THE SPECTRAL THEORY OF QUANTUM CURVES AND TOPOLOGICAL STRINGS

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What is a quantum curve?

Algebraic curves are ubiquitous in modern mathematical physics

$$\Sigma(x, p) = 0$$

0) WKB curve in one-dimensional quantum mechanics

1) Seiberg-Witten curves of supersymmetric gauge theories

2) Mirror manifolds to toric (non-compact) Calabi-Yaus

3) "Spectral curves" for large N matrix models

4) Spectral curves for integrable systems

5) A-polynomials of knots

In all these problems, the “classical” theory is described by the periods of the Liouville one-form

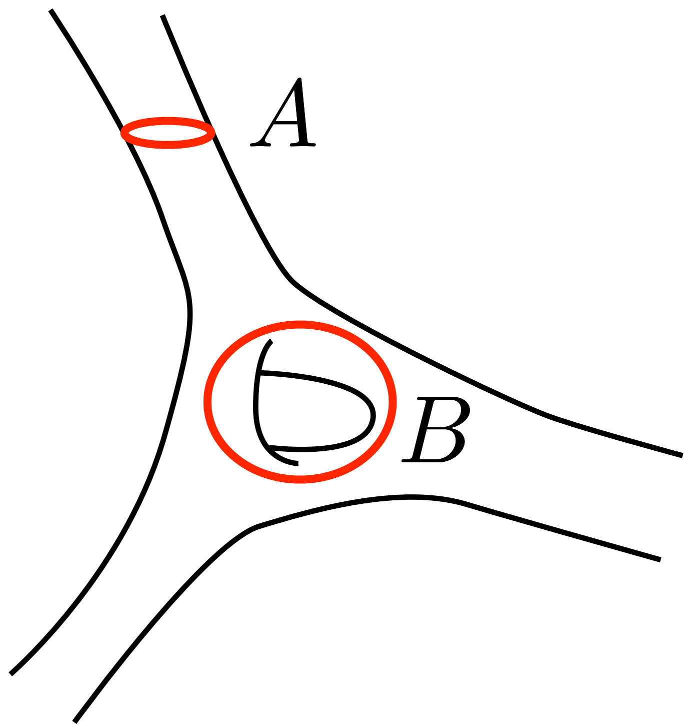
$$p(x)\mathrm{d}x$$

along the one-cycles of the curve

An important example for this talk is **local mirror symmetry**

toric Calabi-Yau manifold X $\longrightarrow$ mirror curve $\Sigma(\mathrm{e}^x, \mathrm{e}^p) = 0$

Here, the periods of the Liouville form determine the "prepotential" of topological string theory on X.



$$e^{x} + e^{p} + e^{-p-x} + u = 0$$

$$t = \oint_A p(x)\mathrm{d}x$$

mirror map

$$\frac{\partial F_0}{\partial t} = \oint_B p(x)\mathrm{d}x$$

$$F_0(t) = \sum_{d\geq 1} N_{0,d}\mathrm{e}^{-dt}$$

genus zero GW invariants

This looks very much like a WKB approximation to a quantum mechanical problem underlying the curve

Therefore, it is tempting to think that the corrections to the "classical" theory can be obtained by "quantizing" the curve in an appropriate way

In the case of local mirror symmetry, this leads to the idea that the higher genus free energies of the topological string

$$F_g(t) = \sum_{d\geq 1} N_{g,d} \mathrm{e}^{-dt}$$

can be obtained by "quantizing" the mirror curve [ADKMV]

Perturbative quantization schemes

There are two perturbative quantization schemes for curves which produce a series of quantum corrections to the prepotential.

The first one is based on a direct quantization of the curve, by promoting the canonically conjugate *x,p* variables to Heisenberg operators

$$[\mathrm{x}, \mathrm{p}] = \mathrm{i}\hbar$$

This leads to a Schroedinger-type equation

$$\Sigma(x, p = -\mathrm{i}\hbar\partial_x)\psi(x,\hbar) = 0$$

One can now use the all-orders WKB method to obtain "quantum" versions of the Liouville form and the periods

$$p(x,\hbar) \sim p(x) + \sum_{n\geq 1} p_n(x)\hbar^{2n}$$

$$t(\hbar) = \oint_A p(x,\hbar)\mathrm{d}x \qquad \frac{\partial F(t,\hbar)}{\partial t} = \oint_B p(x,\hbar)\mathrm{d}x$$

