# Cohomological Hall algebras, instantons and vertex algebras

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My talk is a non-technical overview of recent applications of COHA to the geometric representation theory in the spirit of early works of Nakajima (like in his “Lectures on Hilbert schemes"). Recall that algebro-geometrically the moduli space of framed $U(r)$ instantons of charge $n$ on $S^4 = \mathbb{R}^4 \cup \{\infty\}$ can be identified with the moduli space $\mathcal{M}(r, n)$ of torsion-free sheaves of rank $r$ and $c_2 = n$ on $\mathbb{CP}^2$ together with a fixed trivialization on a given line $l_\infty \simeq \mathbb{CP}^1 \subset \mathbb{CP}^2$ (ADHM, Donaldson, Nakajima). The conjecture of Alday-Gaiotto-Tachikawa (see arXiv: 0906.3212) predicts that on the direct sum over index $n$ of equivariant Borel-Moore homology groups of $\mathcal{M}(r, n)$ there is an action of the $W$-algebra of $gl(r)$. This is a generalization of the result of Nakajima for $r = 1$. It can be considered as an example of the BPS/CFT correspondence of Nekrasov, which predicts that various algebraic structures of $4d$ gauge theories correspond in a non-trivial way to certain algebraic structures of $2d$ CFT's.

One of my goals is to discuss a generalization of the AGT conjecture (now theorem after Maulik-Okounkov, Schiffmann-Vasserot). This generalization is related to the recent works of Nekrasov, Gaiotto, Rapcak and others on one side and my joint works with Rapcak, Yang, Zhao, Creutzig, Chuang, Diaconescu on the other side (see arXiv:1810.10402, arXiv:1907.13005 as well as another work in progress with Rapcak, Yang and Zhao).
In our work we look at the AGT-type results from the perspective of $3CY$ categories rather than $2CY$ categories. As a result, the main part is played not by various versions of quantum groups, but by the Cohomological Hall algebra (COHA) introduced in my joint paper with Maxim Kontsevich (see arXiv: 1006.2706 ) . Quantum algebras (like e.g. quiver Yangians) are secondary objects. I start with a reminder on COHA.

## COHA: definition

For a quiver $Q$ endowed with potential $W$ (=cyclically invariant polynomial in arrows) the Cohomological Hall algebra as a graded vector space is given by:

$$\mathcal{H}^{(Q,W)} = \oplus_{\gamma\in\mathbb{Z}^I_{\geq 0}} H^{BM}_{G_\gamma}(M_\gamma(Q), \phi_{Tr(W)}) := \oplus_\gamma \mathcal{H}_\gamma .$$

Here $M_\gamma(Q)$= space of representations of $Q$ in $\mathbb{C}^\gamma = (\mathbb{C}^{\gamma^i})_{i\in I}$ ($I$=set of vertices of $Q$). The gauge group $G_\gamma$ is the product of general linear groups $GL(\gamma^i, \mathbb{C})$. It acts by changing basis at each vertex. Summands are the equivariant Borel-Moore homology of $M_\gamma$ with coefficients in the sheaf of vanishing cycles of the function $Tr(W)$ (BM homology:=dual to compactly supported cohomology). The associative algebra structure is defined in the usual Hall algebra style via equivariant cohomology of spaces of short sequences of representations of $Q$ and utilizing Thom-Sebastiani theorem for vanishing cycles (see 1006.2706 for details). Can it be computed?

## Quiver with $d$ loops, W=0

For $W = 0$ COHA can be computed for any $Q$ as a shuffle algebra. As an example, let $Q = Q_d$ be now a quiver with just one vertex and $d \geqslant 0$ loops. Then the product is given by

$$(f_1 \cdot f_2)(x_1, \dots, x_{n+m}) :=$$

$$\sum_{\substack{i_1 < \dots < i_n \\ j_1 < \dots < j_m \\ \{i_1,\dots,i_n,j_1,\dots,j_m\} = \\ = \{1,\dots,n+m\}}} f_1(x_{i_1}, \dots, x_{i_n})\, f_2(x_{j_1}, \dots, x_{j_m}) \left( \prod_{k=1}^{n} \prod_{l=1}^{m} (x_{j_l} - x_{i_k}) \right)^{d-1}$$

for symmetric polynomials, where $f_1$ has $n$ variables, and $f_2$ has $m$ variables. The product $f_1 \cdot f_2$ is a symmetric polynomial in $n + m$ variables. It is bigraded by the cohomological degrees and by the dimension vectors.

