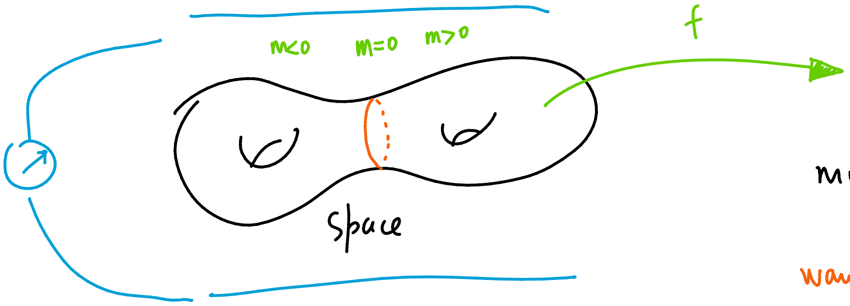
Monday, July 6, 2020 4:57 PM

Inductive construction of elliptic stable envelopes

I teach a course → Wednesday 7:30 am Pacific Time

- stable envelope is an interface

Very Susy QFT in 2+1 dim



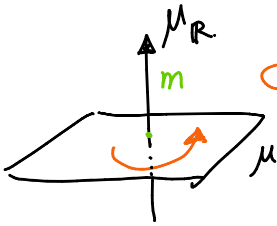
X (external parameters e.g. equivariant mass m)

moduli spaces of vacua

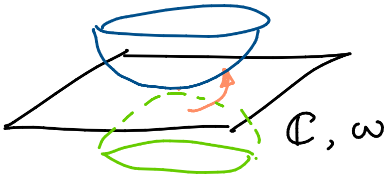
wants to be hyper Kähler / holo sympl

e.g. $U(1) \curvearrowright X$ preserving $\omega_{\mathbb{R}}, \omega_{\mathbb{C}}$

$$\mu_{\mathbb{H}}: X \longrightarrow \mathrm{Lie}\, U(1) \otimes \mathbb{R}^3$$



$\mathbb{C}^*_\hbar$ rotates $\omega_{\mathbb{C}}$ with weight $\hbar$



$\mathbb{C}$, $\omega_{\mathbb{R}} = dx \wedge dy$

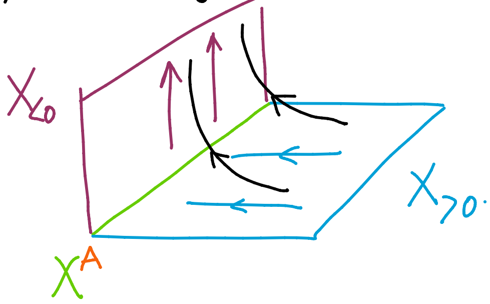
$$\mu_{\mathbb{R}} = m \frac{x^2+y^2}{2}$$

repelling/attracting with respect to

$z \in GL(1) = U(1)_{\mathbb{C}} = A$

$z \to 0$ torus

$X =$ flat symplectic space



interface $\approx$ a Lagrangian in $X(m>0) \times X(m=0)$

$$\begin{matrix} X(m>0) & \times & X(m=0) \\ \parallel & & \parallel \\ X^A & \times & X \\ & \cup & \end{matrix}$$

Lagrangian →

$$\mathrm{Attr} = \{(f,x),\ \lim_{z \to 0} x = f\}$$

for general X, we want a $\overbrace{\text{nice}}^{\text{as nice as possible}}$ correspondence $\subset X \times X^A$

which extends → Attr

$\overline{\mathrm{Attr}} \subset X \times X^A$ has few good properties e.g. quite singular

e.g. $X = T^*Gr(k,n) \circlearrowleft GL(n) \supset A$ maximal torus e.g. $X = T^*Gr(k,n) \circlearrowleft GL(n) \supset A$ maximal torus

$X^A =$ coordinate subspaces, $\binom{n}{k}$ pts.

Attr = conormals to Schubert cells in $Gr(k,n)$

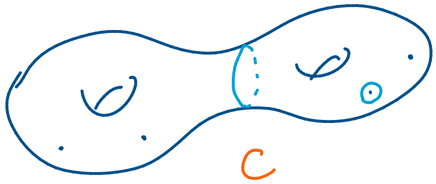
Goal: to have a good class in elliptic cohomology

K- theory

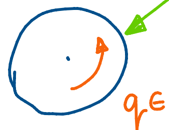
ordinary cohomology, all equivariant

use in the computation of indices

Index (C) ≈ patched together from



C

index ()

$q \in \mathbb{C}^\times_q$.