This procedure was applied to SW/mirror curves by
[Mironov-Morozov, ACDKV]

However, the quantum prepotential obtained in this way

$$F(t,\hbar) = \sum_{n\geq 0} F_n(t)\hbar^{2n}$$

does **not** describe the usual topological string free energy. Rather, it gives the so-called "Nekrasov-Shatashvili free energy"

A different "quantization" scheme is obtained by the topological recursion of Eynard-Orantin

$$\Sigma(x,p) \to \{F_g(t)\}_{g=0,1,2,\cdots}$$

This *does* give the topological string free energies (BKMP conjecture, now a theorem) but there is no obvious relation to conventional quantization

There has been recent progress in relating topological recursion to some sort of quantization of the curve [Bouchard-Eynard, Iwaki, Eynard-Garcia Failde, Orantin-Marchal], but their resulting quantum curves have infinitely many quantum corrections and I will not follow this route

Another shortcoming is that these approaches are perturbative in nature.

We recall that the "total free energy" of topological string theory

$$F(t, g_s) = \sum_{g \geq 0} F_g(t) g_s^{2g-2}$$

does **not** define a function, since it is a factorially divergent series

$$F_g(t) \sim (2g)!$$

Topological recursion does not give much insight on this problem.

A full quantum realization of topological string theory should give a *non-perturbative definition* of the theory, in which the series appears as the asymptotic expansion of a well-defined function.

It turns out that progress along this direction can be made if one looks at the quantum curve as an operator on the natural Hilbert space $L^2(\mathbb{R})$

This might seem unnatural from the point of view of complex geometry, since the spectral theory of operators is very sensitive to reality and positivity issues. However, one can "complexify" afterwards, as we will see.

Operators from curves

By quantizing interesting families of curves one obtains well-known operators on $L^2(\mathbb{R})$, as well as new ones:

1) SW curves of Argyres-Douglas theories lead to Schroedinger operators with polynomial potentials
[e.g. Ito-Shu, Grassi-Gu]

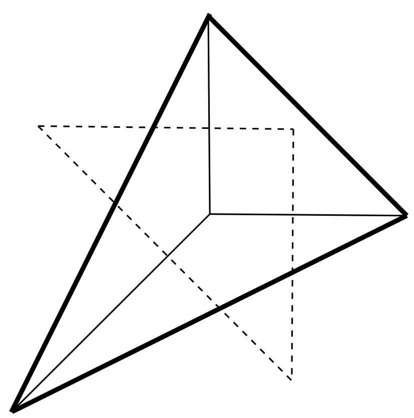
$$p^2 + V_N(x) - E = 0 \longrightarrow \mathsf{H} = \mathsf{p}^2 + V_N(\mathsf{x})$$

2) SW curves of *SU(N)* theories lead to *deformed* Schroedinger operators [Grassi-M.M.]

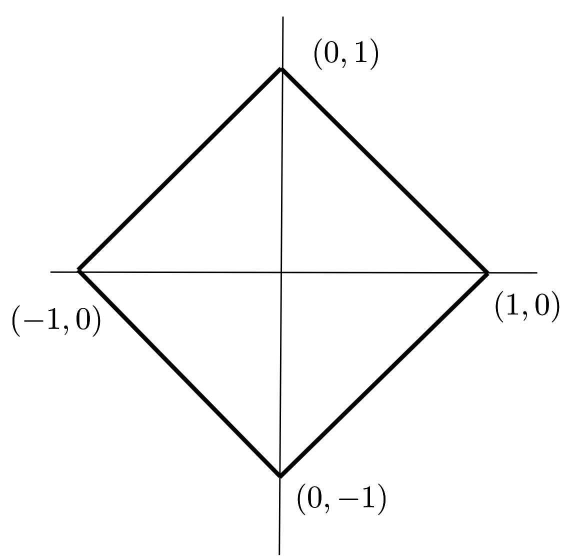
$$2\cosh(p) + V_N(x) - E = 0 \longrightarrow \mathsf{H} = 2\cosh\mathsf{p} + V_N(\mathsf{x})$$

