Noncommutative resolutions of Coulomb branches

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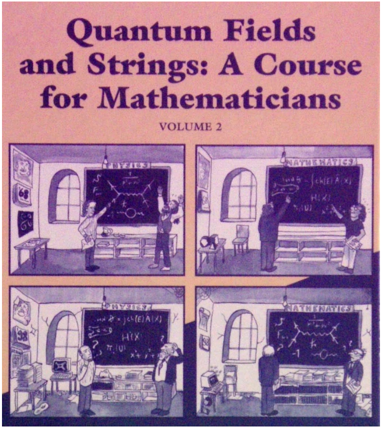
This talk is an attempt to straighten-out and explain some of what I've learned in thinking about 3d $\mathcal{N} = 4$ theories over the past few years. I hope it will not end up like the image below:



Why 3d $\mathcal{N} = 4$?

For many years, I thought I was interested in "symplectic singularities" (spoiler: I still am), but then I found out that all the examples I like come from 3d $\mathcal{N} = 4$ gauge theories.

Every 3d $\mathcal{N} = 4$ SUSY field theory gives us two spaces: the Higgs branch $\mathfrak{M}_H$ and the Coulomb branch $\mathfrak{M}_C$.

I'll be focusing on the latter, though that's an artificial distinction, since there should be a duality switching these.

The main case where we understand are gauge theories associated to a compact gauge group G and matter representation T^*N.

- The Higgs branch is a hyperhamiltonian reduction $T^*N /\!/\!/\!/ G$.
- The chiral ring of (i.e. functions on) the Coulomb branch is the homology of the moduli space of G-principal bundles with an N-section on the raviolo (two formal disks glued away from the origin). We call this the "BFN space." That is:

$$\mathbb{C}[\mathfrak{M}_C] \cong H_*^{BM}\left(\frac{\mathsf{N}}{\mathsf{G}} \times_{\frac{\mathcal{N}}{\mathcal{G}}} \frac{\mathsf{N}}{\mathsf{G}}\right)$$

Taylor series $\mathsf{C} = \mathbb{C}[[t]]$	$\mathsf{G} = G[[t]]$	$\mathsf{N} = N[[t]]$
Laurent series $\mathcal{C} = \mathbb{C}((t))$	$\mathcal{G} = G((t))$	$\mathcal{N} = N((t))$

Both of these are singular affine varieties, with a $\mathbb{C}^*$-action (i.e. a grading on the chiral ring) and carry a Poisson structure of degree -2 given by a secondary product. In most interesting cases, these are symplectic singularities.

We can attempt to smooth these out using parameters of the field theory:

- The FI parameters are valued in $\mathfrak{z}^*(\mathfrak{g}) \otimes \mathbb{R}^3$. We can think of these as adjusting the moment map parameters in the construction of the Higgs branch, so two of the parameters adjust complex structure, and the third deforms the Kähler parameter (and thus gives a partial resolution).
- The mass parameters are valued in $\mathfrak{t}_F \otimes \mathbb{R}^3$, the Cartan of the flavor group $F = N_{U(N)}(G)/G$.

The torus T_F acts on the BFN space, and considering equivariant homology of this action gives the desired deformation of $\mathfrak{M}_C$ over $\mathfrak{t}_F \otimes \mathbb{C}$.

To deform the Kähler parameter, we consider the Coulomb branch for the theory with gauge group $T_F G$, and consider symplectic GIT quotient for $\check{T}_F$.

The structure of the supersymmetry algebra implies the existence of topological twists which we call Q_A and Q_B.

The resulting cohomological field theories are topological (up to homotopy), and so we obtain rings of local operators which are the chiral rings $\mathbb{C}[\mathfrak{M}_C]$ and $\mathbb{C}[\mathfrak{M}_H]$.

An important context for understanding these is the category of line operators $\mathscr{L}_{A/B}$ compatible with these twists.

You can understand the computation of the Coulomb and Higgs branches in terms of junctions of the trivial line operator with itself.