disk

boundary conditions in $K_{eq}(X)$ or $K_{eq}(LX)$

$|q|<1$

regular

$F(qa) = S(a,q,..) F(a)$ a

solutions $a \to 0$

holo for $0<|a|<\varepsilon$

Solution of certain q-difference equations both in equivariant and Kähler variables

$K_{eq}(X)[[z]]$ ↗

record the degree

z^{deg} = character of a torus

$H^2(X,\mathbb{Z}) \otimes \mathbb{C}^\times$

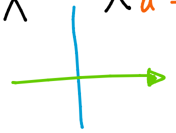
Kähler torus

$\Downarrow$

a

meromorphic function of a

$a \to 0$ $\quad X^a \quad X \quad X^a$ $a \to \infty$



monodromy of the corresponding q- difference equation

$F^{-1}_{a \to \infty} \circ F_{a \to 0}$ = is elliptic in a $\quad /\!/$

Theorem of M.A and I from elliptic stable envelopes paper

fits in the same story as variation of a character θ of a gauge group

monodromy in Kähler variables

$X =$ something else $/\!/\!/_\theta$ gauge group

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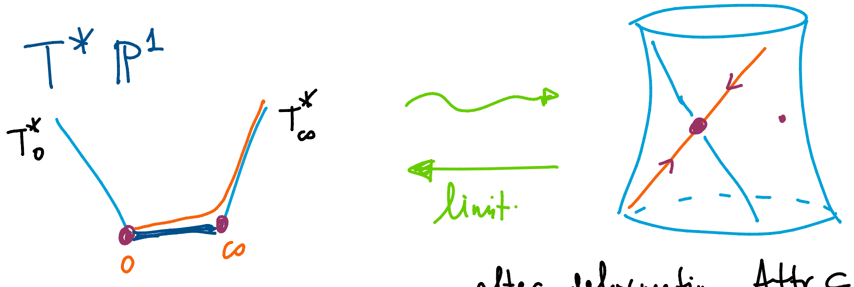
in ordinary equivariant cohomology, construction of stable envelopes is not too hard:



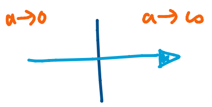
$A \circlearrowright \mathbb{P}^1$

after deformation $\mathrm{Attr} \subset X \times X^A$ is a closed Lagr. submanifold

↑ may be characterized in many different ways including

- equivariant weights of its restriction to fixed pts
- flop-invariance.
- ⋮

won't need anything like this in elliptic theory

many uses 

R-matrix of [MO]

↳ quantum differential equation for cohomology analog of the index.

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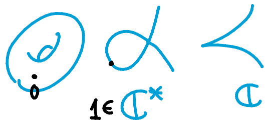
equivariant elliptic cohomology: $G \circlearrowright X \longrightarrow$ complexes sheaves of super-commutative algebras ← (super) schemes

↑ compact Lie group

↓ $\mathrm{Ell}_G(pt)$

what is this?

E = curve of genus 1 with a smooth marked pt.



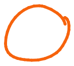
↑ equivariant K-theory

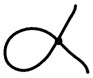
↑ eq. cohomology

↓ Base scheme B ∋ $q, \hbar, \dots$

$K_G(pt)$ = representation ring of G
⊕, ⊗

Spec ↓ $= G_{ss}/\text{conj} = \boxed{T/W}$

semisimple G-local systems on 

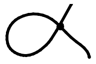
or ss G-bundles of degree 0 

$\mathrm{Ell}_G(pt) =$ s.s. G-bundles of degree 0 on $E^\vee = \mathrm{cochar}(G) \otimes E/W$

for GL(n) $a_1, \dots, a_n \in E = \mathrm{Pic}_0(E^\vee)$ up to permutation for $GL(n)$

up to permutation

or ss G-bundles of degree 0



e.g. $\boxed{\mathrm{Ell}_{U(1)}(pt) = E}$

$\forall$ X can be glued out of $D^n \times G/H$

$$\mathrm{Ell}_G\left(\downarrow\right) = \mathrm{Ell}_H(pt) \longrightarrow \mathrm{Ell}_G(pt).$$

$\mathcal{O}^\bullet_{\mathrm{Ell}_G(X)}$ is obtained from $\mathcal{O}^\bullet_{\mathrm{Ell}_H(pt)}$ periodic with $\mathcal{O}^{\bullet-2} = \mathcal{O}^\bullet \otimes T_0^* E$

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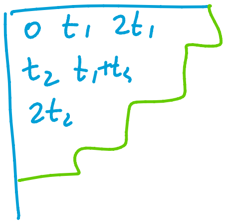
other ways to think about $\mathrm{Ell}_G(X)$: $E = \mathbb{C}^*/q^{\mathbb{Z}}$ $\mathrm{Spec}\, K_{eq}(X)/q$

chern roots of vector bundles now take values in $\mathrm{Ell}_{GL(r)}(pt) = S^r E$ instead of $S^r \mathbb{C}^*$

$$\mathrm{Ell}_T(\mathrm{Hilb}(\mathbb{C}^2, n)) = \bigcup_{|\lambda|=n}$$

$\{x_i\} = \{$weights of Taut at fixed pt $\lambda\}$.



