

# What languages do black holes speak?

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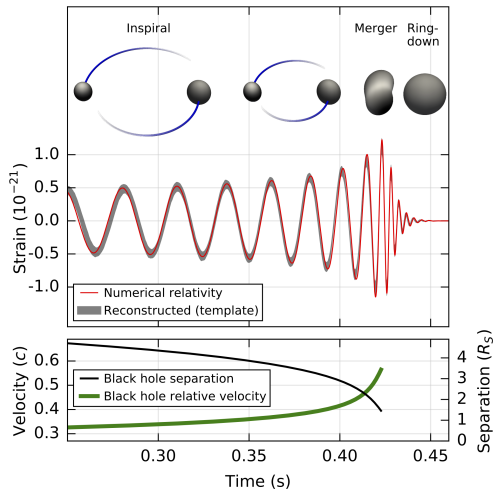
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1. Detectors of gravitational waves
2. Ringdown phase and quasinormal modes (QNMs)
3. First analytic approach to QNMs: instanton corrections
4. Second analytic approach to QNMs: a new underlying recursive structure
5. Multiple polylogarithms and more

- **G. Aminov, A. Grassi, Y. Hatsuda.** "Black Hole Quasinormal Modes and Seiberg-Witten Theory". Ann. Henri Poincaré 23, 1951–1977 (2022). <https://arxiv.org/abs/2006.06111>
- **G. Aminov, P. Arnaudo, G. Bonelli, A. Grassi, A. Tanzini.** "Black hole perturbation theory and multiple polylogarithms". <https://arxiv.org/abs/2307.10141>
- **Gleb Aminov and Paolo Arnaudo.** *Work in progress.*
- **LIGO Scientific Collaboration and Virgo Collaboration.** "GW150914: First results from the search for binary black hole coalescence with Advanced LIGO"
- **NASA** illustration of LISA, taken from <http://lisa.jpl.nasa.gov/gallery/lisa-waves.html>
- **Mr. Gantano**, background from <https://www.pptgrounds.com/3d/8265-space-galaxy-backgrounds>

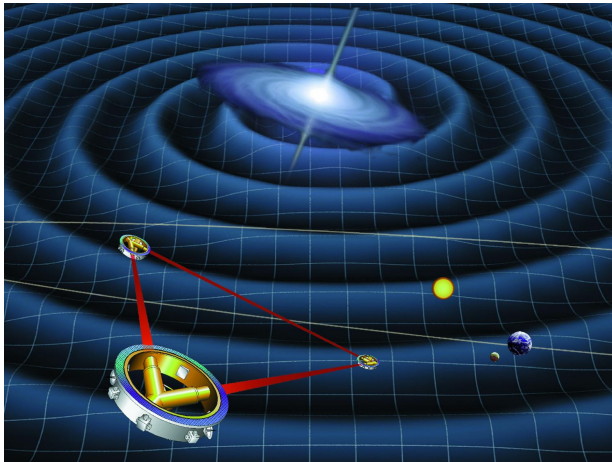
# LIGO and Virgo interferometer

- GW150914:



# Laser Interferometer Space Antenna (LISA)

- Expected to measure and distinguish between different QNMs



# Ringdown phase and quasinormal modes

- QNMs are defined by:

$$\psi''(z) + a_1(z)\psi'(z) + a_0(z)\psi(z) = 0$$

AND

Boundary Conditions (BCs) at  $z = z_1$ ,  $z = z_2$

# Different kinds of black holes

- 4d Schwarzschild black holes in three different backgrounds
- For ordinary black holes:

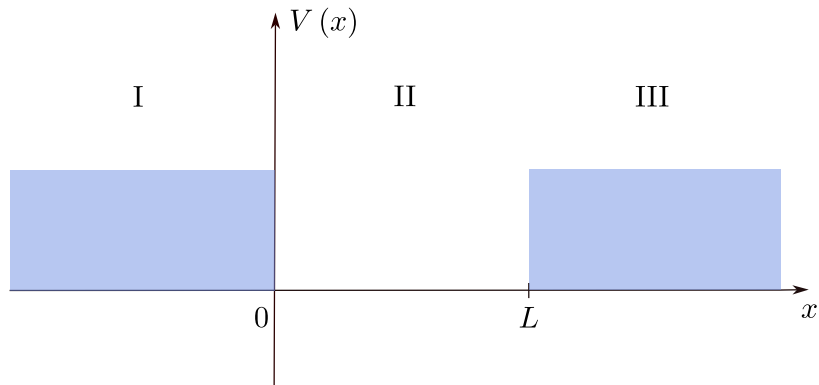
$$\phi''(r) + \frac{f'(r)}{f(r)}\phi'(r) + \frac{\omega^2 - V(r)}{f(r)^2}\phi(r) = 0$$

AND

$$\varphi(r) \sim \begin{cases} e^{-\omega(r+2M \log(r-2M))} & , \quad r \rightarrow 2M \\ e^{+\omega(r+2M \log(r-2M))} & , \quad r \rightarrow \infty \end{cases}$$

# Analogy with time-independent Schrödinger equation

- Simple example in QM:





- QNM ODEs  $\Leftrightarrow$  quantum SW curves:

$$\widehat{H}_{N_f} \psi(z) = E \psi(z).$$

- Not all BCs are equivalent!

Schwarzschild  $\Leftrightarrow N_f = 3$ , same BCs

SdS<sub>4</sub>  $\Leftrightarrow N_f = 4$ , same BCs

SAdS<sub>4</sub>  $\Leftrightarrow N_f = 4$ , different BCs

- Quantum periods in SW theory:

$$a = \Pi_A(E, m_j, \Lambda),$$
$$\frac{\partial}{\partial a} \mathcal{F}^{NS} = \Pi_B(E, m_j, \Lambda).$$

- Quantization condition

$$\Pi_B = \frac{\partial \mathcal{F}^{NS}}{\partial a} = 2\pi\hbar(n+1), \quad n = 0, 1, \dots$$

- Not enough when BCs do not coincide!

- SUSY language needs translation
- Invert the Matone relation

$$E = a^2 - \frac{\Lambda}{2N_c - N_f} \frac{\partial \mathcal{F}^{inst}}{\partial \Lambda}$$

- and use

$$E = -\ell(\ell + 1) + 8M^2\omega^2 - \frac{1}{4}.$$

# An example

- Compute  $a = a(M\omega, \ell, s, \Lambda)$ ,  $\Lambda = 2i\omega$ .
- For  $\ell = s = 0$  and  $M = \frac{1}{2}$ :

$$a = \frac{1}{2} \sqrt{8\omega^2 - 1} + \frac{i\omega\Lambda}{4\sqrt{8\omega^2 - 1}} + O(\Lambda^2).$$

- Exact expression for  $\Pi_b$ :

$$\begin{aligned} \Pi_B = & \frac{\partial \mathcal{F}^{inst}}{\partial a} - 2a \log \left[ \frac{\Lambda}{\hbar} \right] - 2i\hbar \log \left[ \frac{\Gamma\left(1 + \frac{2ia}{\hbar}\right)}{\Gamma\left(1 - \frac{2ia}{\hbar}\right)} \right] \\ & - i\hbar \sum_{j=1}^3 \log \left[ \frac{\Gamma\left(\frac{1}{2} + \frac{m_j - ia}{\hbar}\right)}{\Gamma\left(\frac{1}{2} + \frac{m_j + ia}{\hbar}\right)} \right]. \end{aligned}$$

- We solve perturbatively in  $\Lambda$

$$\Pi_B(M\omega, l, s, \Lambda) = 2\pi\hbar(n+1)$$

- and apply the Padé approximant:

Nb	$2M\omega_0(0,0)$
3	<b>0.21453301 - 0.20342058i</b>
8	<b>0.22088781 - 0.20978038i</b>
12	<b>0.22090951 - 0.20979131i</b>
Num	<b>0.22090988 - 0.20979143i</b>

- For Kerr black holes  $\rightarrow$  two quantization conditions (for angular and radial parts)

