A proposal for nonabelian mirror symmetry

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Berkeley informal string math meeting

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This talk is concerned with non-abelian mirrors. For abelian theories, many things are known. For example:

- Abelian GLSMs are well understood,
- there exists the Batyrev-Borisov construction of mirrors to complete intersections in projective spaces,
- there exists Hori-Vafa construction of mirrors.

For non-abelian theories, much work remains:

- Non-abelian GLSMs are still under active development,
- No known nonabelian mirror construction in physics until '18.

This talk is concerned with the last point.

In this talk, we propose an extension of the Hori-Vafa mirror construction of mirrors of 2d abelian gauge theories, to non-abelian GLSMs

- We will quickly review GLSMs and the Hori-Vafa mirror construction for abelian GLSMs
- We will propose mirrors for nonabelian GLSMs.
- We compute numerous examples to check the ansatz.

Mathematical and physical language for mirror symmetry

- For mirror Calabi-Yau n-folds, $H^{p,q}(X) = H^{n-p,q}(\check{X})$. For 3-folds, we have $\chi(X) = -\chi(\check{X})$.

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- Two Calabi-Yau manifolds are said to be mirror, if the SCFTs are isomorphic, related ultimately by flipping a left $U(1)_R$ sign convention.
- On the worldsheet, mirror symmetry exchanges chiral multiplets and twisted chiral multiplets.

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- **Adjoint fields**: $\mathcal{V}$ is the adjoint representation of G called the gauge field, the field strength is the twisted chiral superfield denoted as Σ.
- **Superpotential**: a holomorphic, G-invariant polynomial $W : V \to \mathbb{C}$, namely $W \in \mathrm{Sym}(V^*)^G$.

(2,2) GLSMs

- **Fayet-Iliopoulos (FI) parameters and theta angles**: a set of FI-parameters r^a and periodic theta angles $\theta^a \in \mathbb{R}/2\pi\mathbb{Z}$ where the index a runs over the number of $U(1)$ sector in G. One can combine them to define $q^a = \exp(-t^a) \in \mathbb{C}^*$, where $t^a = r^a - i\theta^a$.
- **R-symmetry**: a vector $U(1)_V$ and axial $U(1)_A$ R-symmetries that commute with the action of G on V. To (classically) preserve the $U(1)_V$ symmetry the superpotential must have weight 2 under it in our convention:

$$W(\lambda^q \phi) = \lambda^2 W(\phi)$$

where ϕ denotes the coordinates in V.

For abelian group G, many results are known. While nonabelian gauged linear sigma models are still under active development for example, WG w/ Sharpe and Zou '20

(2,2) GLSMs

- The classical potential energy of a GLSM for a degree d hypersurface in $\mathbb{P}^n$ is

$$U = \sum_i \mid \sigma \mid^2 \mid \phi_i \mid^2 + d^2 \mid \sigma \mid^2 \mid p \mid^2 + \frac{e^2}{2} \mid \sum_i \mid \phi_i \mid^2 - d \mid P \mid^2 - t \mid^2 + \mid G(\phi) \mid^2 + \mid p \partial_i G \mid^2 .$$

- GLSMs can RG flow to NLSMs on spaces such as CP^N and quintic. The Kähler parameter is renormalized under RG-flow: r$=\bar{r} + \sum Q_i \log \frac{\mu}{\Lambda}$.
- The twisted chiral rings of these target space can be represented by the gauge invariant functions of σ_s (Witten '93). For example, for projective space $\mathbb{CP}^n$, we only have one σ, which corresponds to the generator of $H^{1,1}$ of the projective space, similar for $\sigma^2 \sim H^{2,2}$

(2,2) GLSMs

- For Fano spaces with a trivial Landau Ginzburg model at the LG point ($r << 0$), quantum ring relations of the target can be obtained from twisted effective superpotential, which are
- $$\widetilde{W}_{eff} = \sum_a \Sigma_a \left[-t_a - \sum_i Q_i^a \left(\log \left(\sum^a Q_i^a \Sigma_a \right) - 1 \right) \right].$$
$$\frac{\partial \widetilde{W}_{eff}}{\partial \sigma_a} = 0.$$
- Notice the quantum potential energy $U \sim \sum_a \mid \frac{\partial \widetilde{W}_{eff}}{\partial \sigma_a} \mid^2$.