Thus, in both twists, the categories of line operators are big categories that contain the trivial line as just one object. These are factorization categories (close to monoidal): Hilburn - Yoo

- In the A-twist, $\mathscr{L}_A$ is the category of D-modules on the loop space $\mathcal{N}/\mathcal{G} = \mathrm{Map}(\mathrm{Spec}\,\mathcal{C}, N/G)$ (i.e. coherent sheaves on the deRham stack $\mathrm{Map}(\mathrm{Spec}\,\mathcal{C}, N/G)_{\mathrm{dR}}$).
- In the B-twist, $\mathscr{L}_B$ should be (ind-)coherent sheaves on the de Rham loop space $\mathrm{Map}((\mathrm{Spec}\,\mathcal{C})_{\mathrm{dR}}, N/G)$

Important constructions of objects:

- In the A-twist, we can fix a subspace $U \subset \mathcal{N}$ invariant under a subgroup $\mathcal{G}_0 \subset \mathcal{G}$, and we can push forward the functions on $U/\mathcal{G}_0$ to $\mathcal{N}/\mathcal{G}$. These are called **vortex lines**. trivial $N[[t]]$ / $G[[t]]$
- In the B-twist, we have **Wilson lines**, that is, the vector bundles associated to G-representations. $T^*N///G \leftarrow V$

Each invertible object $L \in \mathscr{L}_{A/B}$ gives a (partial) resolution $\mathfrak{M}^L_{C/H}$ of the Higgs/Coulomb branch (making the sheaf corresponding that object into an ample line bundle $\mathcal{L}$).

- Invertible vortex lines $\leftrightarrow$ cocharacters $S^1 \to T_F$
- Invertible Wilson lines $\leftrightarrow$ characters $G \to S^1$

Theorem

The derived category of coherent sheaves on $\mathfrak{M}^L_{C/H}$ is the quotient of $\mathscr{L}_{A/B}$ by the subcategory of objects $\mathcal{M}$ such that $\operatorname{Ext}^\bullet(L^{-k}, \mathcal{M}) = 0$ for $k \gg 0$.

We can also think about this as a subcategory $\mathcal{C}_L$: any coherent sheaf on $\mathfrak{M}^L_{C/H}$ can be resolved by $\mathcal{L}^{-k}$ for $k \gg 0$, and we can construct a line operator by replacing these with L^{-k}.

Definition

A **window** for L is a finite set of line operators $X_1, \dots, X_m \in \mathscr{L}_{A/B}$ such that

$$\mathcal{C}_L \cong \mathrm{Ext}^\bullet(\bigoplus X_i)\text{-dgmod} \cong \mathscr{L}_{A/B}/\langle \bigoplus X_i \rangle^\perp$$

If $\{X_i\}$ is a window for L and L' simultaneously, then we obtain an induced equivalence

$$D^b(\mathrm{Coh}(\mathfrak{M}^L_{C/H})) \cong \mathrm{Ext}^\bullet(\bigoplus X_i)\text{-dgmod} \cong D^b(\mathrm{Coh}(\mathfrak{M}^{L'}_{C/H}))$$

Windows have mainly been studied in the case of the B-twist, looking at sets of Wilson lines.

For $G = \mathbb{C}^*$ and $N = \mathbb{C}^n$, $\mathcal{M}_H$ is the minimal nilpotent orbit in $M_{n\times n}(\mathbb{C})$. For any non-trivial Wilson line $L \in \mathbb{Z} \setminus \{0\}$, we have $\mathcal{M}_H^L = T^*\mathbb{CP}^{n-1}$.

Theorem (Beilinson)

For any $k \in \mathbb{Z}$, the Wilson lines with weights $k, \dots, k+n-1$ are a window.

If $L > 0$, these are sent to $\mathcal{O}(k), \dots, \mathcal{O}(k+n-1)$ on $T^*\mathbb{CP}^{n-1}$. If $L < 0$, then to $\mathcal{O}(-k), \dots, \mathcal{O}(-k-n+1)$.

In the B-twist, work of Halpern-Leistner describes (a bit implicitly) windows for all L corresponding to each invertible Wilson line.