$T \subset GL(2)$ max torus

generated by chern classes of taut. vector bundle of rank n

$\frown$ $\mathrm{Ell}_T(pt) \times S^n E$

t_1, t_2 $x_1, \ldots, x_n$

Elliptic cohomology classes = sections of line bundles on $\mathrm{Ell}_G(X)$.

recall we want a class supported $\mathrm{Attr}_{full} \subset X \times X^A$

support of cohomological stable envelope

$\deg_A \mathrm{Attr}$

$=$

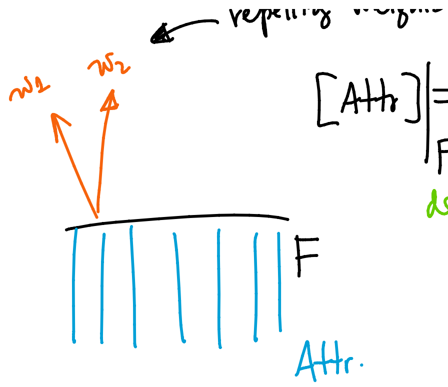
Def. A line bundle $\overset{\mathcal{J}}{\vee}$ on $\mathrm{Ell}_G(X)$ is attractive. if $\deg_A \mathcal{J}\Big|_{\substack{\text{any component } F\\ \text{of } X^A}}$

repelling weights

w_1 w_2

$[\mathrm{Attr}]| = \prod \vartheta(w_i)$

$\mathrm{Ell}_A(F)$



$$[\mathrm{Attr}]\Big|_F = \prod \vartheta(w_i)$$

$\deg = \sum w_i^2$

$$\mathrm{Ell}_A(F)$$
$$\downarrow$$
$$\mathrm{Ell}_A(pt) = E^{\text{rank } A}$$

$$\deg_A \mathcal{S}\Big|_F \in H^2_{alg}(E^{\text{rank } A}) = S^2 \text{ characters of } A.$$

rank $\binom{r+1}{2}$

If X has a polarization, ie. $T^{1/2} \in K_A(X)$ such that $T^{1/2} + (T^{1/2})^\vee = T$

nontrivial weights $\quad T^{1/2}\Big|_F = \pm w_i \qquad (\pm w_i)^2 = w_i^2$

$\Rightarrow \Theta(T^{1/2})$ is an attractive bundle

for any bundle V of rank r we have $\mathrm{Ell}_G(X) \longrightarrow S^r E \ni (x_1, \dots, x_r)$

pull back $\prod \theta(x_i) = \mathcal{O}(D_\Theta)$

theta divisor, i.e. image of $S^{r-1}E$

but so is $\mathcal{S} = \Theta(T^{1/2}) \otimes \underline{\underline{\text{anything of degree } 0}}$

if L is a line bundle on X, then L defines

$$E^\vee \times \mathrm{Ell}_G(X) \longrightarrow \underset{s}{E} \overset{\text{Poincare}}{\times} \underset{z}{E^\vee}$$

$$\frac{\theta(s+z)}{\theta(s)\,\theta(z)}.$$

chern root of L

add to the base of E

$$\downarrow$$

$$B \times \left(\mathrm{Pic}_{eq}(X) \otimes E\right)$$

z_1, z_2 = Kähler variables

Θ divisor

resonant

larger torus

$A \subset T$

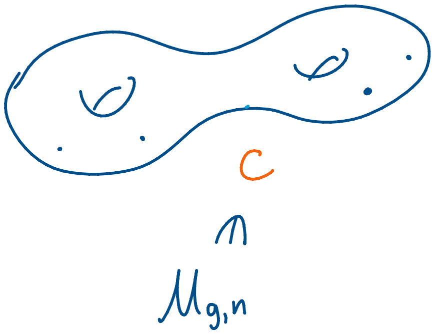
④ divisor

resonant locus

z_1, z_2 = Kähler variables

Definition : Elliptic stable envelope is a section of an attracting line bundle over $\mathrm{Ell}_T(X \times X^A)_{\text{nonres.}}$ which restricts to [Attr] near the diagonal

to be explained later



index $\in K(\mathcal{M}_{g,n} \times X^n)$