## Explicit answer for $d = 0$

For $d = 0$ the above answer gives identification of $\mathcal{H} = \oplus_{n,m}\mathcal{H}_{n,m}$ with a free ferminon algebra, i.e. Grassmann algebra, generated by odd elements $\psi_1, \psi_3, \psi_5, \dots$ of bidegrees $(1,1), (1,3), (1,5), \dots$. Generators $(\psi_{2i+1})_{i\geqslant 0}$ correspond to the additive generators $(x^i)_{i\geqslant 0}$ of

$$H^\bullet(\mathbb{C}P^\infty) = H^\bullet(\mathrm{BGL}(1,\mathbb{C})) \simeq \mathbb{Z}[x] \simeq \mathbb{Z}[x_1]\,.$$

A monomial in the exterior algebra

$$\psi_{2i_1+1}\cdot\dots\cdot\psi_{2i_n+1} \in \mathcal{H}_{n,\sum_{k=1}^n(2i_k+1)}\,, \quad 0 \leqslant i_1 < \dots < i_n$$

corresponds to the Schur symmetric function $s_\lambda(x_1,\dots,x_n)$, where

$$\lambda = (i_n + (1-n), i_{n-1} + (2-n), \dots, i_1)$$

is a partition of length $\leqslant n$.

## Explicit answer $d = 1$

For $d = 1$ algebra $\mathcal{H}$ is isomorphic to the free boson algebra, i.e. to the algebra of symmetric functions in infinitely many variables. It is a polynomial algebra generated by even elements $\phi_0, \phi_2, \phi_4, \dots$ of bidegrees $(1,0), (1,2), (1,4), \dots$. Again, the generators $(\phi_{2i})_{i\geqslant 0}$ correspond to the additive generators $(x^i)_{i\geqslant 0}$ of $H^\bullet(\mathbb{C}P^\infty) \simeq \mathbb{Z}[x]$. For other $d$ we have “Bott periodicity", i.e. the answer depends on $d$ modulo 2. Moreover, as vector spaces for even and odd $d$ COHAs are the same: $\oplus_{n\geq 0} H^\bullet(BGL(n))$. Thus we have a kind of boson-fermion correspondence.

## Relation to refined BPS states in this toy-model example

The Hilbert-Poincaré series $P_d = P_d(z, q^{1/2})$ of bigraded algebra $\mathcal{H}$ twisted by the sign $(-1)^{\text{parity}}$ (motivic DT-series) is: $\sum_{n\geqslant 0, m\in\mathbb{Z}}(-1)^m \dim(\mathcal{H}_{n,m})\, z^n q^{m/2}$, where $q = \mathbb{L} = [\mathbb{A}^1]$.

**Theorem**

*For any $d \geqslant 0$ there exist integers $\delta^{(d)}(n, m)$ for all $n \geqslant 1$ and $m \in (d-1)n + 2\mathbb{Z} = (1-d)n^2 + 2\mathbb{Z}$, such that for a given number $n$ we have $\delta(n, m) \neq 0$ only for* **finitely many** *values of $m$, and*

$$P_d = \prod_{n\geqslant 1}\prod_{m\in\mathbb{Z}} \left(q^{m/2} z^n; q\right)_\infty^{\delta^{(d)}(n,m)}$$

*where we use the standard notation for the q-Pochhammer symbol:*

$$(x; q)_\infty := (1-x)(1-qx)(1-q^2x)\ldots \quad .$$

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## Dimensional reduction

Computation of COHA for $W \neq 0$ is much more difficult. A useful tool for that is the “dimension reduction from 3 to 2" proved in our 1006. 2706 paper. It says that if the potential $W$ is linear with respect to a collection of arrows, then we can replace $\mathcal{H}_\gamma$ by an isomorphic vector space by restricting to the critical set of $W$ with respect to this collection of arrows. Later Ben Davison proved that the naturally defined algebra structure on this “reduced" vector space agrees with the algebra structure on COHA. The term “reduction" refers to the geometric analogy: product of a Calabi-Yau surface with $\mathbb{C}$ is a Calabi-Yau 3-fold. Algebraically: add a loop at each vertex to a symmetric quiver with the “moment map relations" and endow the new quiver with certain cubic potential. The direct sum of equivariant cohomology of the representation spaces of the symmetric quiver with relations is an algebra which is isomorphic to the COHA of the new quiver with potential.