A running theme (at least for me) is that the B-twist seems more familiar, and so it's tempting to think you should work there, but actually the A-twist is at least as good, often better, once you understand it.

So, I'm going to describe for you a large collection of windows for the type A twist given by vortex line operators.

$N((t))/G((t))$ loop rotation

On the quotient $\mathcal{N}/\mathcal{G}$, we have a natural action of the torus $\mathbb{C}^* \times T_F$.

Let $\mu_n \subset \mathbb{C}^*$ be the group of nth roots of unity, and let $\varphi\colon \mathbb{C}^* \to \mathbb{C}^* \times T_F$ be a homomorphism lifting the inclusion:

$$\varphi(t) = (t, \varphi_0(t)) \qquad t \in \mu_n.$$

We can consider the fixed points of the induced action of μ_n on $\mathcal{N}$; this carries an action of the fixed points of μ_n in its action on loop group $\mathcal{G}$.

$$\mathcal{N}^{\mu_n} = t^{-\varphi} N((t^n)) \qquad \mathcal{G}^{\mu_n} = t^{-\varphi} G((t^n)) t^{\varphi}$$

Note, these fixed points only depend on $\phi|_{\mu_n}$, but some of our later constructions depend on ϕ itself.

Consider fixed points of μ_n on $\mathcal{G}/\mathsf{G}$. = Gr

- This is a finite union of orbits of $G((t^n))$, in bijection with orbits of $W \ltimes n\mathfrak{t}_{\mathbb{Z}}$ on $\mathfrak{t}_{\mathbb{Z}}$. ← GL_n orbits unordered residues
- The orbit through $t^\lambda \mathsf{G}$ is isomorphic to $G((t^n))/\mathcal{P}_\lambda$ for the parahoric mod n

$$\mathcal{P}_\lambda = G((t^n)) \cap t^{\varphi+\lambda} G[[t]] t^{-\varphi-\lambda}$$

If the orbit in $\mathfrak{t}_{\mathbb{Z}}$ is free (i.e. generically) then this parahoric is an Iwahori. = all residues different.

The components of $(\mathcal{G} \times^{\mathsf{G}} \mathsf{N})^{\mu_n}$ are of the form

$$G((t^n)) \times^{\mathcal{P}_\lambda} (N((t^n)) \cap t^{\varphi+\lambda} N[[t]])$$

Thus, we have an induced map $X_\phi = (\mathcal{G} \times^{\mathsf{G}} \mathsf{N})^{\mu_n} \to N((t^n))$. Let S_ϕ be the pushforward of the functions by this map (defining a sum of vortex line operators), and let $\mathsf{R} = H_*^{BM, G((t^n))}(X_\phi \times_{N((t^n))} X_\phi)$.

Theorem

If $\mathfrak{M}_C$ is smooth at a generic mass parameter, then S_ϕ gives a window for $\mathfrak{M}_C^L$ for any invertible vortex line L. That is,

$$D^b(\mathrm{Coh}(\mathfrak{M}_C^L)) \cong D^b(\mathsf{R}\text{-mod}),$$

and R is a non-commutative crepant resolution of $\mathfrak{M}_C^L$.

I like think of these morally as the resolutions corresponding to purely imaginary complexified Kähler parameter.

Question to the audience:

What is the physical meaning of this construction?

My motivation was to understand Kaledin's construction of a tilting generator (which is the coherent sheaf corresponding to the line operator S_ϕ), which passes through quantization in characteristic p.

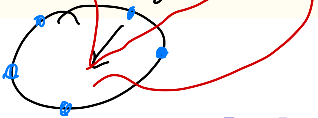
Have no fear! The characteristic p part is only relevant because equivariant cohomology over $\mathbb{F}_p$ localizes to the fixed points of $\mu_p \subset \mathbb{C}^*$, so this was a way of accessing these fixed points without actually knowing about the Coulomb branch.

One primary important application of these windows is a large number of equivalences

$$D^b(\mathrm{Coh}(\mathfrak{M}_C^L)) \cong \mathrm{Ext}^\bullet(S_\phi)\text{-dgmod} \cong D^b(\mathrm{Coh}(\mathfrak{M}_C^{L'}))$$

These are different for different ϕ! In fact, these generate an action of $\pi_1(T_F^\circ)$, the fundamental group of the generic points in flavor torus.

