Quantum Supergroups Extending $\overline{U}_i^H(\mathfrak{sl}(2))$ (in progress with T. Creutzig and T. Dimofte)

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Outline

1. Review of boundary vertex operator algebras of 3d $\mathcal{N} = 4$ abelian gauge theories.
2. Review of simple current extension of VOAs.
3. Extending unrolled restricted quantum group $\overline{U}_i^H(\mathfrak{sl}(2))$.
4. Koszul dualities of quantum groups.

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- Not a complete analysis. A side, requires a computation of BRST cohomology. B side, needs to include monopole operators. Both sides, need to define a braided tensor category carefully.

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- Bilinear form (Garner):

$$\kappa(N_a, N_b) = \sum \rho^i{}_a \rho_{ib}, \ \kappa(N_a, E_b) = \delta_{ab}, \ \kappa(\psi_i^+, \psi_j^-) = \delta_{ij}.$$

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- Ballin-Creutzig-Dimofte-N (to appear): monopole operators corresponds to:

$$\exp\left(\int \sum m^a N_a\right)$$

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$$V_{\{m^a\}} := (\prod_a \sigma_a^{m^a}) V(\mathfrak{g}_*(\rho))$$

- The direct sum:

$$\bigoplus V_{\{m^a\}}$$

has a VOA structure and is identified as $V_{B,\rho}$.

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- Modules $V_{\{m^a\}}$ are called simple currents and $V_{B,\rho}$ is a simple current extension of $V(\mathfrak{g}_*(\rho))$.

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- In general, the action is given by logarithmic intertwiners. For a VOA module, we would like integer moding. This leads to locality condition $R^2 = \mathrm{Id}$:

$$V_{\{m^a\}} \times M \longrightarrow M \times V_{\{m^a\}} \longrightarrow V_{\{m^a\}} \times M$$

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- This idea was rigorously formulated by Creutzig-Kanade-McRae: if $V \to W$ is a simple current extension, then there is a tensor functor:

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- $\mathcal{L}$ identifies a module M with $\sigma_b M$.

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- Creutzig-Rupert: simple current extension for quantum groups leads to quotients of uprolled quantum groups.

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- There are one dimensional representations M_n labelled by $\mathbb{Z}$, such that $E = F = 0$ and $H = 2n$.

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- Conclusion: local $\mathcal{A}$ modules are in one-to-one correspondence with modules of the algebra generated by χ_+, χ_- and K, namely, the exterior algebra of two variables.

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- The quantum group analog of the above extension is:

$$\tilde{\mathcal{A}} = \bigoplus M_n \otimes \mathbb{C}_{0,n},$$

as a module of $\overline{U}_i^H(\mathfrak{sl}(2))\otimes H$.

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- Local $\tilde{\mathcal{A}}$ modules are in one-to-one correspondence with modules generated by $\psi_\pm, E = -kC$ and $N = H/2 - D$. This is the unrolled-restricted quantum group for $\mathfrak{gl}(1|1)$.

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- Conjecture: $\mathrm{Rep}(U_q\mathfrak{g}_*(\rho))$ and $V_{B,\rho}$-Mod are equivalent as BTC.

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- Quantum group perspective gives an explicit BTC structure.

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- Likely to replace $\mathbb{C}[G]$ by $U_q\mathfrak{g}$ (irresponsible statement).

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- BTC structure from VOA is hard, but we have a better understanding now.
- It is equivalent to a category of modules of $\mathfrak{g}_*(\rho)$, with braiding given by $e^{\pi i \Omega}$, and associator given by solutions of KZ equation.
- Proving the equivalence between $\overline{U}_i^H(\mathfrak{sl}(2))$ and $M(2)$ will result in a Drinfeld's isomorphism for these quantum supergroups.

Last word

Thank you!