## Framed quiver with potential

a) Quiver $Q_3$: one vertex, three loops $B_1, B_2, B_3$, potential $W_3 = B_3[B_1, B_2]$. Representation of $(Q_3, W_3)$=critical locus of $Tr(W_3)$ = variety of commuting matrices: $[B_i, B_j] = 0$.

b) More interesting is the framed quiver $Q_3^{fr}$:

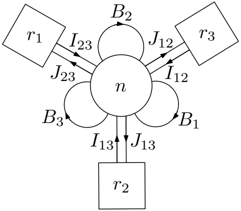


**Figure:** Framed quiver for the spiked instantons moduli space

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$$W_3^{fr} := B_3([B_1, B_2] + I_{12}J_{12}) + B_2([B_1, B_3] + I_{13}J_{13}) + B_1([B_2, B_3] + I_{23}J_{23}) = W_3 + B_1 I_{23}J_{23} + B_2 I_{13}J_{13} + B_3 I_{12}J_{12}.$$

## Quiver description of spiked instantons

We call a representation of $(Q_3^{fr}, W_3^{fr})$ stable if there is no non-trivial $V \subset \mathbb{C}^n$, which is invariant with respect to $B_i, i = 1, 2, 3$, and $I_{ab}(\mathbb{C}^{r_c}) \subset V$, $\{c\} = \{1, 2, 3\} - \{a, b\}$. Stable representations form the moduli space $\mathcal{M}(r_1, r_2, r_3, n)$. This moduli space turns out to be isomorphic to the moduli space of spiked instantons introduced by Nekrasov in arXiv:1608.07272.

If only $B_1, B_2, I_{12}, J_{12}$ are present on the figure we arrive to the ADHM description of the moduli space $\mathcal{M}(r, n) = \mathcal{M}(0, 0, r, n)$ of rank $r$ framed instantons mentioned before.

In general we don't know pure geometric description of $\mathcal{M}(r_1, r_2, r_3, n)$. It should be the moduli space of coherent sheaves on $\mathbb{C}^3$ supported on the union of coordinate planes in $\mathbb{C}^3$, which are pure of rank $r_c$ on the coordinate plane $x_c = 0, c = 1, 2, 3$ and satisfying some additional stability condition. The moduli space $\mathcal{M}(r, n)$ should be thought of as the one associated with the non-reduced divisor $r\mathbb{C}^2 \subset \mathbb{C}^3$.

Let $V_{r_1,r_2,r_3} = \bigoplus_{n\geq 0} H^{BM}_{GL(n)\times GL(r_1)\times GL(r_2)\times GL(r_3)\times \mathbf{T}_2}(\mathcal{M}(r_1,r_2,r_3,n),\phi_{W_3^{fr}})$, where the torus $\mathbf{T}_2=(\mathbb{C}^*)^2\subset(\mathbb{C}^*)^3$ acts by $(B_i,I_{ab},J_{ab})\mapsto(t_iB_i,I_{ab},t_at_bJ_{ab})$ subject to the Calabi-Yau condition $t_1t_2t_3=1$. Various versions of COHA for $(Q_3,W_3)$ act on $V_{r_1,r_2,r_3}$. The most interesting for us is the equivariant spherical COHA $\mathcal{SH}^{(Q_3,W_3)}$. It is defined as follows: first we define the equivariant version of COHA adding the action of $\mathbf{T}_2$ on the loops $B_i$, then we take the subalgebra of the equivariant COHA generated by the graded component with $\gamma=1$. In arXiv:1810.10402 we proved the following result.

**Theorem**

*The algebra $\mathcal{SH}^{(Q_3,W_3)}$ acts on $V_{r_1,r_2,r_3}$ by correspondences. There is a Hopf algebra structure on $\mathcal{SH}^{(Q_3,W_3)}$. The Drinfeld double Hopf algebra $D(\mathcal{SH}^{(Q_3,W_3)})$ acts on $V_{r_1,r_2,r_3}$ as well.*

