q-Opers, QQ-Systems & Bethe Ansatz II

Peter Koroteev

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Literature

[arXiv:2108.04184]
q-Opers, QQ-systems, and Bethe Ansatz II: Generalized Minors
P. Koroteev, A. M. Zeitlin

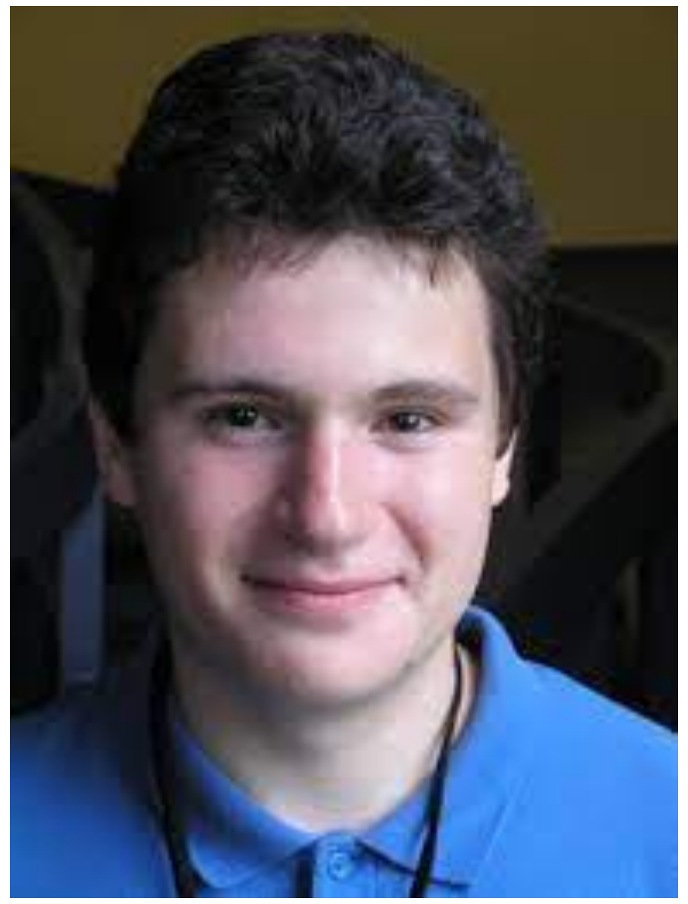
[arXiv:2105.00588]
3d Mirror Symmetry for Instanton Moduli Spaces
P. Koroteev, A. M. Zeitlin

[arXiv:2007.11786] J. Inst. Math. Jussieu
Toroidal q-Opers
P. Koroteev, A. M. Zeitlin

[arXiv:2002.07344] J. Europ. Math. Soc.
q-Opers, QQ-Systems, and Bethe Ansatz
E. Frenkel, P. Koroteev, D. S. Sage, A. M. Zeitlin

[arXiv:1811.09937] Commun.Math.Phys. **381** (2021) 641
(SL(N),q)-opers, the q-Langlands correspondence, and quantum/classical duality
P. Koroteev, D. S. Sage, A. M. Zeitlin

[arXiv:1705.10419] Selecta Math. **27** (2021) 87
Quantum K-theory of Quiver Varieties and Many-Body Systems
P. Koroteev, P. P. Pushkar, A. V. Smirnov, A. M. Zeitlin

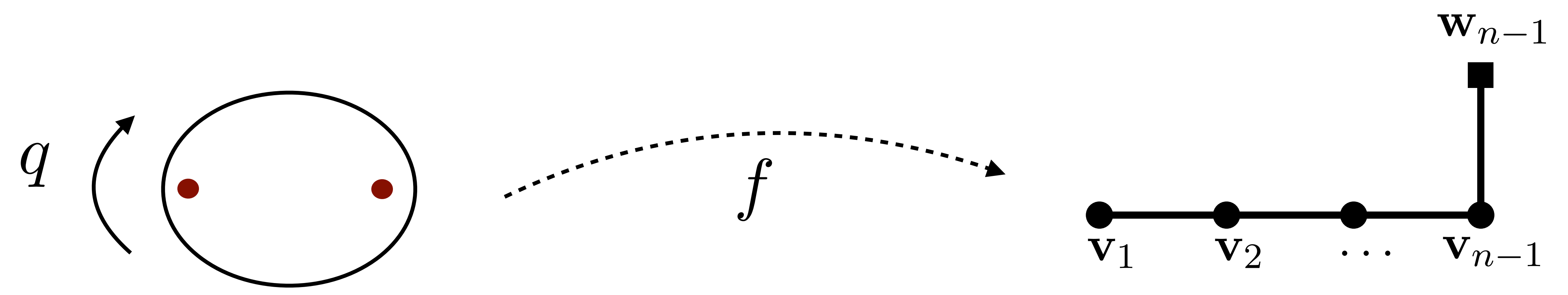


Motivation

Quantum Geometry and Integrable Systems	[Okounkov et al] [Pushkar, Zeitlin, Smirnov] [PK, Pushkar, Zeitlin, Smirnov]
BPS/CFT correspondence	[Nekrasov Shatashvili]
Geometric q-Langlands Correspondence	[Frenkel] [Aganagic, Frenkel, Okounkov]
ODE/IM Correspondence	[Bazhanov, Lukyanov, Zamolodchikov] [Dorey, Tateo]