Nb	$M\omega_0$	$M\omega_1$
3	<b>0.1073438 - 0.1016159 i</b>	<b>0.089515 - 0.330273 i</b>
8	<b>0.1105221 - 0.1047959 i</b>	<b>0.086036 - 0.347811 i</b>
11	<b>0.1105330 - 0.1048013i</b>	<b>0.086216 - 0.347686 i</b>
Num	<b>0.1105331 - 0.1048015 i</b>	<b>0.086203 - 0.347664 i</b>

- Are small  $\Lambda$  and small  $\omega$  expansions the same?

$$\Lambda = 2i\omega.$$

- Poles in the  $\mathcal{F}^{inst}$ :

$$\mathcal{F}^{inst} = \frac{2i\omega^3\Lambda}{(4a^2 + 1)} + O(\Lambda^2)$$

$$\mathcal{F}_2^{inst} = \frac{(14\omega^2 + 5)\Lambda^2}{128(8\omega^2 + 3)} \sim \omega^2$$

$$\mathcal{F}_3^{inst} = -\frac{i(2\omega^2 + 1)\Lambda^3}{512\omega(8\omega^2 + 3)} \sim \omega^2.$$

- Infinitely many instanton corrections need to be resummed!

# The case of SAdS<sub>4</sub>

- Quantum SW periods are not enough!
- Reason: BC is not applied at the singular point
- The singularities are at  $z = 0, 1, t, \infty$
- BCs are

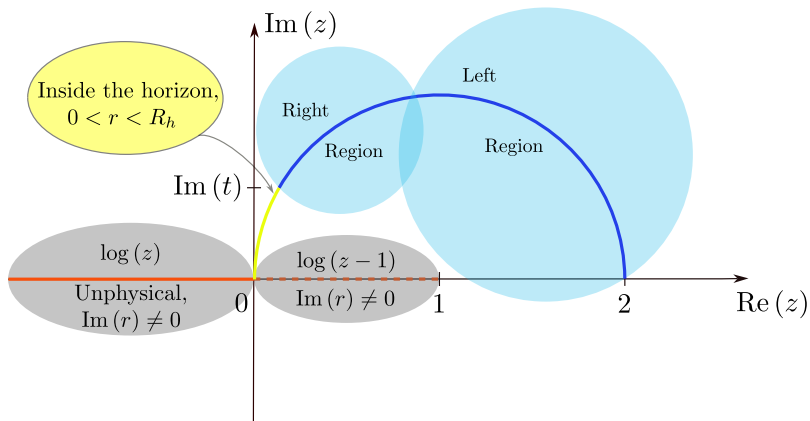
$$\psi(z_\infty) = 0, \quad \psi(z) \sim 1 \quad \text{for } z \sim t,$$

$$z_\infty = 1 + t + \sqrt{1 - t + t^2}.$$

- Two local variables:

$$z^L = z, \quad z^R = \frac{t}{z}.$$

# Left and Right regions around $z = 1$ and $z = t$



**Figure:** Branch cuts (red lines) on the complex  $z$  plane for anti-de Sitter black holes.



# The multi polylog approach

- $\kappa$  - expansion parameter
- Around  $z^L$ ,  $z^R$ :

$$\psi(z) = f_0(z) + \sum_{K \geq 1} f_K(z) \kappa^K.$$

- $f_0(z)$  and  $g_0(z)$  are leading order solutions
- $\kappa$  is such that  $f_0$ ,  $g_0$  are elementary functions
- Wronskian - rational function:

$$W_0 \equiv f_0(g_0)' - (f_0)'g_0.$$

# The multi polylog approach

- The generic solution:

$$f_K(z) = b_K g_0(z) + c_K f_0(z) - g_0(z) \int^z f_0(z') \frac{\eta_K(z')}{W_0(z')} dz' + f_0(z) \int^z g_0(z') \frac{\eta_K(z')}{W_0(z')} dz'.$$

- $c_K$ 's can be set to zero!
- $b_K$ 's are fixed by BCs and continuity of  $\psi(z)$