Review of (2,2) abelian mirrors

For (2,2) abelian GLSMs, the mirrors are Landau-Ginzburg models with fields (Hori, Vafa '00):

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and superpotential

$$W = \sum_a \left(\sum_i Q_i^a Y_i - t^a \right) \Sigma_a + \sum_i e^{-Y_i}.$$

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$$W = \sum_a \left(\sum_i Q_i^a Y_i - t^a \right) \Sigma_a + \sum_i e^{-Y_i},$$

taking the Y_i derivative gives the map between observables

$$Q_i^a \Sigma_a = \exp\left(-Y_i\right).$$

We will use this map later on for concrete examples.

A quick example: $\mathbb{CP}^4$

GLSM: U(1) gauge theory, five chiral superfields of gauge charge 1. The A-model twisted superpotential

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The effective GLSM twisted superpotential is defined from quantum correction of matter fields, there are five vacua. The twisted chiral ring relations

$$\sigma^5 = q.$$

One can also compute the correlation functions are

$$\langle\sigma^{5k+4}\rangle = q^k, \qquad for \quad k \geq 0.$$

The mirror Landau Ginzburg model for $\mathbb{CP}^4$

The mirror LG model superpotential is

$$W = \Sigma \left(\sum_i Y_i - t \right) + \sum_{i=1}^{5} e^{-Y_i}.$$

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$$W_{eff} = \sum_{i=1}^{4} e^{-Y_i} + q \prod_{i=1}^{4} e^{+Y_i}.$$

The LG-model superpotential is defined classically.

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$$\langle X^{5k+4} \rangle = q^k, \qquad \textit{for} \quad k \geq 0.$$

Recall the map for observables is $\sigma \Leftrightarrow X = e^{-Y}$. All match the GLSM result! One can also study mirrors to hypersurfaces in projective spaces similarly (details can be found elsewhere).

Non-abelian mirror proposal

In Gu, Sharpe '18, we propose that the mirror of a non-abelian GLSM with connected gauge group **G** is defined by a Landau-Ginzburg orbifold, which is a Weyl group orbifold of Y_i, X_μ and Σ_a fields with superpotential

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where ρ_i^a are weights of matter representation, α_μ^a are roots of gauge group **G**, and the index a$\in \{1, \ldots, r\}$, and r is the rank of Cartan torus of the non-abelian group.

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- Y_i mirror to matter fields.

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- Σ_a mirror to each $U(1)$ gauge field strength multiplets in Cartan torus which we denote the same notation.
- Y_i mirror to matter fields.
- X_μ mirror to W-bosons ($\sim$ roots of the Lie algebra) .

Weyl group orbifold

The Weyl orbifold maps weights to weights

$$Y_i \mapsto Y_j \qquad \sum_a \Sigma_a \rho_i^a \mapsto \sum_a \Sigma_a \rho_i^a .$$

and roots to roots

$$X_\mu \mapsto X_\nu \qquad \sum_a \Sigma_a \alpha_\mu^a \mapsto \sum_a \Sigma_a \alpha_\nu^a .$$

One can check that the superpotential is invariant under the Weyl group orbifold transformation.

Idea of the proposal

At a generic point on the Coulomb branch, the nonabelian theory becomes an abelian theory with W bosons (lowest component of chiral superfields with vector R-charge 2), it is an effective theory. Our intuition is that the nonabelian proposal is a result of applying abelian (Hori-Vafa) duality at such a point, to both matter fields as well as W bosons, which is why our proposal looks so closely related to abelian duality.

So the logic is that we start from a UV nonabelian GLSM, but we take T-dual of an effective theory that looks pretty like an abelian theory. The effective theory is conjectured to follow to the same NLSM as the original nonabelian GLSM.

If we know the details of the Kahler potential under the RG-flow, one can argue that we have a physical proof of nonabelian mirrors. However, it is still an interesting question to ask whether we can have a framework for nonabelian T-dual, and nonabelian mirrors are expected to be UV fundamental theories. These fundamental theories follow to the same low energy theory as our nonabelin mirrors.

Associated Cartan

- **Gauge group**: gauge group $T = U(1)^{\mathrm{rank}(G)} \rtimes S$, $U(1)^{\mathrm{rank}(G)}$ is the maximal torus of the gauge group G and S is the Weyl group of G.
- **Chiral matter fields**: $\Phi_{i=1,\cdots,N}$ are charged by weights ρ_i^a under the gauge group T, the field space Φ is a Weyl-orbifold free subset of $\mathbb{C}^{N\cdot\mathrm{rank}(G)}$ denoted as V^o. Additional Weyl orbifold free $\dim(\mathfrak{g}) - \mathrm{rank}(\mathfrak{g})$ vector R-charge 2 with gauge charges given by the roots α_μ^a of G.
- **Adjoint fields**: $\mathcal{V}$ is the adjoint representation of $U(1)^{\mathrm{rank}(G)} \rtimes S$, the field strength is the twisted chiral superfield also denoted as Σ.
- with other data

Consistency checks

This proposal satisfies a number of consistency checks, including (but not limited to):

- Matching Witten index
- Matching quantum cohomology rings,
- Matching (topological) correlation functions,

We will see this explicitly in various examples in the rest of this talk.