## Spherical COHA and the affine Yangian

The Drinfeld double of equivariant spherical COHA is isomorphic to the affine Yangian of $gl(1)$. This observation is based on the already mentioned dimensional reduction theorem for COHA. The affine Yangian $Y_{\hbar_1,\hbar_2,\hbar_3}(\widehat{gl(1)})$ is an associative algebra over $\mathbb{C}[\hbar_1,\hbar_2,\hbar_3], \hbar_1+\hbar_2+\hbar_3=0$ subject to explicit relations which I do not recall here (see e.g. Tsymbaliuk, 1404.5240). There is also a shifted version of this affine Yangian $Y_{\hbar_1,\hbar_2,\hbar_3,(z_1,...,z_l;z'_1,...,z'_k)}(\widehat{gl(1)})$ which depends on complex parameters $z_i, z'_j$. If no $z'_j$ are present, it is the shifted affine Yangian $Y_l(z), z=(z_1,...,z_l)$ considered by Kodera-Nakajima in arXiv:1608.00875. We will return to this later.

## Quantum MacMahon

The relation of $\mathcal{SH}^{(Q_3,W_3)}$ with $Y_{\hbar_1,\hbar_2,\hbar_3}(\widehat{gl(1)})$ can be guessed from the comparison of the Hilbert-Poincaré series for the positive part of the affine Yangian with the one of $\mathcal{H}^{(Q_3,W_3)}$, which is the motivic DT-series from Section 5.6 of 1006.2706:

### Proposition

*For the motivic DT-series $A^{(Q_3,W)}$ we have the following formula:*

$$A^{(Q_3,W_3)} = \prod_{n,m\geqslant 1} (1 - q^{m-2}\,\widehat{\mathbf{e}}_1^n)^{-1},$$

*where $\widehat{\mathbf{e}}_1$ is the generator of the motivic quantum torus and $q$ as before denotes the motive of the affine line (Lefschetz motive).*

The series is essentially the quantum MacMahon function (and becomes such for spherical COHA).

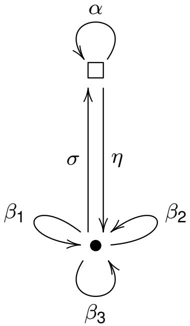
## Further comments on this example

a) As a corollary of the above Theorem we deduced that the Gaiotto and Rapcak vertex algebra at the corner $W_{r_1,r_2,r_3}$ (see arXiv:1703.00982) acts on $V_{r_1,r_2,r_3}$.
b) From physics perspective we study the action of BPS algebra of $D0$ branes on the Calabi-Yau 3-fold $\mathbb{C}^3$ on the space of $D0 - D4$ bound states, where $D4$ branes correspond to stacks of coordinate hyperplanes in $\mathbb{C}^3$.
c) We formulated a conjecture that the above results can be generalized to any toric Calabi-Yau 3-fold defined by a toric diagram with faces colored by non-negative integers $r_c$ (e.g. decorated dimer model). Probably part of this conjecture was proved in the recent paper 2003.08909 by Li and Yamazaki using the crystal melting techniques.

## Geometric approach

A different class of representations of COHA arises when we consider the Hilbert scheme of points of the non-reduced singular divisor $x_1^{r_1} x_2^{r_2} x_3^{r_3} = 0$ of $\mathbb{C}^3$. In the above-mentioned paper with Chuang, Creutzig and Diaconescu arXiv:1907.13005 we studied the simplest case of the “fat divisor" $x_3^{r_3} = 0$. The case $r_3 = 1$ is classical (Hilbert scheme of $\mathbb{C}^2$). In general the study is more difficult, since the structure of the Hilbert scheme of a non-reduced divisor is not known. Main result: the direct sum of BM homology of the Hilbert schemes gives the vacuum representation of the $W$-algebra of $gl(r_3)$. We used the relation of the Hilbert scheme of the fat divisor to the moduli space of Higgs bundles on $\mathbb{CP}^2$ with nilpotent Higgs field, which comes from the multiplication by $x_3$. The corresponding quiver is obtained from the one of Nakajima by adding an additional loop at the framing vertex and fixing the conjugacy class of the loop.