Enumerative AG/Integrable Systems

Quantum equivariant K-theory of Nakajima quiver varieties



$$A \circledast B = A \otimes B + \sum_{d=1}^{\infty} A \circledast_d B z^d$$

$$\mathrm{V}^{(\tau)}(\boldsymbol{z}) = \sum_{\boldsymbol{d}} \mathrm{ev}_{p_2,*}\left(\widehat{\mathcal{O}}^{\boldsymbol{d}}_{\mathrm{vir}} \otimes \tau|_{p_1}, \mathsf{QM}^{\boldsymbol{d}}_{\mathrm{nonsing}\, p_2}\right) \boldsymbol{z}^{\boldsymbol{d}} \in K_{\mathsf{T}\times\mathbb{C}_q^\times}(X)_{loc}[[\boldsymbol{z}]]$$

Saddle point limit yields Bethe equations for XXZ

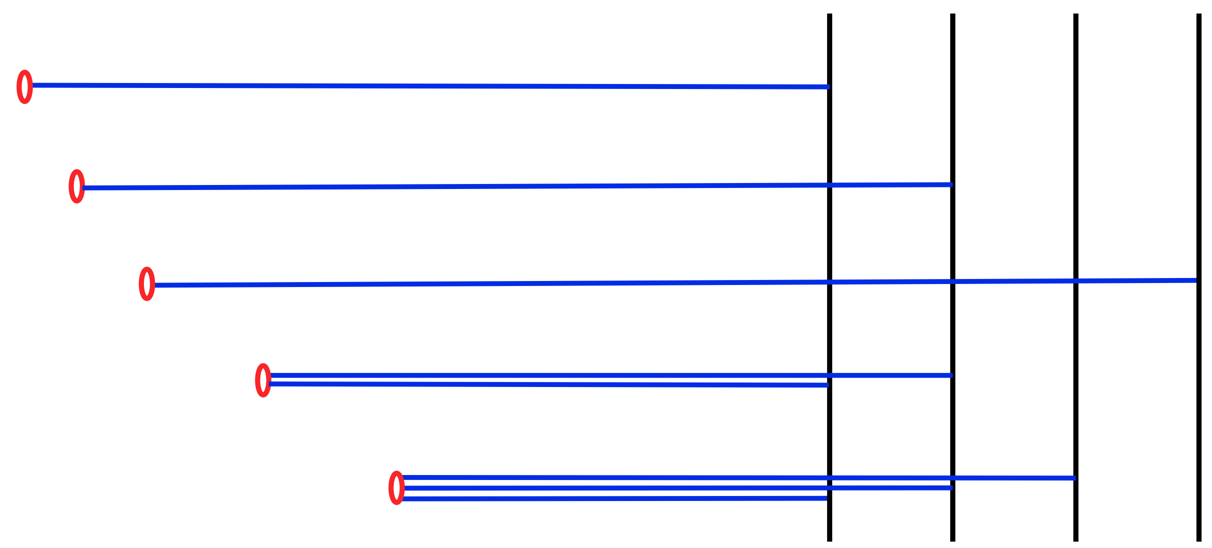
Quantum classes satisfy interesting difference equations in equivariant parameters and Kahler parameters ***qKZ, Dynamical equation*** [Okounkov, Smirnov]

After symmetrization they can be rewritten as eigenvalue equations for ***trigonometric Ruijsenaars-Schneider (tRS)*** system [PK, Zeitlin] [PK]

$$T_r(\mathbf{a}) = \sum_{\substack{\mathcal{I}\subset\{1,\dots,n\} \\ |\mathcal{I}|=r}} \prod_{\substack{i\in\mathcal{I} \\ j\notin\mathcal{I}}} \frac{t\, a_i - a_j}{a_i - a_j} \prod_{i\in\mathcal{I}} p_i$$

$$T_r(\boldsymbol{a})\mathrm{V}(\boldsymbol{a},\vec{\zeta}) = S_r(\vec{\zeta},t)\mathrm{V}(\boldsymbol{a},\vec{\zeta})$$

In terms of string/gauge theory tRS eigenproblem is Ward identity [Gaiotto, PK] [Bullimore, Kim, PK]



This leads to XXZ/tRS duality. Can we generalize it?

(G,q)-Connection

Principal G-bundle $\mathcal{F}_G$ over $\mathbb{P}^1$

$$M_q : \mathbb{P}^1 \to \mathbb{P}^1$$
$$z \mapsto qz$$

U-Zariski open dense set

A meromorphic **(G,q)-connection** on $\mathcal{F}_G$ is a section A of $\mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}_G, \mathcal{F}_G^q)$

Choose U so that the restriction $\mathcal{F}_G|_U$ of $\mathcal{F}_G$ to U is isomorphic to a trivial G-bundle

$A(z) \in G(\mathbb{C}(z))$ on $U \cap M_q^{-1}(U)$

Change of trivialization $A(z) \mapsto g(qz)A(z)g(z)^{-1}$

(G,q)-Opers

A meromorphic (G,q)-oper on $\mathbb{P}^1$ is a triple $(\mathfrak{F}_G, A, \mathfrak{F}_{B_-})$

A is a meromorphic (G,q)-connection

$\mathfrak{F}_{B_-}$ is a reduction of $\mathfrak{F}_G$ to B_-

Oper condition: Restriction of the connection on some Zariski open dense set U

$$A : \mathfrak{F}_G \longrightarrow \mathfrak{F}_G^q \text{ to } U \cap M_q^{-1}(U)$$

takes values in the double Bruhat cell

$$B_-(\mathbb{C}[U \cap M_q^{-1}(U)])cB_-(\mathbb{C}[U \cap M_q^{-1}(U)])$$

Coxeter element: $c = \prod_i s_i$

Locally

$$A(z) = n'(z) \prod_i (\phi_i(z)^{\check{\alpha}_i} \, s_i) n(z)$$

$\phi_i(z) \in \mathbb{C}(z)$ and $n(z), n'(z) \in N_-(z)$

Miura (G,q)-Opers

Definition: A *Miura* (G,q)*-oper* on $\mathbb{P}^1$ is a quadruple $(\mathcal{F}_G, A, \mathcal{F}_{B_-}, \mathcal{F}_{B_+})$, where $(\mathcal{F}_G, A, \mathcal{F}_{B_-})$ is a meromorphic (G,q)-oper on $\mathbb{P}^1$ and $\mathcal{F}_{B_+}$ is a reduction of the G-bundle $\mathcal{F}_G$ to B_+ that is preserved by the q-connection A.