An example

Let us consider a (2,2) supersymmetric pure $SU(2)$ group as an example. Our general ansatz for the mirror superpotential is

$$W = \sum_a \Sigma_a \left(\sum_i \rho_i^a Y_i - \sum_\mu \alpha_\mu^a \log X_\mu - t^a \right) + \sum_\mu X_\mu + \sum_i \exp(-Y_i).$$

For pure $SU(2)$, the mirror has fields Σ, X_1, X_2, and the superpotential reduces to

$$W = 2\Sigma \left(\log X_1 - \log X_2 \right) + X_1 + X_2.$$

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The Weyl group, $\mathbf{Z}_2$, acts on the fields as

$$\Sigma \to -\Sigma, \qquad X_1 \Leftrightarrow X_2,$$

and the superpotential is invariant under this Weyl group action.

Example: Grassmannian $G(k, n)$

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$$W = \sum_a \Sigma_a \left(\sum_{ib} \rho^a_{ib} Y_{ib} + \sum_{\mu\nu} \alpha^a_{\mu\nu} Z_{\mu\nu} - t \right) + \sum_{ia} \exp(-Y_{ia}) + \sum_{\mu\neq\nu} X_{\mu\nu},$$

where $\rho^a_{ib} = \delta^a_b$, $\alpha^a_{\mu\nu} = -\delta^a_\mu + \delta^a_\nu$.

Example: Grassmannian $G(k, n)$

Integrating out the Σ_a, we get constraints

$$\sum_i Y_{ia} - \sum_{\nu \neq a} (Z_{a\nu} - Z_{\nu a}) - t = 0,$$

which we use to eliminate Y_{na}:

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Define

$$\Pi_a = \exp(-Y_{na}) = q \left(\prod_{i=1}^{n-1} \exp(+Y_{ia}) \right) \left(\prod_{\mu \neq a} \frac{X^{a\mu}}{X_{\mu a}} \right),$$

for $q = \exp(-t)$, then the superpotential for the remaining fields, after applying the constraint, reduces to

$$W = \sum_{i=1}^{n-1} \sum_{a=1}^{k} \exp(-Y_{ia}) + \sum_{\mu \neq \nu} X_{\mu\nu} + \sum_{a=1}^{k} \Pi_a.$$

Critical locus for mirror to Grassmannian

The critical locus which corresponding B-model vacuum can be gotten by calculating the following vacuum equations.

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$$\Pi_a = q\left(\frac{1}{\Pi_a}\right)^{n-1}\left(\prod_{\mu\neq a}\frac{-\Pi_a+\Pi_\mu}{-\Pi_\mu+\Pi_a}\right) = q(-)^{k-1}\left(\Pi_a\right)^{1-n},$$

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hence

$$\left(\Pi_a\right)^n = (-)^{k-1}q.$$

The finiteness of the potential requires $X_{\mu\nu} \neq 0$. So it forces the $\Pi_a \neq \Pi_b$, when $a \neq b$.

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$$\frac{n(n-1)\cdots(n-k+1)}{k!}$$

equal to the Euler characteristic of the Grassmannian, which is the number of vacua obtained from the GLSM.

Correlation functions match

One can calculate B-model correlation functions for Grassmannians following (Vafa '90) and compare the results to the A-model results. Correlation functions are computed in terms of the Hessian H which is defined as the determinant of the matrix of second derivatives of the superpotential W.

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Then one can use the B-model correlation function formula

$$\langle O(X) \rangle = \sum_{vacuum} \frac{O(X)}{H} |_{vacuum},$$

to calculate concrete examples.

Correlation functions match

For $G(2,4)$, the first nonzero correlation functions are

$$\langle (\Pi_1)^2 (\Pi_2)^2 \rangle = \frac{2}{2!}, \qquad \langle (\Pi_1) (\Pi_2)^3 \rangle = -\frac{1}{2!} = \langle (\Pi_2) (\Pi_1)^3 \rangle.$$

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These match the classical correlation functions of the A-model. Using the chiral ring relations, we can derive results for more correlation functions, which also match A-model's results. One can also study more complicated cases following the same procedure as for Grassmannians.

Correlation functions match

One can prove the correlation functions match between A-model gauge theory and its corresponding B-model LG in general. Several different ways can be found in Gu, Sharpe '17 and '18.

