Knot homology and sheaves on the Hilbert scheme of points on the plane.

Alexei Oblomkov (joint work with L. Rozansky)

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String Math Seminar, UC Berkeley.

HOMFLY-PT homology

J. Hoste, A. Ocneaunu, K. Millet, P. Freyd, W. Lickorich; J. Przytycki, P. Traczyk; V. Jones 1985

HOMFLY-PT polynomial

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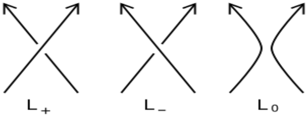

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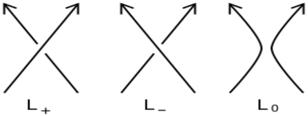

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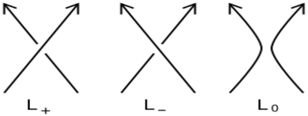

$U_q(\mathfrak{gl}_n)$-quantum invariant from HOMFLY-PT

$$P(L)|_{a=q^{n/2}} = V_n(L).$$

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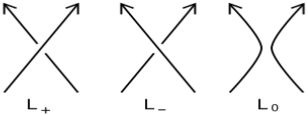

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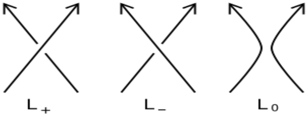

Theorem (Khovanov-Rozansky, 2007, 2008)

For every link L there are doubly graded spaces $H^*_{KhR}(L)$ such that

$$P(L) = \sum_i (-1)^i dim_{q,a} H^i_{KhR}(L).$$

Braids and links

Elements σ_i, $i = 1, \dots, n-1$ generate Br_n

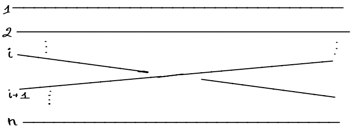

Figure: Generator σ_i

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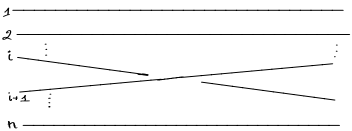

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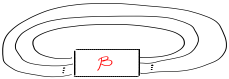

Figure: Closure $L(\beta)$ of the braid β

Hecke algebras and Ocneanu-Jones trace

Hecke algebra $H_n(q)$ is the quotient of Br_n

$$\sigma_i - \sigma_i^{-1} = q^{1/2} - q^{-1/2}.$$

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Theorem (Jones, 1987)

There is a $\mathbb{C}(q, a)$-linear functional Tr_{OJ} on $\bigoplus_n H_n(q)$ such that

- $Tr_{OJ}(\alpha\beta) = Tr_{OJ}(\beta\alpha)$, $\alpha, \beta \in H_n(q)$
- $Tr_{OJ}(1_n) = A^n$, $A = (a^{-1} - a)/(q^{1/2} - q^{-1/2})$.
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$$P(L(\beta)) = a^{?} q^{?} Tr(\beta).$$

Characters and co-characters: OJ trace

$$\begin{tikzcd}
K_{\mathbb{C}_q^*}(MF_n^{st}) \arrow[r, "\sim"] \arrow[d, shift right, "CH"'] & H_n(q) = \bigoplus_{|\lambda|=n} End(V_\lambda) \arrow[d, shift left, "ch"] \arrow[dd, bend left, "Tr_{OJ}"] \\
K_{\mathbb{C}_q^*}(Hilb_n(\mathbb{C}^2)) \arrow[r, "\sim"] \arrow[u, shift right, "HC"] \arrow[d, "\chi_{\mathbb{C}^*}(-\otimes\Lambda^\bullet\mathcal{B})"] & Z(H_n(q)) \arrow[u, shift left, "hc"] \arrow[d] \\
\mathbb{C}(a,q) \arrow[r, equal] & \mathbb{C}(a,q)
\end{tikzcd} .$$

Characters and co-characters: KhR trace

$$
\begin{tikzcd}
MF_n^{st} \arrow[rr, "\sim"] \arrow[d, shift left, "CH"] & & Ho(SBim_n) \arrow[d, shift left, "ch"] \arrow[ddr, bend left, "HH_*"] & \\
D^{per}_{\mathbb{C}^*_q \times \mathbb{C}^*_t}(Hilb_n(\mathbb{C}^2)) \arrow[u, shift left, "HC"] \arrow[r, "\sim"] \arrow[d, "H^*_{\mathbb{C}^*_q \times \mathbb{C}^*_t}(- \otimes \Lambda^\bullet \mathcal{B})"] & & DC(Ho(SBim_n)) \arrow[u, shift left, "hc"] \arrow[d] & \\
\text{3gr. v. sp.} \arrow[rr, equal] & & \text{3gr. v. sp.} &
\end{tikzcd}
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Theorem (O.-Rozansky 2019)

There is a geometric trace map:

$$\mathcal{T}r : Br_n \to D^{per}_{\mathbb{C}^*_q \times \mathbb{C}^*_t}(Hilb_n(\mathbb{C}^2))$$

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5. $\mathcal{T}r(cox_n) = \mathcal{O}_Z$.

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The $\mathbb{C}^*_q \times \mathbb{C}^*_t$ action on $\mathbb{C}^2$ induces the action on $Hilb_n(\mathbb{C}^2)$, hence double grading on $H^i(Z, L^k \otimes \Lambda^m\mathcal{B})$.