# Multiple polylogarithms for SAdS<sub>4</sub>

- Small parameter  $t \sim R_h$
- At order  $t^K \rightarrow$  multiple polylogarithms of weight  $\leq K!$
- First set of words in the BH dictionary:

$$\text{Li}_{s_1, \dots, s_n}(z) = \sum_{k_1 > k_2 > \dots > k_n \geq 1} \frac{z^{k_1}}{k_1^{s_1} \dots k_n^{s_n}}.$$

- $R_h$  expansion of the frequencies:

$$\omega = \sum_{k \geq 0} \omega_k R_h^k, \quad \omega_0 = \pm(2n + \ell + 2).$$

# Results for SAdS<sub>4</sub> QNMs

- $\omega_k \rightarrow$  Euler sums and MZVs:

$$\zeta(s_1, \dots, s_n; \epsilon) = \sum_{k_1 > k_2 > \dots > k_n \geq 1} \frac{\epsilon^{k_1}}{k_1^{s_1} \dots k_n^{s_n}}.$$

- QNMs with  $n = 0$  and  $\ell \geq 1$ :

$$\omega(0, \ell, s) = \ell + 2 - \frac{2^{2\ell+2}}{\pi} \frac{2\ell + s^2}{\ell(\ell + 1)} \frac{((\ell + 1)!)^2}{(2\ell + 2)!} R_h + \mathcal{O}(R_h^2).$$

- Software and full results are available at

[https://github.com/GlebAminov/BH\\_PolyLog](https://github.com/GlebAminov/BH_PolyLog).

# Scalar sector of gravitational perturbations in SAdS<sub>4</sub>

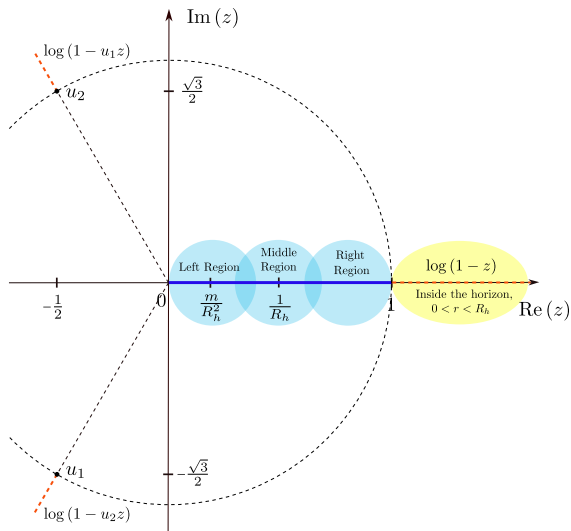
- Small parameter  $\alpha = 1/R_h$
- Regular singularities at  $z \sim 0, 1, u_1, u_2, \infty$
- Robin BC at spatial infinity:

$$\psi(z) \sim 1 \text{ for } z \sim 1,$$
$$\left\{ \frac{d}{dz} \left( \frac{\psi(z)}{z} \right) + \left[ \frac{3(1 + R_h^2)}{m} + \frac{i\omega}{R_h} \right] \frac{\psi(z)}{z} \right\} \Big|_{z=0} = 0.$$

- Apply the multi polylog method in 3 regions:

$$x = \frac{R_h^3}{mr} + \frac{1}{3}, \quad y = \frac{R_h^2}{r}, \quad z = \frac{R_h}{r}.$$

# Left, Middle, and Right regions for scalar sector



# Low-lying quasinormal frequencies

- AdS/CFT  $\rightarrow$  hydrodynamic modes of 3d CFT
- Frequency expansion:

$$\omega = \sum_{k \geq 0} \omega_k \alpha^k.$$

- Second set of words in the BH dictionary:

$$\text{Li}_{s_1, \dots, s_k}(z_1, \dots, z_k) = \sum_{n_1 > n_2 > \dots > n_k \geq 1}^{\infty} \frac{z_1^{n_1} \dots z_k^{n_k}}{n_1^{s_1} \dots n_k^{s_k}},$$

- where  $s_1 = s_2 = \dots = s_k = 1$ .