3) Mirror curves of toric CYs lead to a new family of "Weyl" or exponentiated Heisenberg operators

$$X \to \mathsf{O}_X$$



$$\mathsf{O}_{\mathbb{P}^2} = \mathrm{e}^{\mathsf{x}} + \mathrm{e}^{\mathsf{p}} + \mathrm{e}^{-\mathsf{x}-\mathsf{p}}$$



$$\mathsf{O}_{\mathbb{F}_0} = \mathrm{e}^{\mathsf{x}} + \xi_{\mathbb{F}_0} \mathrm{e}^{-\mathsf{x}} + \mathrm{e}^{\mathsf{p}} + \mathrm{e}^{-\mathsf{p}}$$

For simplicity I will consider polytopes with a single inner point

The spectral problem defined by quantum curves is therefore a fascinating generalization of the spectral theory of one-dimensional Schroedinger operators

The main question we want to address is:
is there a relation between the spectral theory of these operators, and the geometric, “quantum” objects associated to the underlying curves?

I will from now on focus on (local) mirror curves

Spectral theory

Theorem
[Grassi-Hatsuda-M.M., Kashaev-M.M., Laptev-Schimmer-Takhtakjan]

The operator $\rho_X = \mathsf{O}_X^{-1}$ on $L^2(\mathbb{R})$ is positive definite and of trace class

(assuming $\hbar > 0$ and some conditions on "mass parameters")

Discrete spectrum $\mathrm{e}^{-E_n}, \quad n = 0, 1, \cdots$

and all its traces are finite

$$\mathrm{Tr}\, \rho_X^\ell = \sum_{n \geq 0} \mathrm{e}^{-\ell E_n} < \infty, \qquad \ell = 1, 2, \cdots$$

Since the operator is of trace class, we can define its *Fredholm determinant*

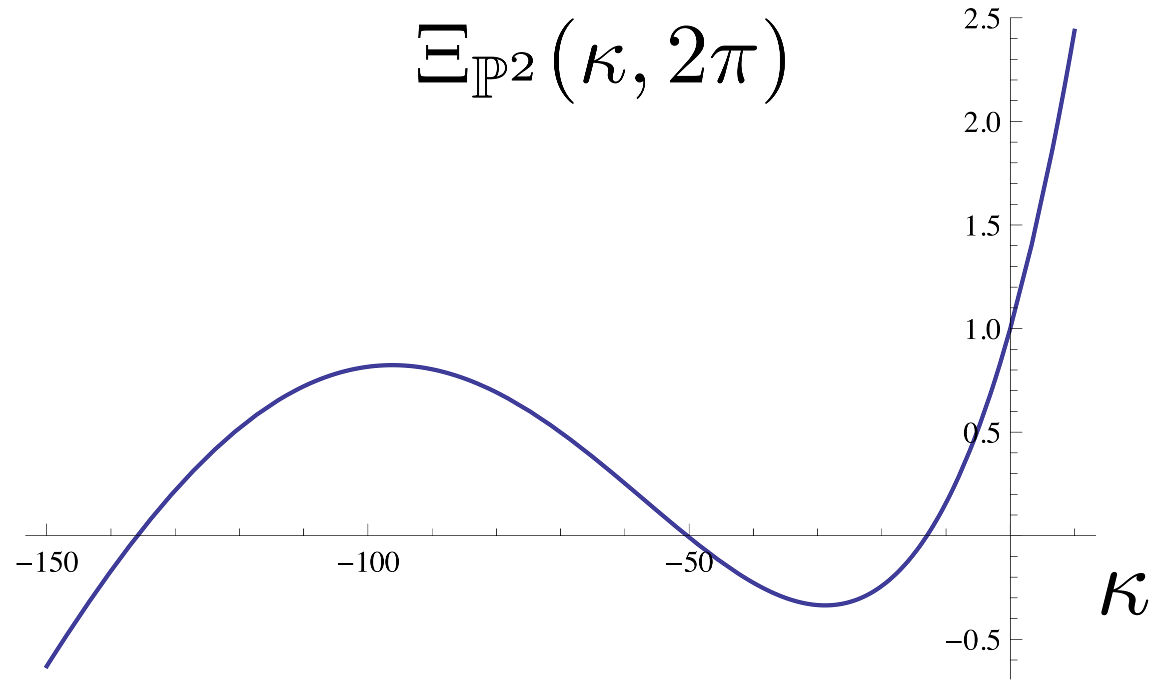
$$\Xi_X(\kappa) = \det(1+\kappa\rho_X) = \prod_{n\geq 0}\left(1+\mathrm{e}^{\mu-E_n}\right)$$