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$$Hilb_2(\mathbb{C}^2) = T^*\mathbb{P}^1 \times \mathbb{C}^2 \to \mathbb{C}^2/\{\pm 1\} \times \mathbb{C}^2.$$

Algebraic homology (after Khovanov and Rozansky)

$$R_n = \mathbb{C}[x_1, \dots, x_n], \quad B_k = R_n \otimes_{R_n^{s_{k,k+1}}} R_n, \quad \deg(x_i) = q^2.$$

Definition (Soergel'90)

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$$[Rouquier'04] \quad Ro : Br_n \to \mathrm{Ho}(SBim_n).$$

Theorem (Khovanov-Rozansky '04)

For $\beta \in Br_n$ the triply graded vector space

$$\mathrm{HHH}_{alg}(\beta) = H^\bullet(HH_*(Ro(\beta))), \quad \deg(\bullet) = t.$$

is an isotopy invariant of $L(\beta)$.

Geometric homology

$$St_n = \{(F_\bullet, F'_\bullet, X) | X(F_i) \subset F_{i-1}, X(F'_i) \subset F'_{i-1}\} \subset Fl \times Fl \times \mathfrak{gl}_n.$$

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Example

$St_2 \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathfrak{gl}(n)$, $\quad St_2 = \mathbb{P}^1 \times \mathbb{P}^1 \bigcup T^*\mathbb{P}^1$. Two components are glued along $\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1$.

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$$[Bezrukavnikov-Riche'13, O-Rozansky'16] \quad \Psi : Br_n \to D^{GL_n}_{\mathbb{C}^*}(St_n).$$

Theorem (O-Rozansky '16)

For $\beta \in Br_n$ the triply graded vector space

$$\mathrm{HHH}_{geo}(\beta) = H^\bullet(Hom(\Psi(\beta), \Psi(1) \otimes \Lambda\mathbb{C}^n)^{GL_n}), \quad \deg(\bullet) = t.$$

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Two realizations of the braids

Algebraic:

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Correction:
$D^{GL_n}_{\mathbb{C}^*}(St_n) \subset D^{GL_n}_{\mathbb{C}^*}(T^*Fl_n \times T^*Fl_n)$,

$$St_n = \{(z_1, z_2) \in T^*Fl \times T^*Fl | \mu(z_1) = \mu(z_2)\}$$

Matrix Factorizations

Better model

$$D^{GL_n}_{\mathbb{C}^*_q}(St_n) = MF^{GL_n}_{\mathbb{C}^*_q \times \mathbb{C}^*_t}(\mathfrak{gl}_n \times T^*Fl \times T^*Fl, Tr(X(\mu(z_1) - \mu(z_2)))).$$

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Matrix Factorizations, Eisenbud 1980

$$W \in \mathbb{C}[x_1, \dots, x_n].$$

$$MF(\mathbb{C}^n, W) = \{\dots \xrightarrow{d_0} M_1 \xrightarrow{d_1} M_0 \xrightarrow{d_0} M_1 \dots\}$$

$$d_0 \circ d_1 = d_1 \circ d_0 = W, \quad M_i = \mathbb{C}[x_1, \dots, x_m] \otimes \mathbb{C}^{m_i}.$$

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Example

If $n = 1$ and $W = x^4$ then following is an element of $MF(\mathbb{C}, W)$:

$$\dots \mathbb{C}[x] \xrightarrow{x} \mathbb{C}[x] \xrightarrow{x^3} \mathbb{C}[x] \xrightarrow{x} \dots$$

Koszul duality

Theorem

$X = Y \times \mathbb{C}^n_z$, $W = \sum_{i=1}^n f(y) z_i$ *then*

$$Kos_z : MF(X, W) \simeq D(f_1(y) = \cdots = f_n(y) = 0).$$

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$$\mathfrak{gl}_n \times T^*Fl_n \times T^*Fl_n = \mathfrak{gl}_n \times GL_n \times \mathfrak{n} \times GL_n \times \mathfrak{n}/B^2,$$

$$W(X, g_1, Y_1, g_2, Y_2) = Tr(X(Ad_{g_1} Y_1 - Ad_{g_2} Y_2))$$

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$$Kos_z : MF(X, W) \simeq D(f_1(y) = \cdots = f_n(y) = 0).$$

$$\mathfrak{gl}_n \times T^* Fl_n \times T^* Fl_n = \mathfrak{gl}_n \times GL_n \times \mathfrak{n} \times GL_n \times \mathfrak{n}/B^2,$$

$$W(X, g_1, Y_1, g_2, Y_2) = Tr(X(Ad_{g_1} Y_1 - Ad_{g_2} Y_2))$$

$$St_n = \{Ad_{g_1} Y_1 - Ad_{g_2} Y_2\} \subset T^* Fl \times T^* Fl$$

$$MF_n = MF^{GL_n}_{\mathbb{C}^* \times \mathbb{C}^*}(\mathfrak{gl}_n \times T^* Fl \times T^* Fl, W) \simeq D^{GL_n}_{\mathbb{C}^*}(St_n).$$

Flag Hilbert schemes.

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$$j^*(W) = Tr(X(Ad_g Y - Y)) = Tr(X[\xi, Y]) + O(\xi^2), \quad g = 1 + \xi, \xi \in \mathfrak{gl}_n.$$

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Torus knots

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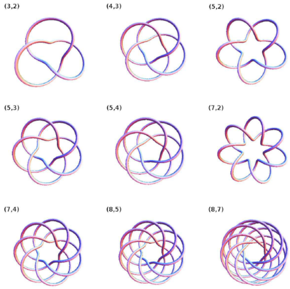

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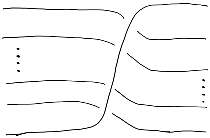

Figure: $cox_n \in Br_n$

$$cox_n = \sigma_1 \cdot \sigma_2 \cdots \sigma_{n-1}.$$

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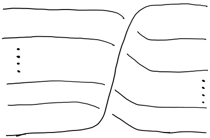

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Corollary (conjectured by Gorsky, O. Rasmussen, Shende, 2012, Aganagic, Shakirov, 2011)

$$\mathrm{HHH}_{alg}(T_{n,1+nk}) = H^0(Z, \Lambda^\bullet \mathcal{B} \otimes \det(\mathcal{B})^k), \quad Z \subset Hilb_n(\mathbb{C}^2).$$

Physics: 3D TQFT with defects

Theorem (O.-Rozansky '18)

There is a gauged topological 3D sigma model with source $\mathbb{R}^2 \times S^1$ *with defect* $\beta \times S^1$ *such that*

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This topological 3D sigma model is an example of Kapustin-Saulina-Rozansky TQFT = Rozansky-Witten theory with defects.