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The Weyl orbifold satisfy this property: each Weyl reflection interchanges Σs with Σs, Ys with Ys and Xs and Xs, so as a result, we have

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so the holomorphic top-form changes by at most a sign. Thus the B-twist is consistent.

Example: pure gauge $SO(3)$

One can discuss the mirror for more general connected gauge groups. Recall that the mirror to the pure $SU(2)$ gauge theory was defined by the superpotential

$$W = 2\Sigma \left(\log X_1 - \log X_2 \right) + X_1 + X_2,$$

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$$X_1 \Leftrightarrow X_2, \qquad \Sigma \rightarrow -\Sigma.$$

Example: pure gauge $SO(3)$

One can find the vacua for the mirror of pure $SO(3)$ group, given by

$$\frac{\partial W}{\partial \Sigma} = \frac{\partial W}{\partial X_1} = \frac{\partial W}{\partial X_2} = 0.$$

It turns out that only for discrete theta angle iπ are there SUSY vacua. The other case breaks SUSY.

One can find more general $SO(k)$ group cases in Gu and Sharpe '18.

Example: pure $Sp(2k)$

One can study other groups following the same ansatz. Recall the general mirror ansatz is

$$W = \sum_a \Sigma_a \left(\sum_i \rho_i^a Y_i - \sum_\mu \alpha_\mu^a \log X_\mu - t^a \right) + \sum_\mu X_\mu + \sum_i \exp(-Y_i).$$

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For pure $Sp(2k)$, the mirror superpotential is given by

$$\begin{aligned} W &= \sum_{a=1}^{k} \Sigma_a \left(\sum_{\mu \leq \nu} (\delta_{\mu,2a} - \delta_{\mu,2a-1} + \delta_{\nu,2a} - \delta_{\nu,2a-1}) Z_{\mu\nu} \right) \\ &+ \sum_\mu X_{\mu\mu} + \sum_{a<b} \left(X_{2a,2b} + X_{2a-1,2b-1} + X_{2a-1,2b} + X_{2a,2b-1} \right). \end{aligned}$$

Example: pure $Sp(2k)$

The critical locus is defined by

$$\frac{\partial W}{\partial X_{2a,2a}} = 0 : \qquad X_{2a,2a} = 2\sigma_a,$$

$$\frac{\partial W}{\partial X_{2a-1,2a-1}} = 0 : \qquad X_{2a-1,2a-1} = -2\sigma_a,$$

$$\frac{\partial W}{\partial X_{2a,2b}} = 0 : \qquad X_{2a,2b} = \sigma_a + \sigma_b \quad \mathit{for} \quad a < b,$$

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Example: pure $Sp(2k)$

In addition, $\partial W / \partial \sigma_a = 0$ implies

$$\left(\frac{\prod_{2a-1 \leq \nu} X_{2a-1,\nu}}{\prod_{2a \leq \nu} X_{2a,\nu}} \right) \left(\frac{\prod_{\mu \leq 2a-1} X_{\mu,2a-1}}{\prod_{\mu \leq 2a} X_{\mu,2a}} \right) = 1,$$

Along the critical locus, each of the ratios appearing in the product above is -1. Since there are manifestly an even number of them, this critical locus equation is trivially satisfied. This suggests the IR limit is a set of k free fields. As a consistent check, note this is consistent with earlier computations for the pure $SU(2) = Sp(2)$ theory.

Hypersurfaces in Grassmannian

Consider an GLSM for a hypersurface of degree d in $G(k,n)$. This is described by a $U(k)$ gauge theory with matter

• n chiral multiplets ϕ_{ia} in the fundamental representation, $i \in \{1, \cdots, n\}$, $a \in \{1, \cdots, k\}$,

• one field p of charge -d under det $U(k)$, and a superpotential

$$W = pG(B),$$

where G is a polynomial of degree d in the baryons,

$$B_{i_1 \cdots i_k} \equiv \epsilon^{a_1 \cdots a_k} \phi_{i_1 a_1} \cdots \phi_{i_k a_k}.$$

We take the chiral superfields ϕ_{ia} to have R-charge zero, and p to have R-charge two.