It can be shown that the two flags $\mathcal{F}_{B_-}$ and $\mathcal{F}_{B_+}$ are in *generic relative position* for some dense set V

The fiber $\mathcal{F}_{G,x}$ of $\mathcal{F}_G$ at x is a G-torsor with reductions $\mathcal{F}_{B_-,x}$ and $\mathcal{F}_{B_+,x}$ to B_- and B_+, respectively. Choose any trivialization of $\mathcal{F}_{G,x}$, i.e. an isomorphism of G-torsors $\mathcal{F}_{G,x} \simeq G$. Under this isomorphism, $\mathcal{F}_{B_-,x}$ gets identified with $aB_- \subset G$ and $\mathcal{F}_{B_+,x}$ with bB_+.

Then $a^{-1}b$ is a well-defined element of the double quotient $B_- \backslash G / B_+$, which is in bijection with W_G.

We will say that $\mathcal{F}_{B_-}$ and $\mathcal{F}_{B_+}$ have a generic relative position at $x \in X$ if the element of W_G assigned to them at x is equal to 1 (this means that the corresponding element $a^{-1}b$ belongs to the open dense Bruhat cell $B_- \cdot B_+ \subset G$).

Structure Theorems

Theorem 1: *For any Miura (G, q)-oper on $\mathbb{P}^1$, there exists a trivialization of the underlying G-bundle $\mathcal{F}_G$ on an open dense subset of $\mathbb{P}^1$ for which the oper q-connection has the form*

$$A(z) \in N_-(z) \prod_i ((\phi_i(z)^{\check{\alpha}_i} s_i) N_-(z) \ \cap \ B_+(z).$$

Theorem 2: *Let F be any field, and fix $\lambda_i \in F^\times, i = 1, \ldots, r$. Then every element of the set $N_- \prod_i \lambda_i^{\check{\alpha}_i} s_i N_- \ \cap \ B_+$ can be written in the form*

$$\prod_i g_i^{\check{\alpha}_i} e^{\frac{\lambda_i t_i}{g_i} e_i}, \qquad g_i \in F^\times,$$

where each $t_i \in F^\times$ is determined by the lifting s_i.

Adding Singularities and Twists

Consider family of polynomials $\{\Lambda_i(z)\}_{i=1,\ldots,r}$

(G,q)-oper with regular singularities can be written as

$$A(z) = n'(z) \prod_i (\Lambda_i(z)^{\check{\alpha}_i} s_i) n(z), \qquad n(z), n'(z) \in N_-(z)$$

Using structure theorem every Miura (G,q)-oper with singularities reads

$$A(z) = \prod_i g_i(z)^{\check{\alpha}_i} e^{\frac{\Lambda_i(z)}{g_i(z)} e_i}, \qquad g_i(z) \in \mathbb{C}(z)^{\times}$$

(G,q)-oper is **Z-twisted** if it is equivalent to a constant element of G $Z \in H \subset H(z)$ Z is regular semisimple. There are W_G Miura (G,q)-opers for each (G,q)-opers

$$A(z) = g(qz) Z g(z)^{-1}$$

Z-twisted Miura (G,q)-oper if gauge transform is from Borel

$$A(z) = v(qz) Z v(z)^{-1}, \qquad v(z) \in B_+(z)$$

Plucker Relations

V_i^+ irrep of G with highest weight ω_i Line $L_i \subset V_i$ stable under B_+

Plucker relations: for two integral dominant weights $L_{\lambda+\mu} \subset V_{\lambda+\mu}$ is the image of $L_\lambda \otimes L_\mu \subset V_\lambda \otimes V_\mu$

under canonical projection $V_\lambda \otimes V_\mu \longrightarrow V_{\lambda+\mu}$

Conversely, for a collection of lines $L_\lambda \subset V_\lambda$ satisfying Plucker relations $\exists B \subset G$ such that L_λ is stabilized by B for all λ

A choice of B is equivalent to a choice of B_+-torsor in G

Let ν_{ω_i} be a generator of the line $L_i \subset V_i$. This is a vector of weight ω_i wrt $H \subset B_+$

The subspace of V_i of weight $\omega_i - \alpha_i$ is one-dimensional and spanned by $f_i \cdot \nu_{\omega_i}$

Thus the 2d subspace spanned by $\{\nu_{\omega_i}, f_i \cdot \nu_{\omega_i}\}$ is a B_+-invariant subspace of V_i

Miura-Pucker (G,q)-Opers

let $(\mathcal{F}_G, A, \mathcal{F}_{B_-}, \mathcal{F}_{B_+})$ be a Miura (G,q)-oper with regular singularities $\{\Lambda_i(z)\}_{i=1,\dots,r}$

Associated vector bundle $\mathcal{V}_i = \mathcal{F}_{B_+} \underset{B_+}{\times} V_i = \mathcal{F}_G \underset{G}{\times} V_i$ contains rank-two subbundle $\mathcal{W}_i = \mathcal{F}_{B_+} \underset{B_+}{\times} W_i$

associated to $W_i \subset V_i$, and $\mathcal{W}_i$ in turn contains a line subbundle $\mathcal{L}_i = \mathcal{F}_{B_+} \underset{B_+}{\times} L_i$