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Rozansky-Witten theory '97

$$\dim M = 3, \quad \dim X = 4n, \quad X \text{ is hyper-Kahler.}$$

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$Z_X(M)$ is a "finite order" Vassiliev-type topological invariant of M.

Kapustin-Rozansky-Saulina: boundary conditions for RW

$$\Phi : M \to X, \quad \Sigma = \partial M \to Y \subset X.$$

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$$L_Y \subset \mathcal{H}_{\Sigma'} \text{ the constrained states}$$

Theorem (Kapustin-Saulina-Rozansky'09)

If L_Y is Lagrangian and preserved by the super symmetries then Y is a holomorophic Lagrangian.

Kapustin-Rozansky-Saulina: enrichment of RW

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If X is compact by theorem of Voisin Y is unobstructed.

Otherwise we need to assume that Y is CY too.

KSR outline

Kapustin, Saulina and Rozansky proposed a realization of the 3D topological field theory, 2008.

Three-category $3Cat_{sym}$

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and $h : H \to X \times W$, $L'' = h(H)$.

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$$Hilb_n(\mathbb{C}^2) = T^*(\mathfrak{gl}_n / GL_n)^{st} \subset T^*(\mathfrak{gl}_n / GL_n).$$

Main Example: Lagrangian

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More geometrically, $L_n \subset Hilb_n(\mathbb{C}^2)$ consists of ideals $I \subset \mathbb{C}[x, y]$ such that $supp(\mathbb{C}[x, y]/I) \subset Sym^n(\{y = 0\})$.

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Steinberg picture

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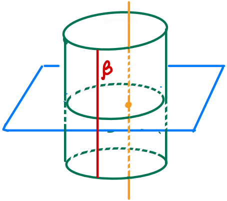
$M = \mathbb{R}^2 \times S^1$



$\beta \in Br_n$

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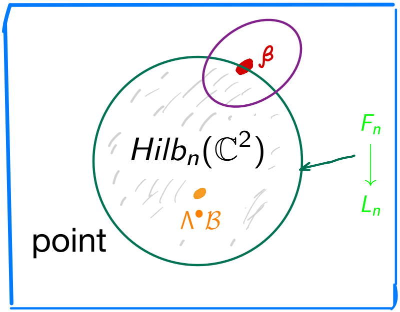

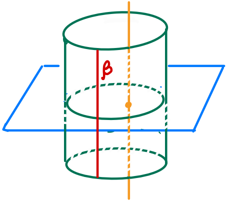

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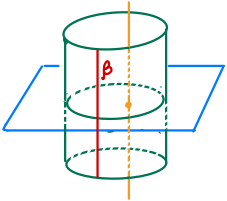

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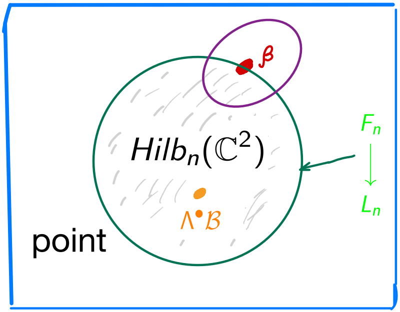
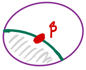


$$Z\left(\right.$$



$$\left.\right) = \mathcal{O}_\beta \in \mathrm{Hom}^\bullet\left(\begin{array}{c} F_n \\ \downarrow \\ L_n \end{array}, \begin{array}{c} F_n \\ \downarrow \\ L_n \end{array}\right)$$

$$\parallel$$

$$D^{per}(St_n)$$

[Bezrukavnikov,Riche 2012]

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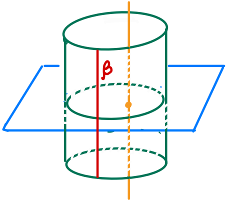

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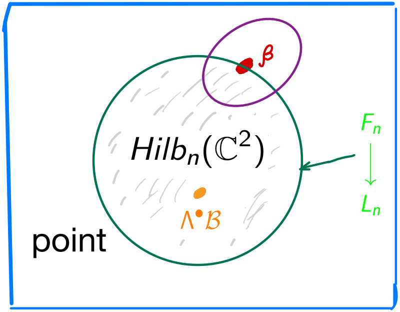
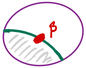


$Z\Big($  $\Big)$

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[Bezrukavnikov,Riche 2012]

$$\| $$

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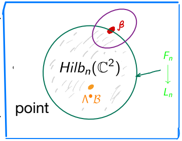
$Z\Big($  $\Big)$

$$= \operatorname{Hom}^\bullet(\mathcal{O}_1 \otimes \Lambda^\bullet \mathcal{B}, \mathcal{O}_\beta) = HHH'_{geo}(\beta).$$

[O. Rozansky 2016]

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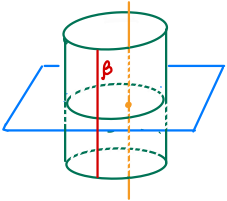