## Quiver with additional loop

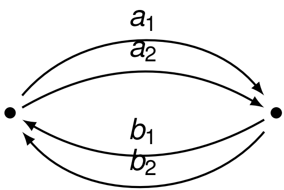


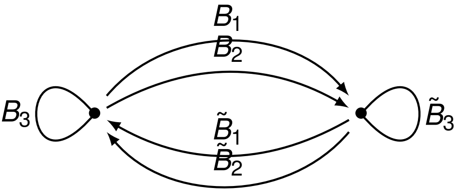

$$W = \beta_3 \circ (\beta_1 \circ \beta_2 - \beta_2 \circ \beta_1) + \beta_3 \circ \eta \circ \sigma - \eta \circ \alpha \circ \sigma$$

## Resolved conifold

The action of COHA of $\mathbb{C}^3$, i.e. COHA of $(Q_3, W_3)$ on the cohomology of framed quiver is not accidental. By general reasons explained in my 1404.1606 this should be expected in many other situations. In the work in progress with Rapcak, Yang and Zhao we consider COHA of the CY 3-folds $X = X_{n,m}$ which is resolution of singularities of the hypersurface $\{xy = z^n w^m\} \subset \mathbb{C}^4$. We study in detail the case $n = m = 1$, which is the resolved conifold as well as as $n = 2, m = 0$ which is $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$ orbifold. For the resolved conifold the corresponding quiver with potential and the chamber structure of the space of stability conditions on $D^b(X)$ were discussed mathematically by Nagao-Nakajima in 0809.2892.

## Quivers with potential for local rational curve




with potentials

$$W_{res} = a_1 b_1 a_2 b_2 - a_1 b_2 a_2 b_1$$

and

$$W_{orb} = B_2 \tilde{B}_1 \tilde{B}_3 - \tilde{B}_2 \tilde{B}_3 B_1 + \tilde{B}_2 B_1 B_3 - B_2 B_3 \tilde{B}_1 .$$

## Shifted Yangians

Without going into details, let me mention that similarly to the case of $\mathbb{C}^3$ which we discussed above there is an action of the COHA of $X_{1,1}$ on the equivariant BM homology of the framed moduli spaces associated with various chambers. At the level of double of COHA of $X_{1,1}$ this action gives rise to a representation of the shifted affine Yangians $Y_l(z), l \in \mathbb{Z}$ mentioned before. We expect that these results can be extended to other $X_{n,m}$. Furthermore we expect appearance of shifted super Yangians, whatever this means.
Appearance of the shifted Yangians naturally brings us to the speculations about possible relation of COHA to Coulomb branches of $3d$ $\mathcal{N} = 4$ gauge theories.

## COHA and Coulomb branch: Jordan quiver gauge theory

Consider the 3$d$ $\mathcal{N} = 4$ gauge theory $\mathcal{T}_{Jordan}$ corresponding to $G = GL(n)$ and the symplectic representation $\mathbf{M} = \mathbf{N} \oplus \mathbf{N}^*$ where $\mathbf{N} = gl(n) \oplus (\mathbb{C}^n)^{\oplus l}$ (adjoint representation of $gl(n)$ plus $l$ copies of the standard one). According no Kodera-Nakajima (see arXiv:1608.00875) there are epimorphisms to the quantized Coulomb branch $\mathbb{C}[\mathcal{M}_{\mathcal{C}}]_{\hbar}$ of $\mathcal{T}_{Jordan}$ of the *shifted* affine Yangian $Y_l(z)$ as well of a certain subalgebra $Y_l$ of the *non-shifted* affine Yangian of $gl(1)$. This observation connects us with the previous discussion.

## Framed quiver gauge theories

COHA should also appear in a class of examples related to framed quiver gauge theories of *ADE* type. In order to fix the theory one fixes the dimension vector $v$ (=positive root of the corresponding quiver $Q$) as well as the framing dimension $w$ (dominant weight of the gauge group $G$). Let $Gr_w$ be the closure of the $G(\mathcal{O})$-orbit in the affine Grassmannian $Gr_G = G(\mathcal{K})/G(\mathcal{O})$ corresponding to $w$ and $Gr_w^v$ be the transversal slice. According to BFN the quantized algebra of functions on $Gr_w^v$ is the quantized Coulomb branch for the corresponding framed quiver gauge theory. BFN also proved that this algebra is isomorphic to the shifted Yangian of $\mathfrak{g} = Lie(G)$, where the shift is determined by $w$ and truncation by $v$.

