zariski open $U$, such that $\bar{A}_i : \mathcal{L}_i/\mathcal{L}_{i+1} \to \mathcal{L}_{i-1}/\mathcal{L}_i$ is an isomorphism on $U \cap {M_\hbar}^{-1}(U)$.

An $(SL(r+1), \hbar)$-*oper* is a $(GL(r+1), \hbar)$-oper with the condition that $det(A) = 1$ on $U \cap {M_\hbar}^{-1}(U)$.

Regular singularities: $\bar{A}_i$ allowed to have zeroes at zeroes of $\Lambda_i(u)$.

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# $SL(2)$ example and $\hbar$-Wronskians



A Z-twisted $(SL(2), \hbar)$-oper on $\mathbb{P}^1$ with regular singularities is a triple $(E, A, \mathcal{L})$:

- $(E, A)$ is a $(SL(2), \hbar)$-connection
- $\mathcal{L}$ is a line subbundle so that $\bar{A} : \mathcal{L} \to (E/\mathcal{L})^\hbar$ is an isomorphism except for zeroes of $\Lambda(u)$.
- A is gauge equivalent to $Z \in H$

Equivalently:

$$s(\hbar u) \wedge A(u)s(u) = \Lambda(u),$$

where $s(u)$ is a section of $\mathcal{L}$.

Choosing trivialization $s(u) = \begin{pmatrix} Q_-(u) \\ Q_+(u) \end{pmatrix}$, we obtain that above condition is the QQ-system:

$$zQ_-(u)Q_+(\hbar u) - z^{-1}Q_-(\hbar u)Q_+(u) = \Lambda(u).$$

# $SL(r+1)$-Wronskians and QQ-systems

More general Wronskians:

$$\mathcal{D}_k(s) = \\ e_1 \wedge \cdots \wedge e_{r+1-k} \wedge s(u) \wedge Z^{-1} s(\hbar u) \wedge \cdots \wedge Z^{1-k} s(\hbar^{k-1} u) = \\ \alpha_k W_k(u) \mathcal{V}_k(u)\,,$$

where

$$\mathcal{V}_k(u) = \prod_{a=1}^{r_k} (u - w_{k,a})\,,$$

and

$$W_k(s) = P_1 \cdot P_2^{(1)} \cdot P_3^{(2)} \cdots P_{k-1}^{(k-2)}, \quad P_i = \Lambda_r \Lambda_{r-1} \cdots \Lambda_{r-i+1}$$

We used the notation $f^{(j)}(u) = D_\hbar^j(f)(u) = f(\hbar^j u)$ above.

One can identify: $\mathcal{V}_k(u) \equiv Q_k^+(u)$ and $Q^-_{i,\ldots,j}(u)$ with other minors.

The bilinear relations for the extended $QQ$-system are nothing but Plücker relations for minors in the $\hbar$-Wronskian matrix.

Natural question is whether generalized minors for simply connected semisimple $G$ describe the extended hierarchy.

## Quantum-classical duality via $(SL(r+1), \hbar)$-opers



Take section of the line bundle $\mathcal{L}_{r+1}$ in complete flag $\mathcal{L}_\bullet$:

$$s(u) = \begin{pmatrix} s_1(u) \\ s_2(u) \\ s_3(u) \\ \vdots \\ s_r(u) \\ s_{r+1}(u) \end{pmatrix} = \begin{pmatrix} Q^-_{1,\dots,r}(u) \\ Q^-_{2,\dots,r}(u) \\ Q^-_{3,\dots,r}(u) \\ \vdots \\ Q^-_r(u) \\ Q^+_r(u) \end{pmatrix}.$$

Interesting case (XXZ chain corresponding to defining representations):

- Polynomials are of degree 1
- Only $\Lambda_1(u) = \prod_i (u - a_i)$ is nontrival

Identification:

- roots of $s_i(u)$ with momenta
- $\xi_i = z_i/z_{i-1}$ with coordinates,

Space of functions on Z-twisted Miura $(SL(r+1), \hbar)$-opers $\leftrightarrow$ space of functions on the intersection of two Lagrangian subvarieties in trigonometric Ruijsenaars-Schneider (tRS) phase space.

Bethe equations $\leftrightarrow \{H_k = f_k(\{a_i\})\}$

Here $H_k$ are tRS Hamiltonians and $f_i$ are elementary symmetric functions of $a_i$.

# $\hbar$-Opers for $\widehat{\widehat{\mathfrak{gl}}}(1)$ and Bethe ansatz



Let us "complete" Miura $(SL(r+1), \hbar)$-opers by $(\overline{GL}(\infty), \hbar)$:

$$A(u) = \prod_{i=+\infty}^{-\infty} g_i^{\check{\alpha}_i}(u) e^{\frac{\Lambda_i(u)}{g_i(u)} e_i}, \quad g_i(u) = z_i \frac{Q_+^i(\hbar u)}{Q_+^i(u)}.$$

Infinite-dimensional $QQ$-system:

$$\xi_{i+1} Q_i^+(\hbar u) Q_i^-(u) - \xi_i Q_i^+(u) Q_i^-(\hbar u) = \Lambda_i(u) Q_{i-1}^+(u) Q_{i+1}^+(\hbar u)\,, i = 1, \dots, r,$$

where $\xi_i = z_i / z_{i-1}$.

Impose periodic condition: $VA(u)V^{-1} = \xi A(pu)$, where $V$ corresponds to automorphism of Dynkin diagram $i \to i+1$.

$V$ can be actually relized as an "infinite" Coxeter element of standard order.

That corresponds to $Q_j^{\pm}(u) = Q^{\pm}(p^j u)$, $\Lambda_j(u) = \xi^j \Lambda(u)$, $\xi_j = \xi^j$:

$$\xi Q^+(\hbar u) Q^-(u) - Q^+(u) Q^-(\hbar u) = \Lambda(u) Q^+(up^{-1}) Q^+(\hbar p u)$$

# Some open problems





- Understanding the $\hbar$-regular singularity structure. "Twisted" $\hbar$-opers?
- Elliptic case.
- Relation to toroidal algebras and double ellliptic systems.
- qDE/IM correspondence? Bridge to ODE/IM correspondence.
- Berenstein-Fomin-Zelevinsky generalized minors and quantum Bäcklund transformations as cluster algebra mutations.
- tRS-type variables and 3D Mirror symmetry.

Thank you!