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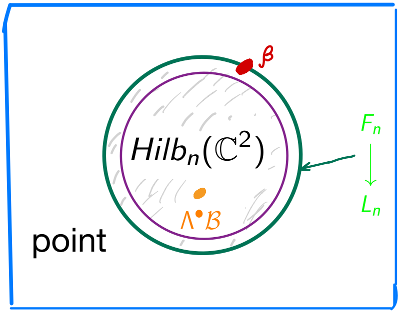

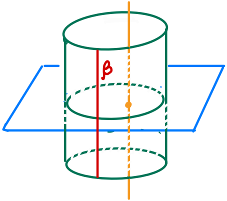

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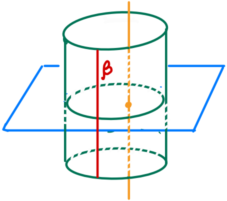

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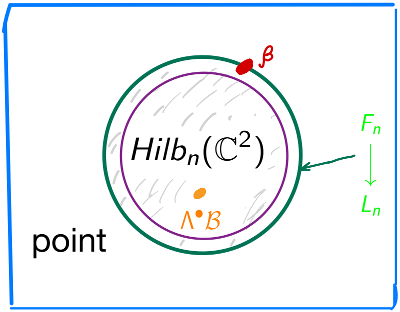
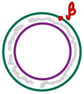
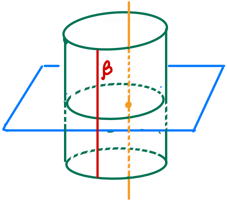


$Z\Big($  $\Big) = \mathcal{F}_\beta \in D^{per}\big(Hilb_n(\mathbb{C}^2)\big).$

Hilb picture $M = \mathbb{R}^2 \times S^1$



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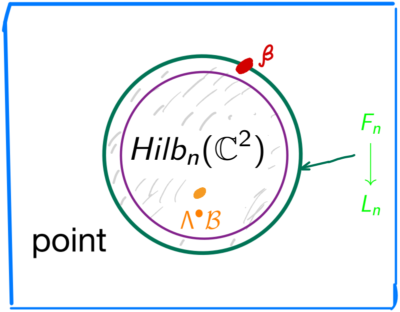
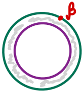
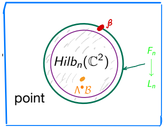


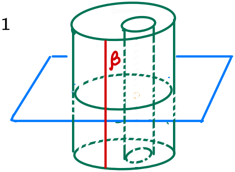
$Z\Big($  $\Big) = \mathcal{F}_\beta \in D^{per}(Hilb_n(\mathbb{C}^2))$

$Z\Big($  $\Big) = \mathrm{Hom}^\bullet(\mathcal{F}_1 \otimes \Lambda^\bullet \mathcal{B}, \mathcal{F}_\beta) = HHH'_{geo}(\beta)$

[O. Rozansky 2018]

Soergel picture

$M = \mathbb{R}^2 \times S^1$



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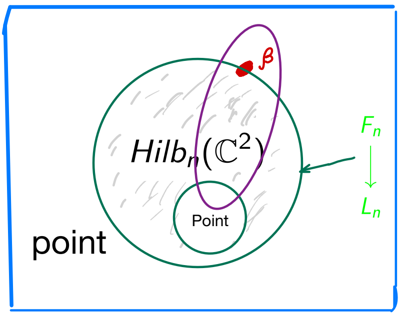

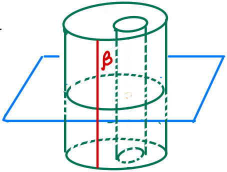

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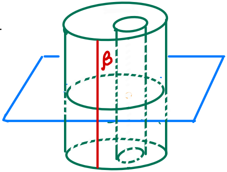

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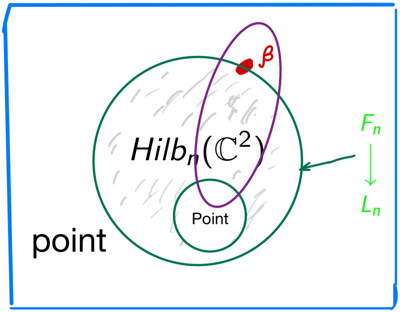
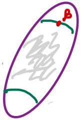


$Z\Big($  $\Big) = S_\beta \in \mathrm{Hom}^\bullet\Big($  ,  $\Big)$

$$\parallel$$

$$D^b(\mathbb{C}^n \times \mathbb{C}^n) \supset SBim_n$$

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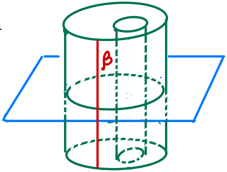

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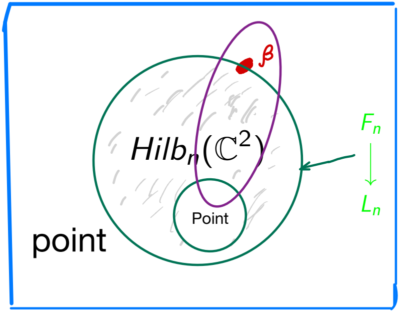
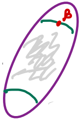


$Z\Big($ 

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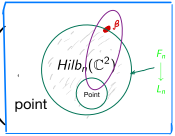
$Z\Big($ 

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||

$$HHH'_{geo}(\beta)$$

[O. Rozansky 2020]

Soergel picture

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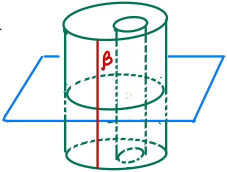

