Inductive construction of elliptic stable envelopes, part II

Monday, August 3, 2020 5:02 PM

① Equivariant elliptic cohomology is a scheme

$$\left(K \text{ compact group (connected)} \circlearrowright X \right) \xrightarrow{\text{contravariant}} H_G^\bullet(X) \quad \text{(super) commutative ring}$$

connected reductive group G

$$\operatorname{Spec} H_G^\bullet(pt) = \mathfrak{h}/W \qquad \text{Cartan} \subset \operatorname{Lie}(G)$$

covariant

(super) schemes

elliptic curve

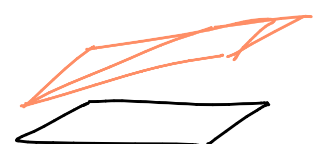
$$\operatorname{Ell}_G(pt) = \operatorname{coweight}(G) \otimes E / W$$

has to do with G-bundles over an elliptic curves

For example:

$$T = \text{a torus}, \quad \operatorname{Ell}_T(pt) = \operatorname{cochar}(T) \otimes E = E^{\operatorname{rank}(T)}$$

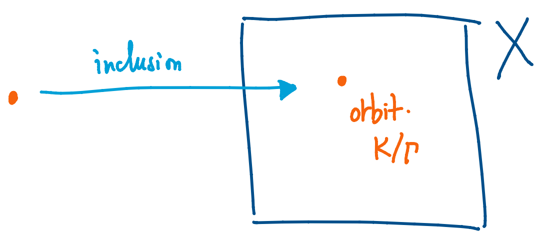
$$G = GL(n), \quad \operatorname{Ell}_{GL(n)}(pt) = E^n / \underset{\underset{S(n)}{\parallel}}{W} = S^n E$$



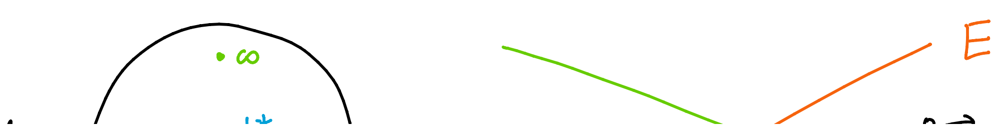
subgroup

$$\operatorname{Ell}_K(K/\Gamma) = \operatorname{Ell}_\Gamma(pt)$$

the smaller the orbit, the bigger its Ell



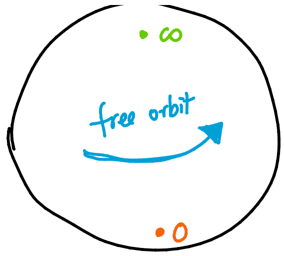
$$\operatorname{Ell}_\Gamma(pt) \longrightarrow \operatorname{Ell}_K(X)$$

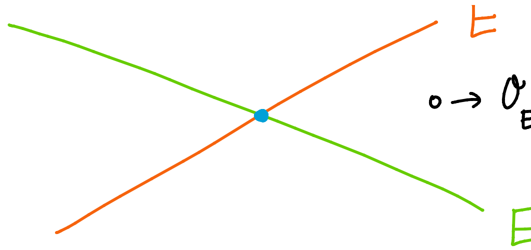


$\mathcal{O}_E$

$\to \mathcal{O} \to \oplus \to \mathcal{O}_{\dots} \to 0.$

$\mathbb{C}P^1 \cong S^2$

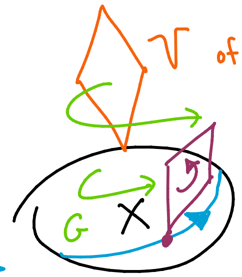




$$0 \to \mathcal{O}_{\mathrm{Ell}_{U(1)}(\mathbb{P}^1)} \longrightarrow \begin{matrix}\mathcal{O}_E \\ \oplus \\ \mathcal{O}_E\end{matrix} \longrightarrow \mathcal{O}_{pt} \longrightarrow 0.$$