which is an *entire* function of $\kappa = \mathrm{e}^{\mu}$

It can be shown that κ is identified with the modulus of the CY, therefore this is an entire function on the CY moduli space

$$\Xi_X(\kappa) = 1 + \sum_{N=1}^{\infty} Z_X(N,\hbar)\kappa^N$$

"fermionic" spectral traces

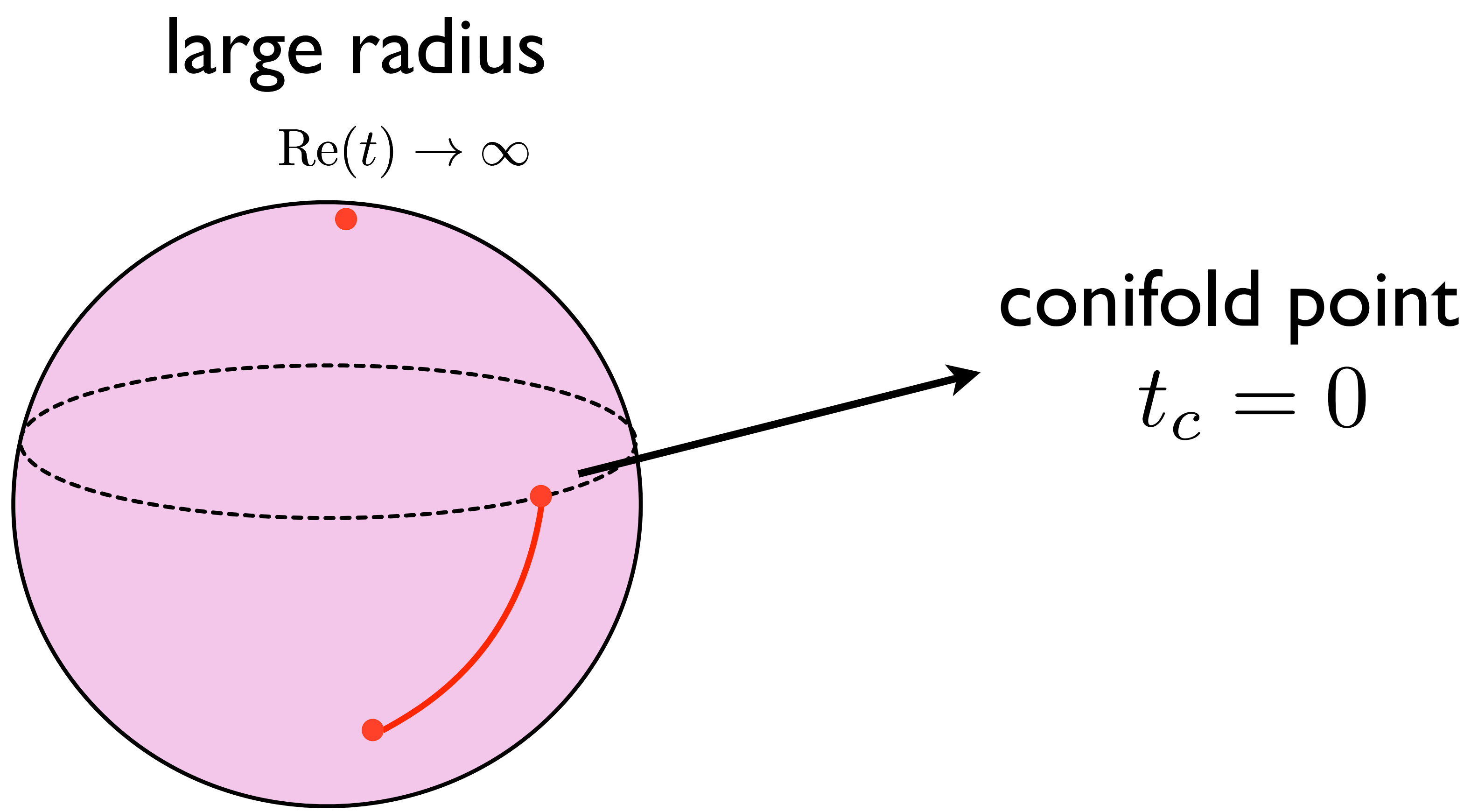


zeroes= spectrum $-\mathrm{e}^{E_n}$

At weak coupling, when $\hbar \to 0$, these quantities are related to the NS topological string

Surprisingly, the strong coupling limit $\hbar \to \infty$ leads to the conventional topological string

The strong coupling result is conveniently formulated by using topological string theory in "conifold" coordinates. We recall that, in the moduli space of the CY, there is a conifold point where the mirror curve becomes singular.