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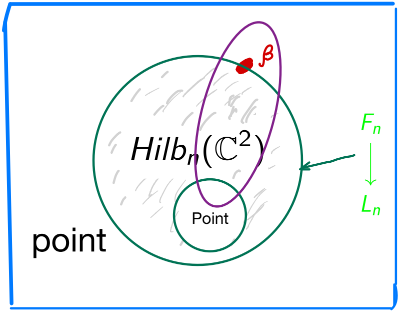
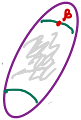
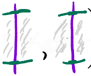


$Z\Big($  $\Big) = S_\beta \in \mathrm{Hom}^\bullet\Big($  $\Big)$

$\|$

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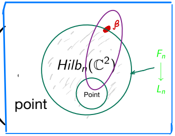
$Z\Big($  $\Big) = \mathrm{Hom}^*(S_1, S_\beta) = \mathrm{HHH}_{alg}(\beta)$

$\|$

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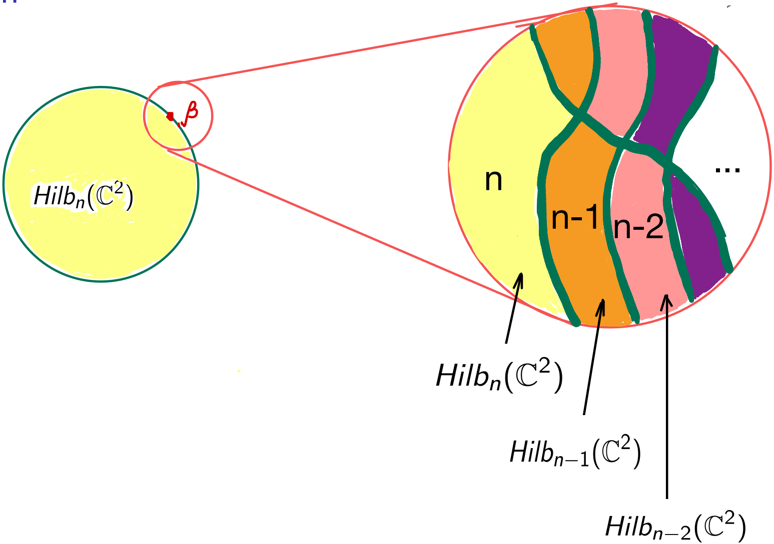
[O. Rozansky 2020]

Zoom in



$Hilb_n(\mathbb{C}^2)$
β
n
n-1
n-2
...
$Hilb_n(\mathbb{C}^2)$
$Hilb_{n-1}(\mathbb{C}^2)$
$Hilb_{n-2}(\mathbb{C}^2)$

$3Cat_{man}$: T^*X is holomorphic symplectic

$$Obj(3Cat_{man}) = \{\text{complex manifolds}\}$$

$$\text{1-Hom}(X, Y) = \{(Z, w) | w : X \times Z \times Y \to \mathbb{C}\}$$

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$$1\text{-Hom}(X, Y) = \{(Z, w) | w : X \times Z \times Y \to \mathbb{C}\}$$

For $(Z, w) \in 1\text{-Hom}(X, Y)$, $(Z', w') \in 1\text{-Hom}(Y, W)$:

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For $(Z, w), (Z', w') \in Hom(X, Y)$ we have

$$\text{2-Hom}((Z, w), (Z', w')) = MF(X \times Z \times Z' \times Y, w' - w).$$

$3Cat_{sym}$ vs $3Cat_{man}$

Functor $3Cat_{man} \to 3Cat_{sym}$

$$X \mapsto T^*X,$$

$$(Z, w) \mapsto (F_w, L_w, \pi)$$

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$$Z \times T^*X \supset F_w \ni (z, x, p) \text{ if } \partial_z w(z, x) = 0, \quad p = \partial_x w(z, x).$$

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Let impose condition on (Z_i, w_i): $Crit_{w_i} \subset \{w_i = 0\}$, then we have

$$MF(X \times Z_1 \times Z_2 \times Y, w_1 - w_2) \to D^{per}(F_{w_1} \times_{T^*(X \times Y)} F_{w_2}).$$

Main example II

3*Cat* $\mathfrak{gl}$

$$Obj = \{\mathfrak{gl}_n, n \in \mathbb{Z}_{\geqslant 0}\},$$

$$\text{1-Hom}(\mathfrak{gl}_n, \mathfrak{gl}_m) = \{Z \text{ with Hamiltonian } GL_n \times GL_m \text{ action}\}$$

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$\mathfrak{gl} \to 3Cat_{man}$

$$\text{1-Hom}(\mathfrak{gl}_n, \mathfrak{gl}_m) \ni Z \mapsto (Z, w(x, z, y) = \mu_n(z)(x) - \mu_m(z)(y)).$$

$$\text{Moment maps: } \mu_n : Z \to \mathfrak{gl}_n^*, \quad \mu_m : Z \to \mathfrak{gl}_m^*$$

Main example

$$Z = T^*Fl = \{(g, Y) \in GL_n \times \mathfrak{n}\}/B \in 1\text{-Hom}(\mathfrak{gl}_n, \mathfrak{gl}_0),$$

$$\mu(g, Y) = Ad_g(Y).$$

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$2\text{ -Hom}(T^*Fl, T^*Fl)$

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Braids with matrix factorizations

$$X = \mathfrak{gl}_n \times GL_n^2 \times \mathfrak{n}^2.$$

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Stable space

$$X^{st} = \{(X, g_1, Y_1, g_2, Y_2, v) | \mathbb{C}\langle Ad_{g_1}^{-1}(X), Y_1 \rangle v = \mathbb{C}^n\}$$