Hypersurfaces in Grassmannian

The mirror of this theory is an orbifold of the Landau-Ginzburg model with fields

- kn chiral superfields Y_{ia}, mirror to ϕ_{ia},
- one chiral superfiel $X_p = \exp(-Y_p)$, mirror to p,
- $X_{\mu\nu} = \exp(-Z_{\mu\nu})$, $\mu, \nu \in \{1, \cdots, k\}$,
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- Σ_a, $a \in \{1, \cdots, k\}$ and superpotential

$$\begin{aligned} W &= \sum_a \Sigma_a \left(\sum_{ib} \rho^a_{ib} Y_{ib} - dY_p - \sum_{\mu\neq\nu} \alpha^a_{\mu\nu} \log X_{\mu\nu} - t \right) \\ &\quad + \sum_{ia} \exp\left(-Y_{ia}\right) + X_p + \sum_{\mu\neq\nu} X_{\mu\nu}, \end{aligned} \tag{1}$$

where

$$\rho^a_{ib} = \delta^a_b, \qquad \alpha^a_{\mu\nu} = -\delta^a_\mu + \delta^a_\nu.$$

Hypersurfaces in Grassmannian

Integrating out Σ_as gives constraints

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Then the superpotential becomes

$$W = \sum_{i=1}^{n-1} \sum_{a} \exp(-Y_{ia}) + \sum_{a} \Pi_a + X_p + \sum_{\mu \neq \nu} X_{\mu\nu},$$

where $\Pi_a = \exp(-Y_{na})$.

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Dimension

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- Instead I will mention a hint for the mirror geometry.
- If the original CY has complex-dimension $k(n-k)-1$, the worldsheet theory flows to a SCFT which has the central charge
$$\frac{c}{3} = k(n-k) - 1$$
- The mirror model should flow to a SCFT with the the same central charge. Indeed
$$\begin{aligned} \frac{c}{3} &= \sum_i (1-q_i) \qquad (2) \\ &= (1)(1-2) + (k^2-k)(1-2) + k(n-1)(1-0) \\ &= k(n-k)-1 \end{aligned}$$
where we assigned the R-charge 2 to fields Π_a and $X_{\mu\nu}$, while we assign R-charge zero to Y fields.

Mirror geometry

Construction of mirror geometries to non-abelian Calabi-Yau's is left for future work.

$O(2)$ case

- It is a non-connected gauge group, the gauge theory has a Z_2 orbifoldand vacuum can intersect with the orbifold fixed points where the twisted sector should be take into account.
- Based on two different projections, we can define two different theories which we called $O_+(2)$ and $O_-(2)$ respectively.
- Consider the mirror to the $O_+(2)$ gauge theory with 3 doublets.
- The mirror Landau-Ginzburg orbifold has six fields Y_a^i as well as one Σ, with a superpotential

$$\begin{aligned} W &= \Sigma\left(-\sum_{i=1}^{3} Y_1^i + \sum_{i=1}^{3} Y_2^i\right) \\ &- \sum_{i=1}^{3} \widetilde{m}_i \left(Y_1^i + Y_2^i\right) + \sum_{i=1}^{3} \exp\left(-Y_1^i\right) + \sum_{i=1}^{3} \exp\left(-Y_2^i\right) \end{aligned}$$

- Z_2 orbifold acting as $\Sigma \mapsto -\Sigma$, $Y_1^i \leftrightarrow Y_2^i$

$O(2)$ case

- One can compute the vacuum equation

$$\prod_{i=1}^{3}(X-\widetilde{m}_i)=\prod_{i=1}^{3}(-X-\widetilde{m}_i)$$

  where

$$X=\frac{1}{2}\left(\exp(-Y_1^i)-\exp(-Y_2^i)\right)$$

- It is symmetric under $X\mapsto -X$, it has roots: 0 and $\pm X_0$. Because the Z_2 orbifold, we should identify the $\pm X_0$ as one single solution. The $X=0$ solution intersects Z_2 fixed point, so we have to include the twisted sector. So the vacuum number is 2+1=3.
- One can consider $SO(2)$ gauge group with three doublets and three singlets with a superpotential.
- This is a propotype in understanding the 2d Hori-Seiberg dual of gauge theories in the mirror, one can see w/ Hadi and Sharpe 1907.06647 for more details.

A brief summary of other relevant development in nonabelian mirrors

- 2005.10845 w/Sharpe and Zou, studied the 2d nonabelian pure gauge theory in mirrors and gave refined IR dynamics.
- 2001.10562 computed the massive orbifold Landau-Ginzburg model's correlation functions and checked the 2d Hori-Seiberg duality in the mirror following 1907.06647.
- 1908.06036 w/Guo and Sharpe studied the 2d (0,2) nonabelian mirrors of Fanos.

Conclusion

- Reviewed the Hori-Vafa construction of mirrors to abelian GLSMs.
- Proposal for mirrors to non-abelian GLSMs.
- Discussed several examples, showing matching correlation functions and twisted/chiral rings.

THANKS!