"By that time, all three of us had already been severely afflicted with the `$q$-disease`, a dangerous mathematical illness whose earliest victim was Euler . . .
The first symptom of the q-disease is that one day you realize that most of the results obtained or acquired during your mathematical life admit a q-deformation.
The second stage is indicated by the idea that the q-case is much more interesting than the classical one."

[P. Etingof, I. Frenkel, A. A. Kirillov, "Lectures on Representation Theory and Knizhnik-Zamolodchikov Equations"]

Work (partly in progress) with Fei Yan.

cf Kontsevich-Soibelman, Fock-Goncharov, Gaiotto-Moore-N, ...

Exact WKB analysis ⟷ q-abelianization ⟷ motivic DT invariants

q-abelianization ⟷ SUSY QFT

q-abelianization ⟷ quantum cluster coords

q-abelianization ⟷ traces / link invariants

THIS TALK

Exact WKB [Voros, Écalle, ..., Iwaki-Nakanishi, ..., Hollands-N]

Let $C = \mathbb{CP}^1$ (or Riemann surface with $\mathbb{CP}^1$-structure)

$P(z)\,dz^2$ meromorphic quadratic differential on C, $\hbar \in \mathbb{C}^\times$

Consider: $$\left(\partial_z^2 + \hbar^{-2} P(z)\right) \psi(z) = 0 \qquad (*)$$ meromorphic Schrödinger eq.

Rewrite it as $$\underbrace{\left[\partial_z + \hbar^{-1}\begin{pmatrix} 0 & -1 \\ P(z) & 0 \end{pmatrix}\right]}_{\nabla_\hbar \text{ flat } SL(2,\mathbb{C})\text{-connection over } C' = C \setminus \{\text{poles of } P\}}\begin{pmatrix} \psi \\ \hbar\partial_z\psi \end{pmatrix} = 0$$

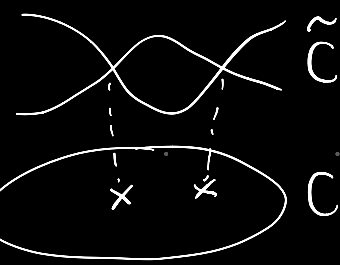
L any loop on C': want to study $\operatorname{Tr} \operatorname{Hol}_L(\nabla_\hbar) \in \mathbb{C}$

Exact WKB method: write solutions of $(*)$ as

$$\psi(z) = \exp\left(\hbar^{-1}\int_{z_0}^{z}\left(\sqrt{-P} + \sum_{n=1}^{\infty}\hbar^n S_n\right)dz\right) \qquad \text{[Borel resummed]}$$

They live naturally on $\tilde{C} = \{y^2 + P(z) = 0\} \subset T^*C$

$$\tilde{C} \xrightarrow{2:1} C$$



$\rightsquigarrow$ <u>Thm</u> $\operatorname{Tr} \operatorname{Hol}_L(\nabla_\hbar) = \sum_{\tilde{L}} \underbrace{\alpha(\tilde{L})}_{\mathbb{Z}\,\ni} X_{[\tilde{L}]}(\hbar)$, where

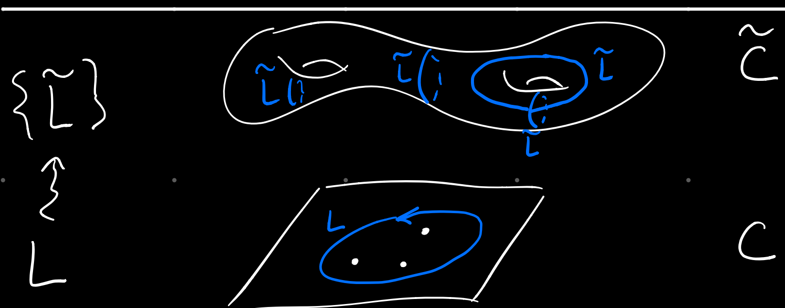
- $\tilde{L}$ runs over loops on $\tilde{C}$ $\qquad [\tilde{L}] \in H_1(\tilde{C})$
- $X_\gamma(\hbar) \sim \exp(\hbar^{-1} Z_\gamma)$ as $\hbar \to 0^+$ $\qquad Z_\gamma = \oint_\gamma y\, dz$
- $X_\gamma(\hbar)$ are Darboux coords on $\mathcal{M}(SL_2\mathbb{C}, C) = \{\text{flat } SL_2\mathbb{C}\text{-conn on } C\}/\sim$ evaluated at $\nabla_\hbar$

[usually "Fock-Goncharov" (cluster) coords]