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$$MF_n^{st} = MF_{GL_n}(X^{st}, W).$$

Theorem (O.-Rozansky, 2017)

For any n there is group homomorphism:

$$\Psi : \quad Br_n \to (MF_n^{st}, \star).$$

Free Hilbert scheme and knot homology

$$FHilb_n^{free} = \{(X, Y, v) \in \mathfrak{b} \times \mathfrak{n} \times V | \mathbb{C}\langle X, Y \rangle v = \mathbb{C}^n\}/B.$$

Free Hilbert scheme and knot homology

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Embedding $j : FHilb^{free} \to X^{st}$

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Main construction

$$\mathcal{S}_\beta = j^*(\Psi(\beta)) \in MF(FHilb^{free}, 0) = D^{per}_{\mathbb{C}_q^* \times \mathbb{C}_t^*}(FHilb^{free}).$$

Free Hilbert scheme and knot homology

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Theorem (O.-Rozansky 2016)

The triply graded vector space

$$\mathrm{HHH}'_{geo}(\beta) = \mathbb{H}(\mathcal{S}_\beta \otimes \Lambda^* \mathcal{B}) \textit{ is an isotopy invariant of } L(\beta).$$

Defects

$$X = \mathbb{R}^2 \times S^1.$$

Defect surfaces: $D_\beta = \beta \times S^1 \subset \mathbb{R}^2 \times S^1.$

$$\mathbb{R}^2_\beta = \{D_\beta \subset \mathbb{R}^2 \times S^1 | \text{ with monotonous marking of } \mathbb{R}^2 \times S^1 \backslash D_\beta\}.$$

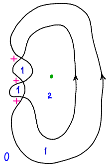

Figure: $\mathbb{R}^2_\beta$ for $\beta = \sigma_1^3$.

Partition function I

$$Z(p_n \to p_n) = T^* GL_n \in \text{1-Hom}(\mathfrak{gl}_n, \mathfrak{gl}_n)$$

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$$Z(I_{n|m}) = T^* Hom(\mathbb{C}^n, \mathbb{C}^m).$$

Composition

$$Z_{nm} \in 1\text{-Hom}(\mathfrak{gl}_n, \mathfrak{gl}_m), \quad Z_{mk} \in 1\text{-Hom}(\mathfrak{gl}_m, \mathfrak{gl}_k).$$

$$Z_{nm} \circ Z_{mk} = Z_{nm} \times Z_{mk} /_{\det} GL_m.$$

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$$MF^{st}_{GL_n}(\mathfrak{gl}_n^3, Tr(Y[X, Z])) = D^{per}(Hilb_n(\mathbb{C}^2))$$

Partition function II

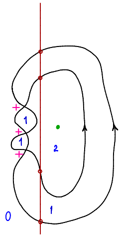

Figure: Plane $\mathbb{R}^2_{\sigma_1^3}$ is cut by $\mathbb{R}_{0|1|2|1|0}$ on two connected components D_β^{ste} and D_1^{ste}.

Partition function II

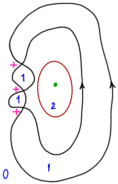

Figure: Plane $\mathbb{R}^2_{\sigma_1^3}$ is cut by S^1_n on two connected components $D^{hilb}_{L(\beta)}$ and $D^{hilb}_{\varnothing}$.

Partition function III

$$Z(D_\beta^{ste}) \in MF_n^{st} = Z(S^1_{0|1|\dots|n|n-1|\dots|1}), \quad Z(D_\beta^{ste}) = \Psi(\beta).$$

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$$\text{3-Hom}(Z(D_\beta^{ste}), Z(D_1^{ste})) = HHH_{geo}(\beta) = \text{3-Hom}(Z(D^{hilb}_{\varnothing}), Z(D^{hilb}_{L(\beta)})).$$

Characters and co-characters: OJ trace

$$
\begin{tikzcd}
K_{\mathbb{C}_q^*}(MF_n^{st}) \arrow[rr, "\sim"] \arrow[d, shift left, "CH"] & & H_n(q) = \bigoplus_{|\lambda|=n} End(V_\lambda) \arrow[d, shift left, "ch"] \arrow[dd, bend left, "tr_{OJ}"] \\
K_{\mathbb{C}_q^*}(Hilb_n(\mathbb{C}^2)) \arrow[rr, "\sim"] \arrow[u, shift left, "HC"] \arrow[d, "\chi_{\mathbb{C}^*}(-\otimes\Lambda^\bullet\mathcal{B})"] & & Z(H_n(q)) \arrow[u, shift left, "hc"] \arrow[d] \\
\mathbb{C}(a,q) \arrow[rr, equal] & & \mathbb{C}(a,q)
\end{tikzcd}
.
$$

Characters and co-characters: KhR trace

$$\begin{tikzcd}
MF_n^{st} \arrow[rr, "\sim"] \arrow[d, shift left, "CH"] & & Ho(SBim_n) \arrow[d, shift left, "ch"] \arrow[dd, bend left=60, "HH_*"] \\
D^{per}_{\mathbb{C}_q^* \times \mathbb{C}_t^*}(Hilb_n(\mathbb{C}^2)) \arrow[u, shift left, "HC"] \arrow[rr, "\sim"] \arrow[d, "H^*_{\mathbb{C}_q^* \times \mathbb{C}_t^*}(- \otimes \Lambda^\bullet \mathcal{B})"] & & DC(Ho(SBim_n)) \arrow[u, shift left, "hc"] \arrow[d] \\
\text{3gr. v. sp.} \arrow[rr, equal] & & \text{3gr. v. sp.}
\end{tikzcd}.$$

CH and HC

Theorem (O.-Rozansky 2018)

$$
\begin{tikzcd}
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CH and HC

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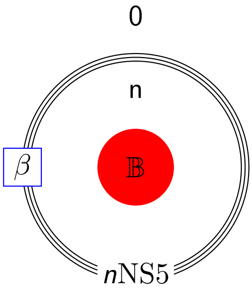
Torus links

Theorem (O.-Rozansky 2018)

1. $CH(\Psi(FT) \star \mathcal{F}) = \det(\mathcal{B}) \otimes CH(\mathcal{F})$.
2. $CH(\Psi(cox)) = \mathcal{O}_Z$
3. $CH(\Psi(1)) = \mathcal{P}|_{y=0}$.