- QNM frequencies of the M2-brane:

$$\mathfrak{w} = \frac{2\omega}{3R_h}, \quad R_h \rightarrow \infty, \quad \ell \rightarrow \infty, \quad \frac{2\ell}{3R_h} \rightarrow \mathfrak{q}.$$

- Upon taking the limit:

$$\mathfrak{w} = \sum_{k \geq 1} \mathfrak{w}_k \mathfrak{q}^k.$$

- $\mathfrak{w}_{1,2,3}$  agree with the known results
- Expressions for  $\mathfrak{w}_4 - \mathfrak{w}_7$  are new!



# Results for $\mathfrak{w}_k$

- Hydrodynamic frequencies  $\rightarrow$  colored multiple zeta values
- For  $\mathfrak{w}_4$ :

$$\begin{aligned}\mathfrak{w}_4 = & -\frac{\sqrt{3}}{16} [\text{Li}_{1,1}(u_1, u_1) + u_1 \text{Li}_{1,1}(u_2, u_1) - u_2 \text{Li}_{1,1}(u_1, u_2)] \\ & + \frac{72 i \sqrt{3} + 24 i \pi + \pi^2}{384 \sqrt{3}} - \frac{12 i \sqrt{3} + i \pi}{64 \sqrt{3}} \log(3) \\ & + \frac{\sqrt{3}}{128} (u_2 - 3 u_1) \log(3)^2.\end{aligned}$$

- Full results are available at

[https://github.com/GlebAminov/BH\\_PolyLog](https://github.com/GlebAminov/BH_PolyLog).

# Advantages compared to numerical methods

- Speed and Accuracy
- Coefficients can be computed with the desired precision fast

$$w_1 = \frac{1}{\sqrt{2}},$$

$$w_2 = -\frac{i}{4},$$

$$w_3 = 0.155473446153645\dots,$$

$$w_4 = 0.067690388847266\dots \cdot i,$$

$$w_5 = -0.010733416957692\dots,$$

$$w_6 = 0.013959543659902\dots \cdot i,$$

$$w_7 = -0.016615814626711\dots$$

# Ordinary Schwarzschild black holes

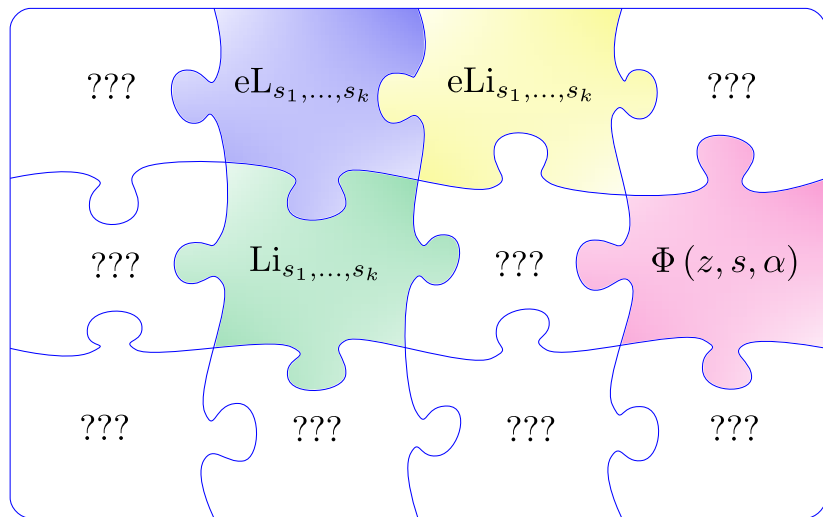
- Small parameter  $\omega$
- Regular singularities at  $z \sim 0, 1$
- Irregular singularity at  $z \sim \infty$
- Near-horizon region:

$$\text{Li}_{s_1, \dots, s_n}(z)$$

- Near-spatial infinity region:

$$eL_{s_1, \dots, s_n}(z) = \sum_{k_1 > k_2 > \dots > k_n \geq 1} \frac{1}{k_1^{s_1} \dots k_n^{s_n}} \frac{z^{k_1}}{k_1!}.$$

There is more to explore!



**Thank you!**