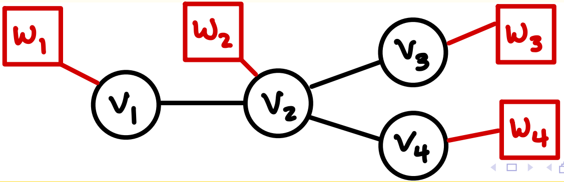
These exactly correspond to complexified Kähler parameters for $\mathfrak{M}_C$, and you can think of the induced equivalences as coming from a path varying these parameters.



We'll specialize to the case of a quiver gauge theory for a quiver Γ:

$$G = \prod_{i \in \Gamma} GL(v_i) \qquad N = \bigoplus_{i \to j} \mathrm{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{v_j}) \oplus \bigoplus_{i \in \Gamma} \mathrm{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{w_i})$$

We can only smooth $\mathfrak{M}_C$ using mass parameters coming from framing (square) nodes if Γ is type ADE and w_i is only non-zero on minuscule nodes, so assume this for now. (Affine type A can be smoothed using mass parameters coming from the cycle.)



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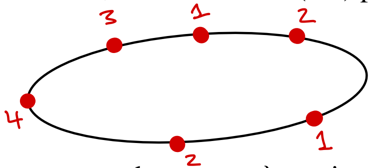
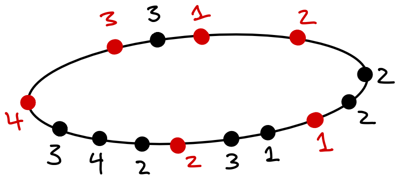
In this case, we can interpret $\mathcal{N}/\mathcal{G}$ and N/G as moduli spaces of quiver representations over $\mathcal{C}$ and C respectively, and the BFN space as QR over $\mathcal{C}$ with a choice of two invariant lattices.

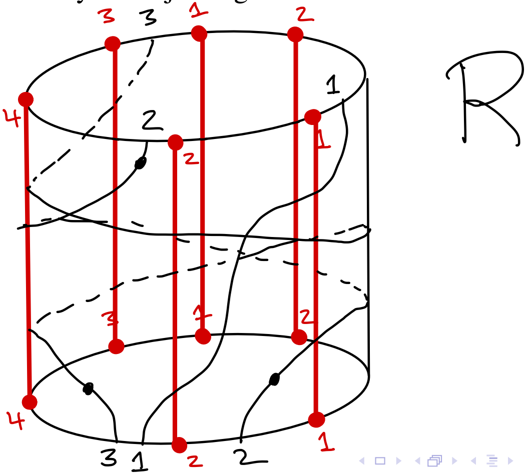
Cocharacter φ lands in $F = \prod GL(\mathbb{C}^{w_i})$, so corresponds to weights $c_{i,1}, \dots, c_{i,w_i}$ for our cocharacter. To consider mod n, embed $\mathbb{Z}/n\mathbb{Z}$ as the n-torsion points of S^1, and think of as (red) points on the circle.



Each component corresponds to some λ, again considered mod n, and up to action of W: unordered v_i-tuples $x_{i,1}, \dots, x_{i,v_i}$, also drawn on circle.

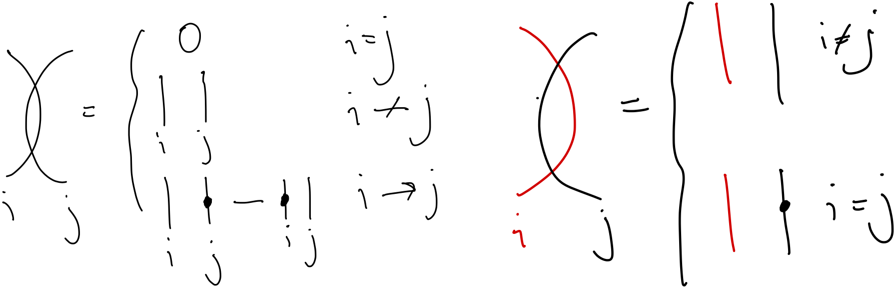


Homology classes in $X_\phi \times_{N((t^n))} X_\phi$ can be represented by "KLRW" diagrams" drawn on a cylinder joining two such decorated circles.