More traces



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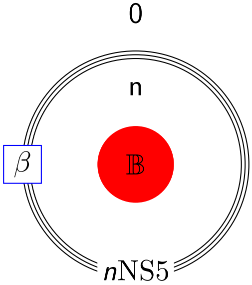

$\mathbb{B}$	$\Lambda^\bullet V_n$	$(\Lambda^\bullet V_n, d_{m\|k})$	$\mathrm{NS5}^{(n)}$	$\mathrm{D5}^{(k)} + \mathrm{D5}^{(n-k)}$
β	Br_n	Br_n	$Br_n^\flat$	Br_n
r	1	1	1	0
Z	$\mathrm{HHH}(\beta)$	$\mathbb{H}_{m\|k}(\beta)$	$\mathrm{HHH}_{alg}(\beta)$	$\mathcal{T}r(\beta)[H^\lambda_{1^n,\nu+k}]$

$gl(m|k)$ homology

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$\phi_{m|k} \in H^0(\mathrm{Hilb}_n(\mathbb{C}^2), \mathcal{B}^\vee)$, $\phi_{m|k}(X, Y, v) = X^m Y^k v$

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Differential
$i_{\phi_{m|k}} : \Lambda^i \mathcal{B} \to \Lambda^{i-1}\mathcal{B}$, $d_{m|k} : \mathcal{F}_\beta \otimes \Lambda^i \mathcal{B} \to \mathcal{F}_\beta \otimes \Lambda^{i-1}\mathcal{B}$.

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Let $d_{\mathcal{F}}$ be the differential of $\mathcal{T}r(\beta) = S_\beta \in D^{per}(Hilb_n(\mathbb{C}^2))$ and

$$\mathbb{H}_{m|k}(\beta) := H(S_\beta \otimes \Lambda^\bullet \mathcal{B}, d_{\mathcal{F}} + d_{m|k})$$

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$gl(m|k)$ homology

Theorem (O., Rozansky 2022)

The doubly graded vector space

$$\mathbb{H}_{m|k}(\beta)$$

is an isotopy invariant of $L(\beta)$ that categorifies quantum $gl(m|k)$ polynomial.

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Conjecture [O., Rozansky 2016]

$$\mathbb{H}_{m|0}(\beta) = H^{KhR}_{gl(m)}(L(\beta)).$$

$gl(m|n)$ holomogy of unknot

$$\mathrm{Hilb}_1(\mathbb{C}^2) = \mathbb{C}_x \times \mathbb{C}_y, \quad \mathcal{B} = \mathbb{C}, \quad \mathcal{T}r(1) = \mathcal{O}_{\mathbb{C}_x}$$

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$$\dim_{\mathrm{Q,T}} \mathbb{H}_m(1) = \frac{\mathrm{Q}^m - 1}{\mathrm{Q} - 1}, \quad \dim_{\mathrm{Q,T}} \mathbb{H}_{m|k}(1) = \frac{\mathrm{T}\mathrm{Q}^{m-k} + 1}{1 - \mathrm{Q}}.$$

NS5, D5

[Nakajima-Taniyama '17]:$\mathscr{R}_\lambda = p^{-1}(\mathcal{N} \cap \mathcal{S}(\lambda)) \subset T^*Fl$

$$\mathscr{R}_\lambda = Q^{bow}(\lambda)/G_\lambda \times G_{1^n}, \quad G_\lambda = \prod_i GL_{L_i}, \quad \lambda_i = L_i - L_{i-1}.$$

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$$\begin{tikzcd}
\mathbb{C} \arrow[dr] & \mathbb{C} \arrow[drr] & \cdots \arrow[dr] & \mathbb{C} \arrow[dr] & & & & \\
\mathbb{C}^{L_1} \arrow[u] \arrow[r] & \mathbb{C}^{L_2} \arrow[u] \arrow[r] & \cdots \arrow[r] & \mathbb{C}^{L_{l-1}} \arrow[u] \arrow[r] & \mathbb{C}^n \arrow[r, shift left] & \mathbb{C}^{n-1} \arrow[l, shift left] \arrow[r, shift left] & \cdots \arrow[l, shift left] \arrow[r, shift left] & \mathbb{C}^1 \arrow[l, shift left]
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$$D5^{(\lambda_1)} D5^{(\lambda_2)} \cdots D5^{(\lambda_{l-1})} NS5^{(1)} NS5^{(1)} \cdots NS5^{(1)}.$$

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Let V_ν is a Verma module for $U_q(sl_2)$ and $L_1 \simeq \mathbb{C}^2$.

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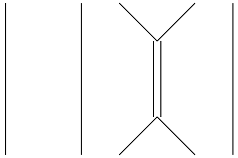
Conjecture $\mathcal{T}r : Br_n \to 2-\mathrm{grVect}$ categorifies $Tr_{k,n-k}$.