Computations made by Yang and Zhao show that in this case there are two homomorphisms of the equivariant preprojective algebra of $Q$ to this quantized Coulomb branch. The equivariant preprojective algebra is the dimensional reduction of the equivariant COHA associated with the tripled quiver $Q$ (tripled=add opposite for each arrow and add new loop at each vertex) endowed with a cubic potential $\sum_{i,a} l_i[a, a^*]$, where $l_i$ is a loop at the vertex $i$ and $a$ (resp. $a^*$) is an arrow (resp. opposite arrow). Hopefully these two homomorphisms can be combined into a homomorphism of the double of COHA which we discussed before.Then we obtain a homomorphism of the double of equivariant spherical COHA to the quantized Coulomb branch of a framed quiver gauge theory.

## COHA and Nakajima proposal for the Coulomb branch

One can ask why COHA should appear in the story with Coulomb branches. Recall that COHA is defined as the convolution algebra for the BM-homology of the stack of objects of a $3CY$ category. In the first part of the talk it was the $3CY$ category associated with the pair $(Q, W)$. In arXiv:1006.2706 we speculated about bigger framework in which COHA (or its derived version) could be defined. In particular, one can try to think of COHA of a compact CY 3-fold $X$ endowed with the holomorphic CS functional $CS_{\mathbb{C}}(A) = \int_X Tr(\overline{\partial}A + \frac{1}{3}A^3)\Omega^{3,0}$, where $A$ is a $(0,1)$-connection and $\Omega^{3,0}$ is a holomorphic volume form. Critical locus of $CS_{\mathbb{C}}$ consists of holomorphic vector bundles. We can think of the corresponding COHA as of $H_{\bullet}^{BM}(Coh_X, \phi_{CS_{\mathbb{C}}})$, where $Coh_X$ is the stack of coherent sheaves. In the case of the complexified Chern-Simons this was made more precise in the last section of arXiv:1006.2706.

## Nakajima's proposal

In arXiv:1503.03676 Nakajima propose non-rigorous but intuitively appealing approach to $\mathbb{C}[M_{\mathcal{C}}]$ via the BM-homology with coefficients in the sheaf of vanishing cycles of the CS-type functional. One can combine that with ideas of motivic Donaldson-Thomas theory (see arXiv:0811.2435, arXiv:1006.2706). His analog of the holomorphic CS functional depends on some choices: group and symplectic representation $(G, \mathbf{M})$, compact Riemann surface $C$ with fixed spin structure $K_C^{1/2}$, principal $G$-bundle $P$. Nakajima's functional depends on fields which are: $\overline{\partial} + A$=the $(0,1)$-connection on $P$, and the Higgs field $\Phi \in \Gamma(C, K_C^{1/2} \otimes (P \otimes_G \mathbf{M}))$. The actual formula $CS_{Nak} = CS_{Nak}(A, \Phi) = \frac{1}{2}\int_C \omega_{\mathbf{M}}^{2,0}((\overline{\partial} + A)(\Phi) \wedge \Phi)$. Here the wedge product is multiplied by the holomorphic symplectic form on the vector space $\mathbf{M}$. This gives a $C^{\infty}$ section of $(T^{0,1})^* \otimes K_C = (T^{1,1})^*$, which can be integrated over $C$.

## Critical points

Critical points of $CS_{Nak}$ are solutions to the system of equations

$$(\overline{\partial} + A)(\Phi) = 0,$$

$$\mu(\Phi) = 0,$$

where $\mu$ is the moment map for in the definition of the symplectic reduction of **M** with respect to the symplectic action of $G$. Then the Borel-Moore homology (with certain infinite shifts) of the critical set or set of all fields with coefficients in the sheaf of vanishing cycles $\phi_{CS_{Nak}}$ should be $\mathbb{C}[M_{\mathcal{C}}]$ if we take $C = \mathbb{P}^1$.

Nakajima's proposal cannot be taken literally: it deals with infinite-dimensional spaces of solutions and infinite-dimensional gauge groups (this is why we have infinite shifts in the BM homology). Furthermore, as in any definition which involves the sheaf of vanishing cycles one should choose the so-called orientation data, the notion introduced in our arXiv:0811.2435. Hopefully the latter can be constructed using methods developed by Joyce with collaborators (see arXiv:1811.01096, 1908.03524). As for the former (more conceptual) difficulty, one try can to factorize the infinite-dimensional neighborhood of the critical set into the product of a finite-dimensional manifold and the "transversal" infinite-dimensional slice along which the $CS_{Nak}$ has quadratic behavior. This idea was discussed in 1006.2706 in the case of holomorphic CS and complexified CS functionals. Then one still has to show that the multiplication is well-defined. How to make all that rigorous is an interesting open problem.