① VOA

Def
- Vector space V (usually over $\mathbb{C}$)
- $|0\rangle \in V$
- $T : V \to V$
- $Y : V \to \mathrm{End}(V)[[z^{\pm 1}]]$

such that:
- $Y(|0\rangle, z) = \mathrm{Id}_V$
- $Y(v, z)|0\rangle = v + zV[[z]]$
- $[T, Y(v,z)] = \frac{d}{dz} Y(v,z)$
- $(z-w)^N [Y(v,z), Y(v',w)] = 0 \quad N \gg 0$

Main Example: Affine VOAs + W-algebras

$\mathfrak{g}$ Lie superalgebra

$B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ invariant, supersymmetric bilinear

$\hat{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}K$ affine Lie superalgebra

$$[x \otimes t^n, y \otimes t^m] = [x,y] \otimes t^{n+m} + B(x,y)\, n\, \delta_{n+m,0} K$$

K central

2

$\hat{g}_+ := g \otimes \mathbb{C}[t] \oplus \mathbb{C} K$

$\mathbb{C}_k := \mathbb{C}|0\rangle$ 1-dim. $\hat{g}_+$ - module with

$K|0\rangle = k|0\rangle$ and $x \otimes t^n |0\rangle = 0 \quad n \geq 0$

$V^k(g) := \mathrm{Ind}_{\hat{g}_{\geq 0}}^{\hat{g}} \mathbb{C}_k$ universal affine VOA of g at level $k \in \mathbb{C}$.

$X(z) := Y(x \otimes t^{-1}|0\rangle, z) \quad x \in g$

$$x(z)\, y(w) = \frac{k\, B(x,y)}{(z-w)^2} + \frac{[x,y](w)}{(z-w)} \quad \text{operator product}$$

$$x(z) = \sum_{n \in \mathbb{Z}} x_n z^{-n-1}, \quad x_n := x \otimes t^n$$

$f \in g$ nilpotent, even $\to \exists$ complex $V^k(g) \otimes C_f$

with $H(V^k(g) \otimes C_f, d) = W^k(g, f)$ W-algebra

<u>Modules</u> ρ g-module $\Rightarrow$ ρ $\hat{g}_+$-module via

$K|_\rho = k \cdot \mathrm{Id}$ and $x_n|_\rho = 0 \quad n > 0$

$V^k(\rho) = \mathrm{Ind}_{\hat{g}_+}^{\hat{g}} \rho$

$W^k(\rho, f) = H(V^k(\rho) \otimes C^f, d)$

③ Why VOAs?

1.) $V^\natural$ proves monster moonshine

2.) V-mod for "nice" V is a rigid braided tensor category

$\Rightarrow$ modular functor + applications to low dimensional topology

3.) V chiral algebra of 2-dim conformal field theory

4.) V and V-mod appear in 4-dim and 3-dim gauge theories (and also the quantum geometric Langlands correspondence)

Idea:

$\mathcal{B}_1, \mathcal{B}_2$: 3-dim. top. boundary conditions

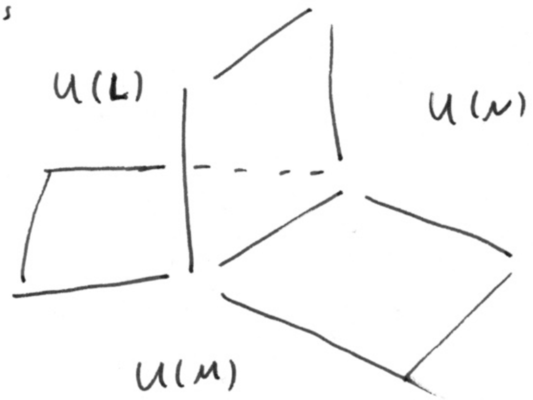
$\mathcal{C}_1, \mathcal{C}_2$: V-module categories

V: VOA at the 2-dim corner of $\mathcal{B}_1 \cap \mathcal{B}_2$.