Braid-graphs

The monoid of braid-graphs $Br_n^\flat$ is generated by the elements $\sigma_\bullet^{(i,i+1)}$:



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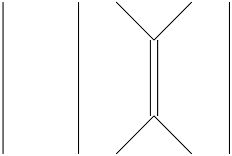

$$\Psi(\sigma_\bullet^{(1,2)}) = \mathcal{O}_{St_2}, \quad \Psi : Br_n^\flat \to D_{\mathbb{C}^*}^{GL_n}(St_n).$$

$$\Psi^{(k,k)} : Br_{2k}^\flat \to Hom(Z(I^{(k,k),1^{2k}}), Z(I^{(k,k),1^{2k}})).$$

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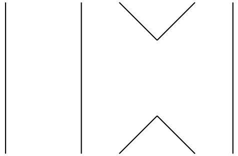
The monoid of tangles $Tang^{\flat}_{2k}$ is generated the elements of braid group Br_{2k} and by the elements $cup^{(i,i+1)} \circ cap^{(i,i+1)}$:



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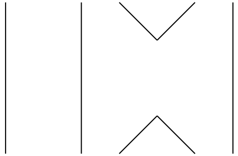

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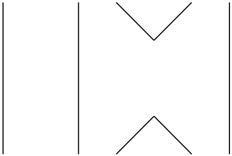

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O-Rozansky' 20 The construction of the trace functor $\mathcal{T}r_{(k,k)}$ can be extended to affine tangles and the corresponding invariant provides a realization of the sl_2 annular Khovanov homology.

Thanks

THANK YOU!

Dualizable CH and HC

Theorem (O.-Rozansky 2022)

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Dualizable triply-graded link homology

Theorem (O.-Rozansky 2019,2022)

There is a geometric trace map:

$$\underline{\mathcal{T}r} : Br_n \to D^{per}_{\mathbb{C}^*_q \times \mathbb{C}^*_t}(Hilb_n(\mathbb{C}^2))$$

such that

1. $\mathrm{HXY}(\beta) = \oplus_i H^*(\underline{\mathcal{T}r}(\beta) \otimes \Lambda^i \mathcal{B})$ *is an isotopy invariant of the braid closure* $L(\beta)$
2. $\underline{\mathcal{T}r}(\beta \cdot FT_n) = \underline{\mathcal{T}r}(\beta) \otimes \det(\mathcal{B})$.
3. $\underline{\mathcal{T}r}(\beta) = \underline{\mathcal{T}r}(\beta)|_{q \to t^2/q}$
4. $\mathrm{HHH}_{geo}(\beta) = \mathrm{HXY}(\beta) \otimes_{R_{x,y}(\beta)} R_x(\beta)$, $R_{x,y}(\beta) = \mathbb{C}[x,y]^I$, $R_x(\beta) = \mathbb{C}[x]^I$, $I = \pi_0(L(\beta))$

Nakajima functors

Nested Hilbert scheme

$$\mathrm{Hilb}_{n+1,n}(\mathbb{C}^2) \subset \mathrm{Hilb}_{n+1}(\mathbb{C}^2) \times \mathrm{Hilb}_n(\mathbb{C}^2), \quad \{(I,J) | I \subset J\}$$

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The functor

$$P_{1,k}[\mathcal{G}] : D^{per}_{\mathbb{C}^*_q \times \mathbb{C}^*_t}(\mathrm{Hilb}_n(\mathbb{C}^2)) \to D^{per}_{\mathbb{C}^*_q \times \mathbb{C}^*_t}(\mathrm{Hilb}_{n+1}(\mathbb{C}^2))$$

$$\mathcal{G} \in D^{per}_{\mathbb{C}^*_q \times \mathbb{C}^*_t}(\mathbb{C}^2), \quad k \in \mathbb{Z}.$$

Nakajima functors

Nested Hilbert scheme

$$\mathrm{Hilb}_{n+1,n}(\mathbb{C}^2) \subset \mathrm{Hilb}_{n+1}(\mathbb{C}^2) \times \mathrm{Hilb}_n(\mathbb{C}^2), \quad \{(I,J) | I \subset J\}$$

$$\pi_{n+1} : \mathrm{Hilb}_{n+1,n}(\mathbb{C}^2) \to \mathrm{Hilb}_n(\mathbb{C}^2), \quad \pi_n : \mathrm{Hilb}_{n+1,n}(\mathbb{C}^2) \to \mathrm{Hilb}_n(\mathbb{C}^2)$$

$$\mathcal{L}_{(I,J)} = J/I, \quad p : \mathrm{Hilb}_{n+1,n}(\mathbb{C}^2) \to \mathbb{C}^2$$

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$$P_{1,k}[\mathcal{G}](S) = \pi_{n+1*}(\pi_n^*(S) \otimes p^*(\mathcal{G}) \otimes \mathcal{L}^k).$$

Skein algebra vs sperical DAHA_∞ aka elliptic Hall algebra.

$$\beta \in Br_n = \langle \sigma_1, \dots, \sigma_{n-1} \rangle$$

Theorem (O-Rozansky,2022)

For any $k \in \mathbb{Z}$

$$\underline{\mathcal{T}r}(\beta \cdot \sigma_n^{2k}) = P_{1,k}[\mathcal{O}_{\mathbb{C}^2}](\underline{\mathcal{T}r}(\beta))$$

$$\underline{\mathcal{T}r}(\beta \cdot \sigma_n^{2k+1}) = P_{1,k}[\mathcal{O}_{(0,0)}](\underline{\mathcal{T}r}(\beta))$$

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Functors $P_{1,k}$ and $\otimes \det(\mathcal{B})$ generate elliptic Hall algebra.

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Expectation: if β is a periodic braid then $\mathrm{HHH}(\beta)$ has an explicit formula in terms of the elliptic Hall algebra.