<u>Conj</u>
- The $X_\gamma(\hbar)$ can be obtained by solving $\int$ eq. of TBA type [Dorey-Tateo, Voros, $\to$ Gaiotto-Moore-N, Gaiotto, Allegretti, Bridgeland]
- Similar story for $N > 2$, eg $N = 3$ involves

$$\left[\partial_z^3 + \hbar^{-2} P_2(z)\,\partial_z + \left(\hbar^{-3} P_3(z) + \tfrac{1}{2}\hbar^{-2} P_2'(z)\right)\right]\psi(z) = 0$$

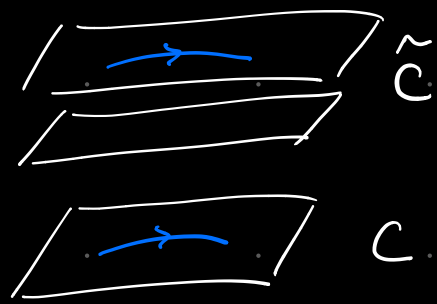
and $\tilde{C} = \{y^3 + P_2(z)y + P_3(z) = 0\} \subset T^*C$



What are the loops $\tilde{L}$ in the Thm?

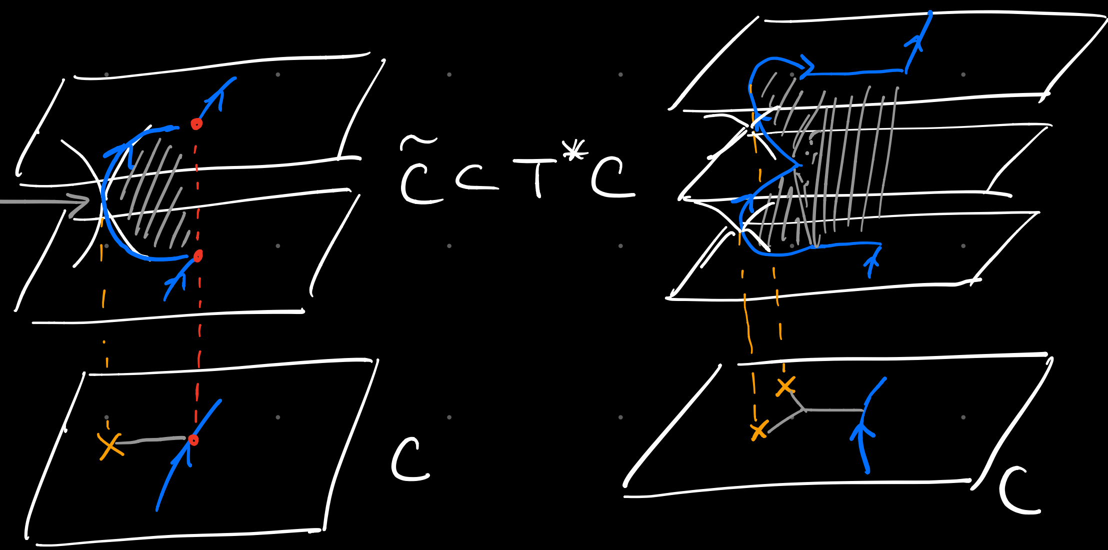
Built from pieces:

1) <u>direct lifts</u>



2) <u>detours</u>

almost sLag disc in T^*C



Next goal: <u>quantize</u>.

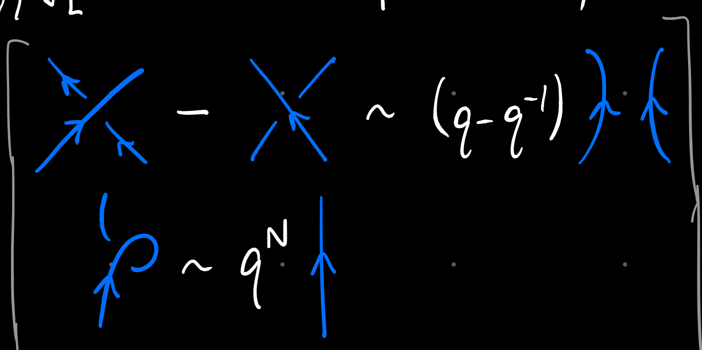
$$\mathrm{Tr}\,\mathrm{Hol}_L \longmapsto \sum_{\tilde{L}} \alpha(\tilde{L})\,\mathrm{Hol}_{\tilde{L}}$$