Using structure theorems we obtain **r** Miura (GL(2),q)-opers

$$A_i(z) = \begin{pmatrix} g_i(z) & \Lambda_i(z) \prod_{j>i} g_j(z)^{-a_{ji}} \\ 0 & g_i^{-1}(z) \prod_{j\neq i} g_j(z)^{-a_{ji}} \end{pmatrix}$$

Z-twisted Miura-Plucker (G,q)-oper is meromorphic Miura (G,q)-oper on P1 such that for each Miura (GL(2),q)-oper

$$A_i(z) = v(zq) Z v(z)^{-1}|_{W_i} = v_i(zq) Z_i v_i(z)^{-1}$$

where $v_i(z) = v(z)|_{W_i}$ and $Z_i = Z|_{W_i}$

QQ-System

Theorem: *There is a one-to-one correspondence between the set of nondegenerate Z-twisted Miura-Plücker (G,q)-opers and the set of nondegenerate polynomial solutions of the QQ-system*

$$\widetilde{\xi}_i Q^i_-(z) Q^i_+(qz) - \xi_i Q^i_-(qz) Q^i_+(z) = \Lambda_i(z) \prod_{j>i} \left[Q^j_+(qz)\right]^{-a_{ji}} \prod_{j<i} \left[Q^j_+(z)\right]^{-a_{ji}}, \qquad i = 1, \dots, r,$$

$$\widetilde{\xi}_i = \zeta_i \prod_{j>i} \zeta_j^{a_{ji}}, \qquad \xi_i = \zeta_i^{-1} \prod_{j<i} \zeta_j^{-a_{ji}}$$

Proof uses

$$v(z) = \prod_{i=1}^{r} y_i(z)^{\check{\alpha}_i} \prod_{i=1}^{r} e^{-\frac{Q^i_-(z)}{Q^i_+(z)} e_i} \dots, \qquad g_i(z) = \zeta_i \frac{Q^i_+(qz)}{Q^i_+(z)}$$

XXZ Bethe Ansatz Equations for G

roots of Q+

$$\frac{Q^i_+(q w^k_i)}{Q^i_+(q^{-1} w^k_i)} \prod_j \zeta_j^{a_{ji}} = -\frac{\Lambda_i(w^i_k)\prod_{j>i}\left[Q^j_+(q w^i_k)\right]^{-a_{ji}}\prod_{j<i}\left[Q^j_+(w^i_k)\right]^{-a_{ji}}}{\Lambda_i(q^{-1}w^i_k)\prod_{j>i}\left[Q^j_+(w^i_k)\right]^{-a_{ji}}\prod_{j<i}\left[Q^j_+(q^{-1}w^i_k)\right]^{-a_{ji}}}$$

Space of nondegenerate solutions of QQ-system for G

Nondegenerate ***Z-twisted Miura-Plucker*** (G,q)-opers with regular singularities

?

Space of nondegenerate solutions of XXZ for G

?

Nondegenerate ***Z-twisted Miura*** (G,q)-opers with regular singularities

SL(2) Example

$$A(z) = \begin{pmatrix} g(z) & \Lambda(z) \\ 0 & g(z)^{-1} \end{pmatrix}$$

Z-twisted q-oper condition

$$A(z) = v(zq) Z v(z)^{-1}, \qquad Z = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$$

Gauge transformation reads

$$v(z) = \begin{pmatrix} y(z) & 0 \\ 0 & y(z)^{-1} \end{pmatrix} \begin{pmatrix} 1 & -\frac{Q_-(z)}{Q_+(z)} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} y(z) & -y(z)\frac{Q_-(z)}{Q_+(z)} \\ 0 & y(z)^{-1} \end{pmatrix}$$

We find

$$g(z) = \zeta_i y(zq) y(z)^{-1}$$

$$\Lambda(z) = y(z) y(zq) \left(\zeta \frac{Q_-(z)}{Q_+(z)} - \zeta^{-1} \frac{Q_-(zq)}{Q_+(zq)} \right)$$

The q-oper condition becomes the **SL(2) QQ-system**

$$\zeta Q_-(z) Q_+(zq) - \zeta^{-1} Q_-(zq) Q_+(z) = \Lambda(z)$$

To get Bethe equations

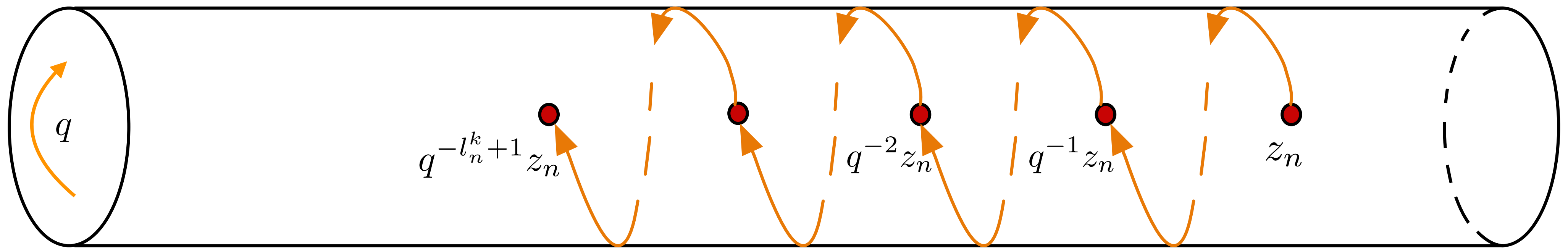
$$Q_+(z) = \prod_{k=1}^{m} (z - w_k)$$

evaluate at roots of Q

$$\frac{\Lambda(w_k)}{\Lambda(q^{-1}w_k)} = -\zeta^2 \frac{Q_+(qw_k)}{Q_+(q^{-1}w_k)}, \qquad k = 1, \ldots, m.$$