We can parametrize the moduli space by a special vanishing coordinate at the conifold, t_c

A conjecture

Let us consider the following 't Hooft limit of the fermionic spectral traces

$$N \to \infty \qquad \hbar \to \infty \qquad \frac{N}{\hbar} = t_c$$

Then,

$$\log Z_X(N, \hbar) \sim \sum_{g \geq 0} F_g(t_c) \hbar^{2-2g}$$

note that $\hbar = \frac{1}{g_s}$

Corollary: a "volume conjecture"

Explicit calculations show that these spectral traces are very similar to state integrals for hyperbolic three-manifolds (or to partition functions of 3d susy theories)

It turns out that the volume conjecture for state integrals [Kashaev, Andersen-Kashaev] has a counterpart in this context:

$$Z_X(N, \hbar) \sim \exp\left(-\hbar V\right)$$

$\hbar \to \infty$

N fixed

V : value of the Kahler parameter at the conifold point

In order to obtain the traditional Gromov-Witten free energies, one can instead consider the following limit of the Fredholm determinant

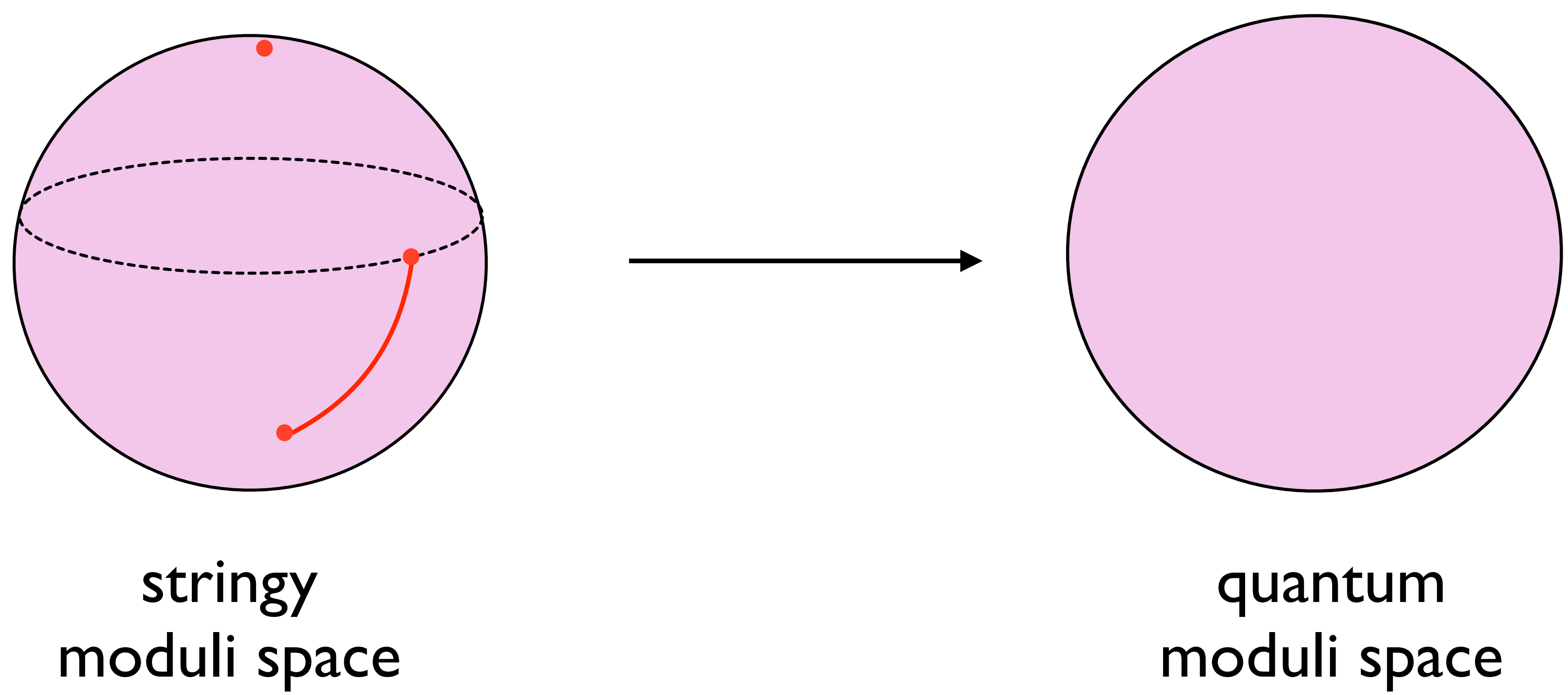
$$\hbar \to \infty, \qquad \frac{\mu}{\hbar} = t \qquad \text{fixed}$$