X can be written as a cell complex with cells $D^n \times (K/\Gamma)$, there is an analog

Chern classes:



elliptic Chern roots

$$\begin{array}{ccccc} K/\Gamma & \longrightarrow & G \circlearrowright X & \longrightarrow & \bullet/GL(r) \\ \mathrm{Ell}_\Gamma(pt) & \longrightarrow & \mathrm{Ell}_G(X) & \longrightarrow & \mathrm{Ell}_{GL(r)}(pt) = S^r E \end{array}$$

records the representation of Γ in the fiber of $\mathcal{V}$

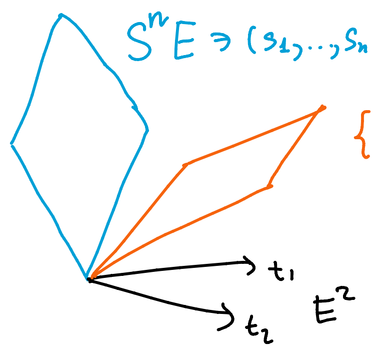
$$X = \mathrm{Hilb}(\mathbb{C}^2, n) \circlearrowleft T^2 = \left\{\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix}\right\}.$$

$\mathrm{Hilb}(\mathbb{C}^2, n) = \{ I \subset \mathbb{C}[x_1, x_2] \}$

$\mathcal{V} = \mathbb{C}[x_1, x_2]/I$

tautological bundle of rank n

$S^n E \ni (s_1, \dots, s_n)$



$\{s_i\} =$

1	t_1^{-1}	t_1^{-2}	..
t_2^{-1}			
.			
⋮			

$\bigcup$ $p(n)$ copies of E^2

$p(n)$ = # of partitions of n = rank $H^\cdot(\mathrm{Hilb}(\mathbb{C}^2, n))$

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(2) An elliptic cohomology class is a section of a line bundle on $\mathrm{Ell}_G(X)$.

recall: $\forall\ \mathcal{V}$ on X of rank r we get

$$\mathrm{Ell}_G(X) \longrightarrow S^r E \ni (s_1, \dots, s_r)$$

has interesting line bundles / divisors.

$\Theta(\mathcal{V}) \xleftarrow{\text{pullback}} \Theta = \{\text{one of the } s_i = 0\} = (\prod \vartheta(s_i))$

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line bundle

$\vartheta(s_i) = 0$

$\Theta(\mathcal{V}_1 \oplus \mathcal{V}_2) = \Theta(\mathcal{V}_1) \otimes \Theta(\mathcal{V}_2)$ gives a map $K_G(X) \to \mathrm{Pic}(\mathrm{Ell}_G(X))$

has degree and also has Pic_0

Suppose $\mathcal{L} \in \mathrm{Pic}_G(X)$ and take another $\mathbb{C}^\times \ni z$

operation in K ?

Consider $\Theta((\mathcal{L}-1)(z-1))$ is a line bundle over $\mathrm{Ell}_G(X) \times \mathrm{Ell}_{\mathbb{C}^\times}(pt)$

$\mathcal{L}z - \mathcal{L} - z + 1$ $\qquad \dfrac{\theta(sz)}{\theta(s)\,\theta(z)}$ looks like Poincaré line bundle, degree 0 in each variable

$\Rightarrow$ over $\mathrm{Ell}_G(X) \times (\mathrm{Pic}_G(X) \otimes_{\mathbb{Z}} E)$ lives $\mathcal{U}$ "universal such line bundle" of degree 0 along

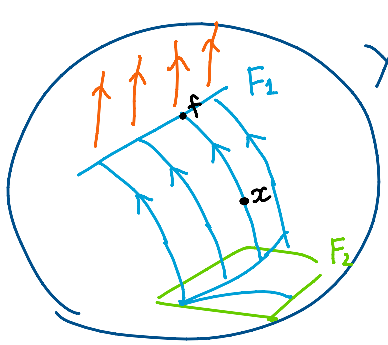
elliptic cohomology classes which we want are sections of $\mathcal{L} \otimes \mathcal{U}$