Singularities

$$\Lambda(z) = \prod_{p=1}^{L} \prod_{j_p=0}^{r_p-1} (z - q^{-j_p} z_p)$$



XXZ Bethe equations

$$q^r \prod_{p=1}^{L} \frac{w_k - q^{1-r_p} z_p}{w_k - q z_p} = -\zeta^2 q^m \prod_{j=1}^{m} \frac{q w_k - w_j}{w_k - q w_j}, \qquad k = 1, \ldots, m.$$

Quantum Backlund Transformation

Theorem: Consider the following q-gauge transformation

$$A \mapsto A^{(i)} = e^{\mu_i(qz)f_i} A(z) e^{-\mu_i(z)f_i}, \quad \text{where} \quad \mu_i(z) = \frac{\prod_{j\neq i}\left[Q^j_+(z)\right]^{-a_{ji}}}{Q^i_+(z)Q^i_-(z)}$$

changes the set of Q-functions

$$\begin{aligned} Q^j_+(z) &\mapsto Q^j_+(z), & j &\neq i, \\ Q^i_+(z) &\mapsto Q^i_-(z), & Z &\mapsto s_i(Z) \end{aligned}$$

$$\{\widetilde{Q}^j_+\}_{j=1,\dots,r} = \{Q^1_+, \dots, Q^{i-1}_+, Q^i_-, Q^{i+1}_+ \dots, Q^r_+\}$$

$$\{\widetilde{z}_j\}_{j=1,\dots,r} = \{z_1, \dots, z_{i-1}, z_i^{-1}\prod_{j\neq i} z_j^{-a_{ji}}, \dots, z_r\}$$

Now the strategy is to successively apply Backlund transformations according to the reduced decomposition of the element of the Weyl group

Consider longest element $w_0 = s_{i_1}\dots s_{i_\ell}$

Theorem: Every Z-twisted Miura-Plucker (G,q)-oper is Z-twisted Miura (G,q)-oper

The proof based on properties of double Bruhat cells addresses existence of the diagonalizing element v(z) (to be constructed later)

(SL(N),q)-Opers

The QQ-system $$\xi_i\phi_i(z) - \xi_{i+1}\phi_i(qz) = \rho_i(z)$$

$$\phi_i(z) = \frac{Q_i^-(z)}{Q_i^+(z)}\,, \qquad \rho_i(z) = \Lambda_i(z)\frac{Q_{i-1}^+(qz)Q_{i+1}^+(z)}{Q_i^+(z)Q_i^+(qz)}$$

q-Oper condition $$v(qz)^{-1}A(z) = Zv(z)^{-1}$$

Diagonalizing element

Polynomials $Q^-_{i,\dots,j}(z)$

form extended QQ-system

$$v(z)^{-1} = \begin{pmatrix} \frac{1}{Q_1^+(z)} & \frac{Q_1^-(z)}{Q_2^+(z)} & \frac{Q_{12}^-(z)}{Q_3^+(z)} & \dots & \frac{Q^-_{1,\dots,r-1}(z)}{Q_r^+(z)} & Q^-_{1,\dots,r}(z) \\ 0 & \frac{Q_1^+(z)}{Q_2^+(z)} & \frac{Q_2^-(z)}{Q_3^+(z)} & \dots & \frac{Q^-_{2,\dots,r-1}(z)}{Q_r^+(z)} & Q^-_{2,\dots,r}(z) \\ 0 & 0 & \frac{Q_2^+(z)}{Q_3^+(z)} & \dots & \frac{Q^-_{3,\dots,r-1}(z)}{Q_r^+(z)} & Q^-_{3,\dots,r}(z) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \dots & \dots & \frac{Q_{r-1}^+(z)}{Q_r^+(z)} & Q_r^-(z) \\ 0 & \dots & \dots & \dots & 0 & Q_r^+(z) \end{pmatrix}$$