One has, conjecturally, the asymptotic expansion

$$\log \Xi_X(\kappa) \sim \sum_{g \geq 0} F_g(t) \hbar^{2-2g}$$

(this asymptotics has however oscillatory corrections)

One striking consequence of this picture is the following. The topological string free energies $F_g(t)$ have a branch cut structure and are singular at the conifold point. However, in the non-perturbative version, the Fredholm determinant is an entire function with no singularity!



Therefore, branch cuts and singularities are *artifacts* of the asymptotic expansion (i.e. of perturbation theory)

[cf. Maldacena-Moore-Seiberg-Shih]

Although I have emphasized asymptotic results, we have conjectured an exact expression for the Fredholm determinant of quantum curves, in terms of (refined) BPS invariants of the toric CY. Roughly,

$$\Xi_X(\kappa) = \sum_{n\in\mathbb{Z}} \exp\left(\sum_{g\geq 0} F_g(t + 2\pi \mathrm{i}\hbar n)\hbar^{2g-2} + \mathrm{WKB} \right)$$

This provides in principle a complete solution of the spectral problem for these “Weyl-type” operators

Note that this spectral problem is different from the one addressed by Nekrasov and Shatashvili: we quantize *curves*, not integrable systems. The two problems are identical in curves of genus one with a 5d susy gauge theory realization; for curves of genus $g>1$ they are different, albeit not completely unrelated.

However, our solution of the spectral problem gives non-perturbative corrections to the quantization conditions of NS

Some applications and extensions

In an independent development, it has been found that tau functions of Painleve functions can be written in terms of 4d instanton partition functions by using the same type of Zak transform [Gamayun-Iorgov-Lissovvy]

$$\tau = \sum_{n \in \mathbb{Z}} \exp \left(\sum_{g \geq 0} F_g(a + 2\pi \mathrm{i} \hbar n) \hbar^{2g-2} \right)$$

SW/4d limit

This fits in our framework, after taking a "geometric engineering"/4d limit of the appropriate Calabi-Yau. It leads to a reinterpretation of the tau function as a spectral determinant [Bonelli-Grassi-Tanzini]

Complexification

So far we have required $\hbar$ to be real and positive, but it is clearly interesting to complexify it.

It turns out that the integral kernel of ρ_X can be computed explicitly in many cases [Kashaev-M.M.] . It involves Faddeev's quantum dilogarithm

$$\Phi_{\mathsf{b}}(x) \qquad\qquad \mathsf{b}^2 \propto \hbar$$

which can be analytically continued to the complex plane. This makes it possible to reformulate topological string partition functions in the language of (double) q-series, study their resurgent properties in the complex $\hbar$ plane, and so on [in progress with Jie Gu].

Open problems

I have presented a precise "duality" between (conventional) topological strings and the spectral theory of trace class operators. This provides a non-perturbative definition of topological strings (in the spirit of AdS/CFT) and has many implications in both fields.

What is lacking is a more physical understanding of this duality. Formally, it suggests a deep relationship with a 3d SUSY theory, or with one-dimensional fermions, but this has not been made more concrete

Mathematically, most of our conjectures are unambiguous and well-defined, but probably very hard to prove (it also seems to be difficult to find mathematicians who know well both sides of the conjectures, i.e. spectral theory and Gromov-Witten theory!)

Thank you for your attention!