$\mathrm{Pic}_G(X) \otimes_{\mathbb{Z}} \mathbb{C}^*$

the Kähler variables in enumerative context

③ What is elliptic stable envelope?

components



$X \supset X^A = \bigsqcup F_i$

extend $\mathrm{Attr} = \{(x,f),\ \lim_{a\to 0} a\cdot x = f\} \subset X \times X^A$

to a (canonical) elliptic cohomology class

more general category of questions like $X_{ss} \subset X$ (open)

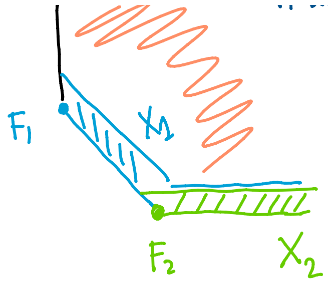
Means an interpolation problem for sections of $\mathcal{L} \otimes \mathcal{U}$ ...

open $= X \setminus \bigcup \mathrm{Attr}(F_i)$

F_i below F_1

$X_i = \bigcup$ neighborhoods of $\mathrm{Attr}(F_j)$.

grading in cohomology



$X_i = \bigcup_{j > i}$ neighborhoods of $\mathrm{Attr}(F_j)$.

grading in cohomology

$$\to \mathrm{Th\mathring{o}m}(N_{\mathrm{Attr}(F_i)})^{\cdot} \to \mathcal{O}^{\bullet}_{\mathrm{Ell}(X \setminus X_2)} \xrightarrow{\text{restriction}} \mathcal{O}^{\cdot}_{\mathrm{Ell}(X \setminus X_1)} \to$$

lift.

$\mathcal{S} = \mathcal{L} \otimes \mathcal{U}$

$$\to$$

$$\to \mathrm{Th\mathring{o}m}(N_{\mathrm{Attr}(F_i)})^{\cdot} \otimes \mathcal{S} \to \mathcal{O}^{\bullet}_{\mathrm{Ell}(X \setminus X_2)} \otimes \mathcal{S} \xrightarrow{\text{restriction}} \mathcal{O}^{\cdot}_{\mathrm{Ell}(X \setminus X_1)} \otimes \mathcal{S} \to$$

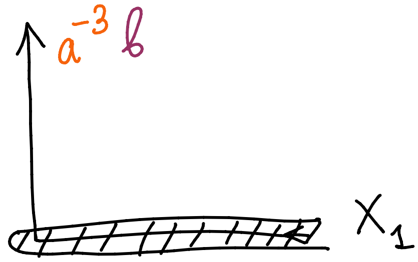
$$\to$$

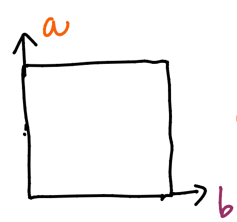
vanish.

other group action

$a^{-3}b$

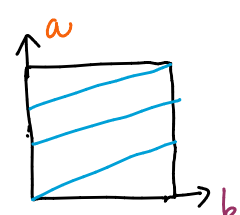
Example. $X = \mathbb{C}^2$





$\otimes \mathcal{S}$

restrict to a curve.



in this variable we allow poles

$$\mathrm{Ell}_{eq}(X) = \mathrm{Ell}_{eq}(pt) = \underset{a}{E} \times \underset{b}{E} \times \ldots$$

$$\mathrm{Ell}_{eq}(X \setminus X_1) = \mathrm{Ell}_{eq}(\mathbb{C}^*)$$

stabilizer = kernel of $a^{-3}b$

if $\mathcal{S}$ is of degree 3

degree $[\mathrm{Attr}]$ $=$ $\vartheta(a^{-3}b)$.

$$\to \mathcal{L}(-3pts) \to \mathcal{L} \to \mathcal{L}\big|_{3pts} \to$$

$$H^i(\mathcal{L}(-3pt)) = 0$$

unless this bundle is <u>trivial</u>.

unless this bundle is trivial.

pole in

Applications:

(0) gives construction in more general setting than [Aganagic - O.]