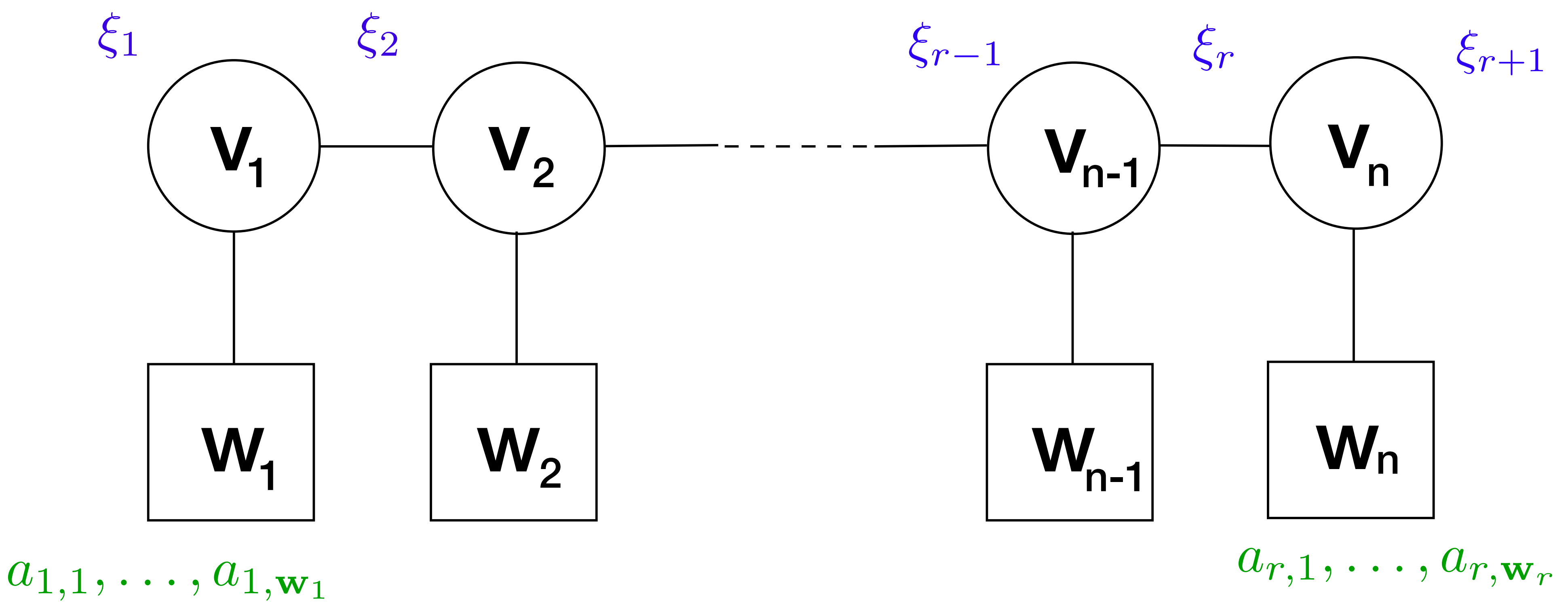
Quantum Wronskians

(SL(N),q)-oper can also be constructed from flag of subbundles $(E, A, \mathcal{L}_\bullet)$ such that the induced maps $\bar{A}_i : \mathcal{L}_i/\mathcal{L}_{i-1} \longrightarrow \mathcal{L}^q_{i+1}/\mathcal{L}^q_i$ are isomorphisms

The quantum determinants $\mathcal{D}_k(s) = e_1 \wedge \cdots \wedge e_{r+1-k} \wedge Z^{k-1}s(z) \wedge Z^{k-2}s(qz) \wedge \cdots \wedge Zs(q^{k-2}) \wedge s(q^{k-1}z)$

vanish at q-oper singularities $W_k(s) = P_1(z) \cdot P_2(q^2 z) \cdots P_k(q^{k-1}z), \qquad P_i(z) = \Lambda_r \Lambda_{r-1} \cdots \Lambda_{r-i+1}(z)$

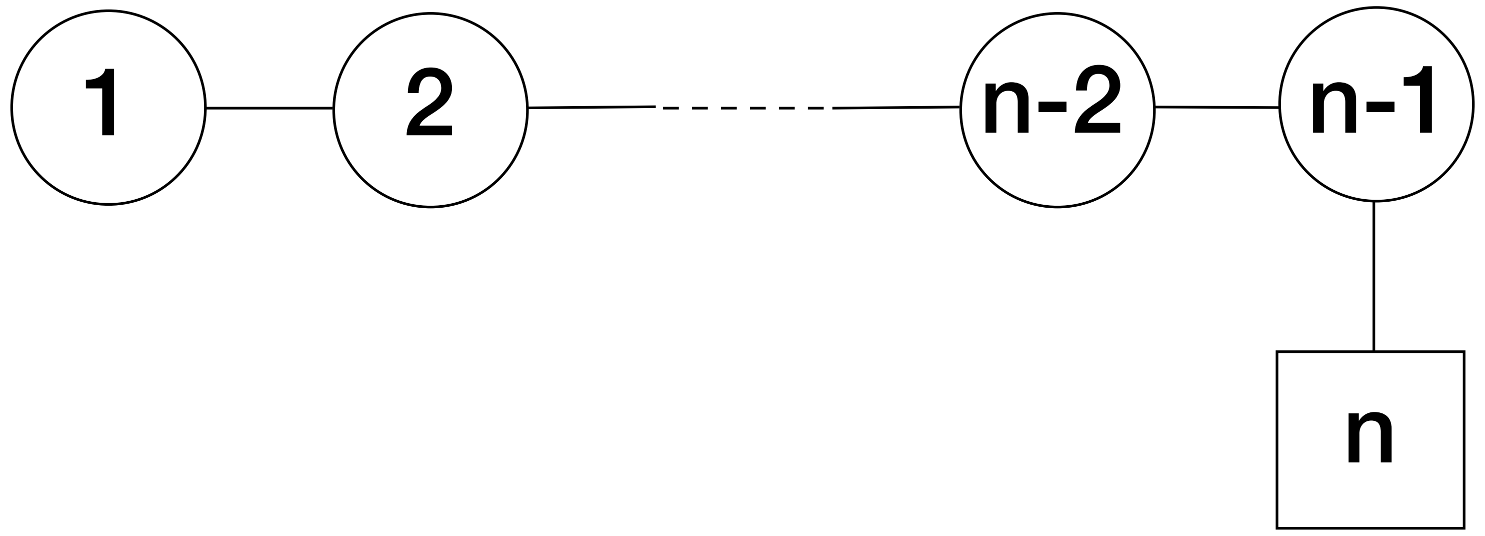
Diagonalizing condition $\det_{i,j}\left[\xi^{k-j}_{r+1-k+i} s_{r+1-k+i}(q^{j-1}z)\right] = \alpha_k W_k \mathcal{V}_k$