④

Gaiotto - Rapčák Y - algebras

3- interfaces



- GL-twisted $N=4$ super Yang Mills with gauge groups $U(L), U(N), U(M)$ and coupling Ψ.
- Categories of VOA-modules end on the interface
- A VOA, the $Y_{L,N,M}[\Psi]$-algebra at the junction.
- S_3 - invariance $\Rightarrow$ triality of $Y_{L,N,M}$-algebras

(5)

Ex

$$Y_{N,0,0}[\psi] = \mathrm{Com}\left(V^{\psi-N}(sl_N),\, V^{\psi-N-1}(sl_N) \otimes L_1(sl_N)\right)$$

$$\text{“}=\text{”} \quad \frac{\hat{sl}^{(N)}_{\psi-N-1} \oplus \hat{sl}_1(N)}{\hat{sl}(N)_{\psi-N}}$$

$$Y_{0,N,0}[\psi] = W^{-\psi+N+1}(sl_N, f_{prim})$$

the W_N-algebra, e.g.

W_2 is Virasoro algebra

W_3 is Zamolodchikov's W_3-algebra

$$Y_{0,0,N}[\psi] = W^{\psi-N}(sl_N, f_{prim})$$

Feigin-Frenkel duality: $W^{\psi-N}(sl_N, f_{prim}) \cong W^{\psi^{-1}-N}(sl_N, f_{prim})$

Arakawa-C-Linshaw: $\frac{1}{\psi} + \frac{1}{\psi'} = 1 \Rightarrow$

$$\mathrm{Com}\left(V^{\psi-N}(sl_N),\, V^{\psi-N-1}(sl_N) \otimes L_1(sl_N)\right) \cong$$

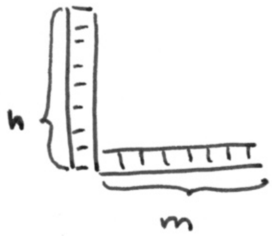
$$W^{\psi'-N}(sl_N, f_{prim})$$

6

- Nilpotent elements in sl_N $\longleftrightarrow$ partitions of N

$\longleftrightarrow$ Young tableaux with N-boxes

Let $f_{n,m}$:



- Nilpotent elements in $sl(n|m)$ $\longleftrightarrow$ pairs of partitions of (n,m) $\longleftrightarrow$ pairs of Young tableaux with n-boxes and m-boxes

Let $f_{n|m}$:

- $$C^{\psi}(n,m) := \mathrm{Com}\left(V^{\psi-m-1}(gl_m), W^{\psi-n-m}(sl_{n+m}, f_{n,m})\right)$$

$$\text{“}=\text{”} \quad \frac{W^{\psi-n-m}(sl_{n+m}, f_{n,m})}{\widehat{gl}(m)_{\psi-m-1}}$$

$$D^{\psi}(n,m) := \mathrm{Com}\left(V^{-\psi-m+1}(gl_m), W^{\psi-n+m}(sl(n|m), f_{n|m})\right)$$

$$\text{“}=\text{”} \quad \frac{W^{\psi-n+m}(sl(n|m), f_{n|m})}{\widehat{gl}(m)_{-\psi-m+1}}$$

$$\mathcal{D}^{\psi}(N, L) = Y_{L,0,N}[\psi]$$

$$\mathcal{C}^{\psi}(N-M, M) = Y_{0,M,N}[\psi] \qquad M \leq N$$

<u>Thm</u> (C-Linshaw) ψ generic *

$$\mathcal{D}^{\psi}(n, m) \cong \mathcal{C}^{\psi^{-1}}(n-m, n) \qquad n \geq m$$

$$\begin{array}{c} \cong \\ \mathcal{D}^{\psi'}(m, n) \end{array} \qquad \frac{1}{\psi} + \frac{1}{\psi'} = 1$$

i.e triality for $Y_{M,N,0}$-algebras holds.

Outlook

- $KL^{\mathrm{rev}}_{-\psi-m+1}(\mathfrak{sl}_m) \boxtimes KL^{\mathrm{rev}}_{\frac{1}{1-\psi}-n}(\mathfrak{sl}_n) \overset{??}{\longleftrightarrow} \mathrm{Rep}\, D^{\psi}(n,m)$

- $Y_{L,M,N}$-algebras are ~~W~~ cosets of W-superalgebras by affine vertex superalgebras. The Duflo-Serganova functor relates different Lie superalgebras.
  There seems to be a DS-functor on W-superalgebras that could explain relations between $Y_{L,M,N}$ and $Y_{L-1,M-1,N-1}$ algebras

- Gluing Y-algebras, beyond $\mathfrak{sl}_N$, ...