(1) Monodromy problems

with a choice of stability cond.

$$\text{Vertex function} = \sum_d z^d \; ev_{\infty,*}\left(QM_d(\mathbb{P}^1 \to X/\!\!/G)_{\text{nonsing. at } \infty},\ \hat{\mathcal{O}}_{vir} \right) \in K_{G_{aut} \times \mathbb{C}^*}(X/\!\!/G)_{loc}[[z]]$$

$q \in \mathbb{C}^*$

G_{aut}

variable in $\mathrm{Pic}_G(X) \otimes \mathbb{C}^*$.

// I-function

disc partition function.

very fancy brother of q-hypergeom. function

in particular satisfies a q-difference equations (regular) z

and also $A \subset G_{aut}$

[AO] for Nakajima quiver varieties

$$\text{Monodromy} = \Phi^{-1}_{z=\infty} \cdot \Phi_{z=0}$$

fund solution

q-periodic in z

$$\mathrm{Ell}_G(X_{stable,-}) \xrightarrow{\mathrm{Stab}_-} \mathrm{Ell}_G(X) \xleftarrow{\mathrm{Stab}_+} \mathrm{Ell}_G(X_{stable,+})$$

sections of $\Theta(T^{1/2}X) \otimes \mathcal{U}$

$$\text{Monodromy} := \mathrm{Stab}_-^{-1} \circ \mathrm{Stab}_+$$

for $G = \prod GL(V_i)$,
$X =$ quiver

[Bethe]

in reduces to $\mathbb{C}^*$-stable envelopes

quotient of lci by reductive.

(2) 3d mirror partners:

$$X \qquad X^\vee \curvearrowleft A^\vee \qquad X^A = \{p_i\} \longleftrightarrow \{p_i^\vee\} = (X^\vee)^{A^\vee}$$

② 3d mirror partners.

$A \curvearrowright X \longleftrightarrow X^\vee \circlearrowleft A^\vee$ $\qquad X^A = \{p_i\} \longleftrightarrow \{p_i^\vee\} = (X^\vee)^{A^\vee}$

Provisional sufficient condition: Vertex function $(p_i) \approx$ Vertex function $(T_{p_i^\vee})$

depends only on $z_i \in Z$ and $\hbar =$ weight of ω_X

presented as a quotient

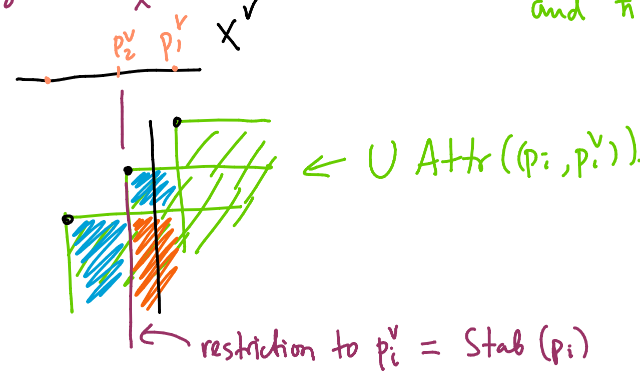
as a vector space with an action of $A^\vee \cong Z$ and $\hbar' = q/\hbar$

Mother class / duality interface

elliptic cohomology class

p_1

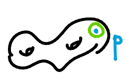
$X \quad p_2$



explicit formula for hypertoric but hard in general

poles

$(Z \quad) \longleftrightarrow$ weights of $(A^\vee, \hbar^\vee)$ in $T_{p_i^\vee} X^\vee$

$QM($  $p) \xrightarrow{ev} QM(\mathrm{Spec}\, \mathbb{C}[x^{-1}; x]] \to X/\!\!/G)$.