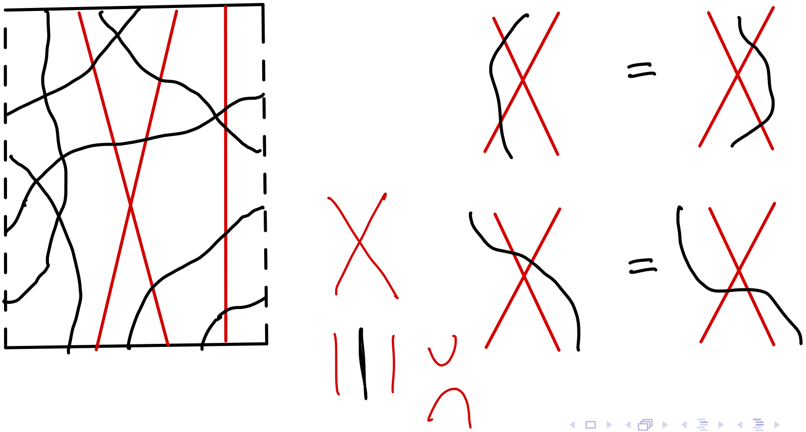


Theorem

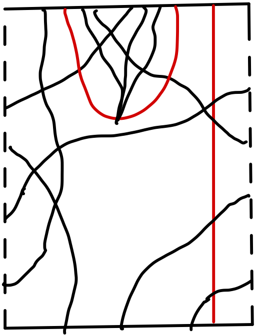
The convolution algebra R *has a presentation by cylindrical KLRW diagrams with the same local relations.*

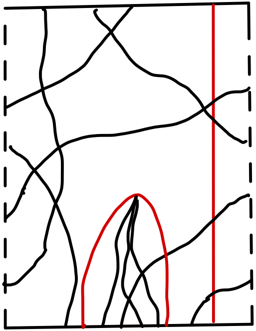


Wall-crossing functors correspond to same type of diagrams, but we let red strands cross and wrap around the cylinder.



Wall-crossing functors correspond to same type of diagrams, but we let red strands cross and wrap around the cylinder. Can extend to knot homology by adding in cups and caps.

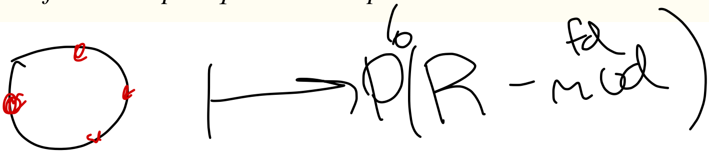




Theorem

These braid and cup/cap functors define a functor

- *from the category of oriented affine ribbon tangles, labeled with minuscule fundamentals,*
- *to the category of dg-categories with morphisms given by functors up to quasi-isomorphism.*



Making a labeled ribbon link annular in the boring way, this gives a link homology $\mathscr{D}_{coh}(K)$.

Theorem

The following link homologies are all the same:

- $\mathscr{D}_{coh}(K)$*, constructed from the affine tangle action above.*
- *the invariant constructed in my older knot homology work (which matches Khovanov-Rozansky in type A).* $\leftarrow \mathbb{C}$
- *Aganagić's physical construction.*

Of course, this gives an *annular* knot invariant as well.

Conjecture

In type A, this agrees with annular Khovanov-Rozansky homology (as defined by Queffelec and Rose).

The categories of R-mod for all possible labelings by fundamentals should carry an action of annular foams (by the web bimodules defined by Mackaay-W.)

This reduces to the check that a single unknot looped around the cylinder has the right value. I can do this calculation in $\mathfrak{sl}_2$, and am one ugly complex away from doing so in $\mathfrak{sl}_n$.

Thanks for listening.