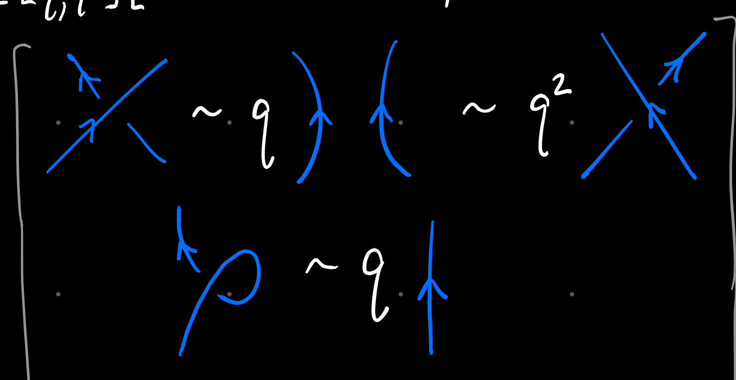
hom $F: \mathcal{O}(\mathcal{M}(C, GL_N\mathbb{C})) \longrightarrow \mathcal{O}(\mathcal{M}(\tilde{C}, GL_1\mathbb{C}))$

$$\rightsquigarrow \qquad\qquad\qquad \rightsquigarrow$$

hom F_q: $gl(N)$ skein algebra of $C \longrightarrow gl(1)$ skein algebra of $\tilde{C}$ "quantum torus"

$\mathbb{Z}[q,q^{-1}][\text{framed oriented loops in } C\times\mathbb{R}]/\sim$ $\qquad$ $\mathbb{Z}[q,q^{-1}][\text{framed oriented loops in } \tilde{C}\times\mathbb{R}]/\sim$





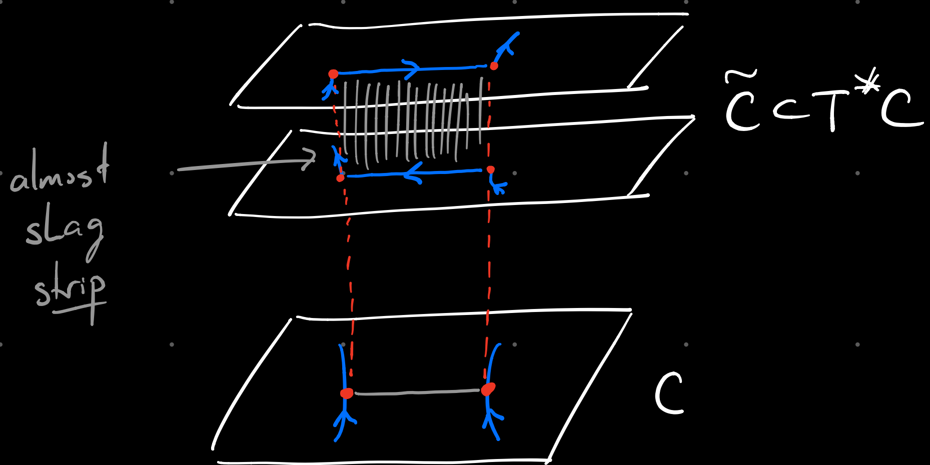
$$\left[\ \bigcirc \sim \frac{q^N - q^{-N}}{q - q^{-1}}\ \right] \qquad \left[\ \bigcirc \sim 1\ \right]$$

<u>Thm (in prog for $N>2$)</u> [AN–Yan, cf. Bonahon–Wong, Gaiotto–Longhi–Moore, Gabella]

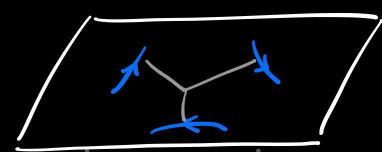
Such an F_q exists, of form $F(L) = \sum_{\tilde{L}} \alpha(\tilde{L})\, \tilde{L}$, with $\alpha(\tilde{L}) \in \mathbb{Z}[q, q^{-1}]$

$\tilde{L}$ built from pieces:

1) <u>direct lifts</u> as above
2) <u>detours</u> as above

3) 

and higher webs for $N>2$ eg



$\alpha(\tilde{L})$ built from various factors:

- $(q - q^{-1})^{n-1}$ for a web with n ends on L
- $q^{\text{winding \#}}$
- a few others related to <u>framing</u> of links

<u>Uses</u> of F_q:

- If $C = \mathbb{C}$ and $\tilde{C}$ is given by <u>constant</u> 1-forms then $gl(1)$ skein algebra of $\widetilde{C} \times \mathbb{R}$ is $\mathbb{Z}[q, q^{-1}]$

and $F_q(L) = P_{HOMFLY}(L, z = q - q^{-1}, a = q^N)$

But the method of computation is new. [Show demo page]

- For general C, $L \subset C \times \mathbb{R}$, $F_q(L)$ is a "knot invariant" (depends on $\widetilde{C}$, has wall-crossing): protected spin character for $\frac{1}{2}$-BPS line defects and $\frac{1}{4}$-BPS "fat line defects" in class S theories

- We can use F_q to construct representations of skein algebra. (related to Chern-Simons theory)

- (At least for $N=2$) our construction has 3-dimensional covariance so can imagine extending it to more general M.

This needs one new ingredient: "$gl(1)$ skein algebra with hypermultiplets"

roughly,

Related to [Freed-N] for $q=1$.

Q: what is the q-deformation of exact WKB?

(Barbieri-Bridgeland-Stoppa?)