Components of the section of the
line subbundle are the Q-polynomials! $s_{r+1}(z) = Q^+_r(z)\,, \qquad s_r(z) = Q^-_r(z)\,, \qquad s_k(z) = Q^-_{k,\ldots,r}(z)$

Quantum/Classical Duality

Consider T*G/B



Construct the corresponding space of (SL(N),h)-opers

Specify components of the section of L1

$$s_1(z) = z - p_1, \quad , \dots , \quad s_{k+r}(z) = z - p_{k+l}$$

$$p_{k+l+1-p} = -\frac{Q_p^+(0)}{Q_{p-1}^+(0)}$$

Then the space of functions on the space of such h-opers

$$\mathrm{Fun}(\hbar \mathrm{Op})(F\mathbb{F}l_L)) \cong \frac{\mathbb{C}(\{\xi_i\},\{a_i\},\{p_i\},\hbar)}{\{H_i(\{p_j\},\{\xi_j\},\hbar) = e_i(a_1,\dots,a_L\})_{i=1,\dots,L}}$$

is described by trigonometric Ruijsenaars-Schneider model with n particles

$$H_k = \sum_{\substack{\mathcal{I}\subset\{1,\dots,L\}\\ |\mathcal{I}|=k}} \prod_{\substack{i\in\mathcal{I}\\ j\notin\mathcal{I}}} \frac{\xi_i - \hbar\,\xi_j}{\xi_i - \xi_j} \prod_{m\in\mathcal{I}} p_m$$

Generalized Wronskians

Consider big cell in Bruhat decomposition

$$G_0 = N_- H N_+$$

$$g = n_- \ h \ n_+$$

V_i^+ irrep of G with highest weight ω_i

$$h\nu^+_{\omega_i} = [h]^{\omega_i}\nu^+_{\omega_i}$$

Define **principal minors** for group element g

For SL(N) they are standard minors of matrices

$$\Delta^{\omega_i}(g) = [h]^{\omega_i}, \quad i = 1, \ldots, r$$

Then **generalized minors** are regular functions on G

$$\Delta_{u\omega_i, v\omega_i}(g) = \Delta^{\omega_i}(\tilde{u}^{-1} g \tilde{v}) \qquad u, v \in W_G$$

Proposition *Action of the group element on the highest weight vector in*

$$g \cdot \nu^+_{\omega_i} = \sum_{w \in W} \Delta_{w \cdot \omega_i, \omega_i}(g) \tilde{w} \cdot \nu^+_{\omega_i} + \ldots,$$

where dots stand for the vectors, which do not belong to the orbit $\mathcal{O}_W$.

Generalized Minors and QQ-system

The set of generalized minors $\{\Delta_{w\cdot\omega_i,\omega_i}\}_{w\in W;i=1,\dots,r}$ creates a set of coordinates on G/B^+, known as *generalized Plücker coordinates.* In particular, the set of zeroes of each of $\Delta_{w\cdot\omega_i,\omega_i}$ is a uniquely and unambiguously defined hypersurface in G/B.

Proposition *For a W-generic Z-twisted Miura-Plücker (G,q)-oper with q-connection $A(z)=v(qz)Zv(z)^{-1}$, where $v(z)\in B_-(z)$ we have the following relation:*

$$\Delta_{w\cdot\omega_i,\omega_i}(v^{-1}(z)) = Q_+^{w,i}(z)$$

for any $w\in W$.

Proof: Since $\Delta^{\omega_i}(v^{-1}(z)) = Q_+^i(z)$ Diagonalizing gauge transformation $v^{-1}(z) = \prod_{i=1}^{r} e^{\frac{Q_-^i(z)}{Q_+^i(z)}f_i} \prod_{i=1}^{r} \left[Q_+^i(z)\right]^{\check{\alpha}_i}\dots$

$$v^{-1}(z)\nu_{\omega_i}^+ = Q_+^i(z)\nu_{\omega_i}^+ + Q_-^i(z)f_i\nu_{\omega_i}^+ + \dots$$

Fundamental Relation for Generalized Minors

[Fomin Zelevinsky]

Proposition 4.8. *Let,* $u, v \in W$*, such that for* $i \in \{1, \ldots, r\}$, $\ell(uw_i) = \ell(u) + 1$, $\ell(vw_i) = \ell(v) + 1$. *Then*

$$\Delta_{u\cdot\omega_i, v\cdot\omega_i}\Delta_{uw_i\cdot\omega_i, vw_i\cdot\omega_i} - \Delta_{uw_i\cdot\omega_i, v\cdot\omega_i}\Delta_{u\cdot\omega_i, vw_i\cdot\omega_i} = \prod_{j\neq i}\Delta_{u\cdot\omega_j, v\cdot\omega_j}^{-a_{ji}}, \tag{4.7}$$

Can we make sense of this relation using our approach of q-Opers?

Generalized Wronskians

The approach is similar to Miura-Plucker q-Opers

Let $\nu^+_{\omega_i}$ be a generator of the line $L^+_i \subset V^+_i$ $\quad$ V^+_i irrep of G with highest weight ω_i

The subspace $L^+_{c,i}$ of V_i of weight $c^{-1} \cdot \omega_i$ is one-dimensional and is spanned by $s^{-1}\nu^+_{\omega_i}$

Associated vector bundle $\quad \mathcal{V}^+_i = \mathcal{F}_{B_+} \underset{B_+}{\times} V^+_i = \mathcal{F}_G \underset{G}{\times} V^+_i$

Contains line subbundles $\quad \mathcal{L}^+_i = \mathcal{F}_H \underset{H}{\times} L^+_i, \quad \mathcal{L}^+_{c,i} = \mathcal{F}_H \underset{H}{\times} L^+_{c,i}$

Define **generalized Wronskian** on $\mathbb{P}^1$ as quadruple $(\mathcal{F}_G, \mathcal{F}_{B_+}, \mathscr{G}, Z)$

$\mathscr{G}$ is a meromorphic section of a principle bundle $\mathcal{F}_G$

s.t. for sections $\{v^+_i, v^+_{c,i}\}_{i=1,\dots,r}$ of line bundles $\{\mathcal{L}^+_i, \mathcal{L}^+_{c,i}\}_{i=1,\dots,r}$ on $U \cap M_q^{-1}(U)$

$$\mathscr{G}^q \cdot v^+_i = Z \cdot \mathscr{G} \cdot v^+_{c,i}$$

Adding Singularities

Effectively the above definition means that the Wronskian, written as an element of G(z), satisfies

$$Z^{-1}\mathscr{G}(qz)\;\nu^+_{\omega_i} = \mathscr{G}(z)\cdot s_\phi(z)^{-1}\cdot \nu^+_{\omega_i}$$

$$s_\phi(z) = \prod_i \phi_i^{-\check{\alpha}_i} s_i$$

Define **generalized Wronskian with regular singularities** if

$$s_\Lambda(z)^{-1} = \prod_i^{\mathrm{inv}} s_i \Lambda_i^{\check{\alpha}_i}$$

Fomin-Zelevinsky relations then read

$$\Delta_{\omega_i,\omega_i}\Delta_{w_i\cdot\omega_i,c^{-1}\cdot\omega_i} - \Delta_{w_i\cdot\omega_i,\omega_i}\Delta_{\omega_i,c^{-1}\cdot\omega_i} = \prod_{j<i=i_l}\Delta^{-a_{ji}}_{\omega_j,c^{-1}\cdot\omega_j}\prod_{j>i=i_l}\Delta^{-a_{ji}}_{\omega_j,\omega_j}, \qquad i=1,\dots,r,$$

q-Opers and q-Wronskians

Theorem 1:

Nondegenerate generalized q-Wronskians with regular singularities $\{\Lambda_i\}_{i=1,\dots,r}$ $\longleftrightarrow$ Nondegenerate Z-twisted Miura (G,q)-opers with regular singularities $\{\Lambda_i\}_{i=1,\dots,r}$

Theorem 2: *For a given Z-twisted (G,q)-Miura oper, there exists a unique generalized q-Wronskian*

$$\mathscr{W}(z) \in B_-(z)w_0B_-(z) \cap B_+(z)w_0B_+(z) \subset G(z),$$

satisfying the system of equations

$$\mathscr{W}(q^{k+1}z)\nu^+_{\omega_i} = Z^k\mathscr{W}(z)s^{-1}(z)s^{-1}(qz)\dots s^{-1}(q^kz)\nu^+_{\omega_i}\,,$$

$$i = 1,\dots,r, \qquad k = 0,1,\dots,h-1, \tag{4.32}$$

where h is the Coxeter number of G.

Examples: SL(2)

$$\mathscr{W}(qz)\nu_\omega^+ = Z\mathscr{W}(z)s^{-1}(z)\nu_\omega^+$$

$$s^{-1}(z) = \tilde{s}^{-1}\Lambda(z)^{\check{\alpha}} = \begin{pmatrix} 0 & \Lambda(z)^{-1} \\ \Lambda(z) & 0 \end{pmatrix}, \qquad \nu_\omega^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$$

In terms of Q-polynomials

$$\mathscr{W}(z) = \begin{pmatrix} Q^+(z) & \zeta^{-1}\Lambda(z)^{-1}Q_+(qz) \\ Q^-(z) & \zeta\Lambda(z)^{-1}Q^-(qz) \end{pmatrix}$$

$$\zeta Q^+(z)Q^-(qz) - \zeta^{-1}Q^+(qz)Q^-(z) = \Lambda(z)$$

is equivalent to $\det\mathscr{W}(z) = 1$.

Examples SL(N)

$$\mathscr{W}(z) = \left(\Delta_{\mathbf{w}\omega,\omega}\middle|\Delta_{\mathbf{w}\omega,s^{-1}\omega}\middle|\dots\middle|\Delta_{\mathbf{w}\omega,s^{r+1}\omega}\right)(\mathscr{G}(z))$$

Lift for standard ordering along the Dynkin diagram

$$s_\Lambda^{-1}(z) = \tilde{s}^{-1}\prod_i \Lambda_i^{d_i}$$

$$d_i = \textstyle\sum_{j=1}^{i}\check{\alpha}_j$$

$$\tilde{s}^{-1} = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \dots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

$$\mathscr{W}(z) = \left(Q^{\mathbf{w}\cdot\omega}(z)\middle|ZF_1(z)Q^{\mathbf{w}\cdot\omega}(qz)\middle|\dots\middle|Z^{r-1}F_{r-1}(q^{r-1}z)Q^{\mathbf{w}\cdot\omega}(q^{r-1}z)\right)$$

where $F_i(z) = \prod_{j=1}^{i}\Lambda_j(z)^{-1}$.

Lewis Carroll Identity

In Type A FZ relation reduces to

$$\Delta_{u\omega_i,v\omega_i}\Delta_{us_i\omega_i,vs_i\omega_i} - \Delta_{us_i\omega_i,v\omega_i}\Delta_{u\omega_i,vs_i\omega_i} = \Delta_{u\omega_{i-1},v\omega_{i-1}}\Delta_{u\omega_{i+1},v\omega_{i+1}}$$

$$M_1^1 M_i^2 - M_i^1 M_1^2 = M_{1i}^{12} M$$