K3 Metrics

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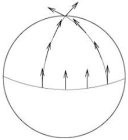
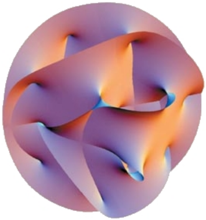
Introduction

- Calabi-Yau (CY) compactification has played a central role in string theory. Reduced holonomy $\Rightarrow$ low-energy SUSY
- Type II compactifications preserve 4d $\mathcal{N} = 2$ and are the setting of mirror symmetry
- Heterotic and orientifold compactifications preserve 4d $\mathcal{N} = 1$ and provide semi-realistic starting points for string phenomenology
- Setting in which much of our non-perturbative understanding of string theory has been developed



K3

- K3 has played a particularly important role
- $SU(2) = Sp(1)$, so in 4d Calabi-Yau = hyper-Kähler. Only compact examples are K3 and T^4
- A concrete way to think about K3 is as T^4/Z_2 orbifold.



Introduction (continued...)

- Since K3 is hyper-Kähler, preserves even more SUSY (e.g. K3$\times T^2$ has 4d $\mathcal{N}=4$)
- Heterotic (on T^4) - type IIA (on K3) duality plays an essential role in our understanding of how the various perturbative superstring theories are related. Can fiber this duality over a $\mathbb{P}^1$ base to find dual 4d $\mathcal{N}=2$ theories
- Earliest example of black hole microstate counting in string theory

Introduction (continued...)

- Remarkably, all of this was achieved without an explicit form of the metric! Indeed, no smooth (compact, non-toroidal) Ricci-flat Calabi-Yau metric is (was) known!
- Why might this matter to a string theorist? Supposedly, (tree-level) string vacuum from CFT, such as non-linear sigma model with action

$$\frac{i}{8\pi\alpha'}\int (g_{ij} - B_{ij})\partial x^i \bar{\partial} x^j \, d^2 z - 2\pi \int \Phi R^{(2)} \, d^2 z + \dots$$

(where the ... involve fermions). But, in reality since we don't have the metric, this formulation is rather useless.

K3 Non-Linear Sigma Models

- This question is particularly well-motivated for K3 (as opposed to other Calabi-Yaus) because the β function of the non-linear sigma model is exactly 0 – not just to leading order in α'
- As an example of our ignorance, even for K3 the worldsheet partition function is not known at almost all points in moduli space.

Explicit K3 metrics

Based on recent work (1810.10540, 2006.02435, 2009.xxxx) with

Shamit Kachru, Arnav Tripathy

Indeed, we have not one, but two constructions!

Overview of math results: Higgs branch construction

- Construction of K3 as a hyper-Kähler quotient of an infinite-dimensional affine space
- One construction for each T^4 orbifold limit in moduli space of Ricci-flat K3 metrics
- K3 arises as moduli space of singular equivariant instantons on the dual torus
- Differential-geometric analogue of derived McKay correspondence
- Explicit perturbative description of moduli space near orbifold limit
- Certain moduli spaces of singular Higgs bundles on $\mathbb{P}^1$ coincide with moduli spaces of singular equivariant Higgs bundles on T^2

Overview of math results: Coulomb branch construction

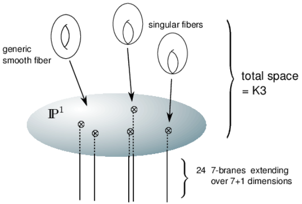
- Formula for K3 metrics in terms of certain K3 open string reduced Gromov-Witten invariants. SYZ for K3 at level of metric
- Instead, for now we'll compare these two formalisms in order to extract these invariants from K3 metrics
- Reformulations of this problem:
  - Tropical limit – at special loci, combinatorial flat surface problem
  - Generalized Donaldson-Thomas theory of an auxiliary non-compact Calabi-Yau threefold
  - For K3 surfaces which arise as generalized Kummer varieties, analogues of these problems

Little string theory

- Heterotic small instanton 5-branes have a decoupling limit
- From supergravity perspective, this works because the corresponding soliton is so singular. In particular, an infinite throat with diverging g_s develops.
- It is *not* a QFT – it has T-duality, for example, so there is no unique stress-energy tensor.

Geometrizing the moduli space, I: heterotic / F-theory duality

- Strong-weak duality (for $SO(32)$ heterotic theory, for concreteness) takes us to D5-brane in type I. Now, to study the moduli space of the theory on T^2, use T-duality twice to replace D5 by D3.
- Heterotic $(T^2) \leftrightarrow$ type IIB orientifold on $T^2/Z_2 \rightarrow$ F-theory on K3



Geometrizing the moduli space, II: heterotic / M-theory duality

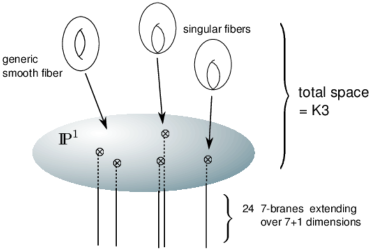
- Similarly, to study the theory on T^3, use T-duality three times to replace D5 by D2. An extra dimension is provided by the M-theory circle.
- Heterotic (T^3) $\leftrightarrow$ M-theory on K3

Parameters of LST

- Moduli of the heterotic string theory become parameters of the LST. Similarly, gauge symmetry in spacetime descends to global symmetry of LST.

Compactification of the 4d theory

- Study little string theory on T^2, further compactified on S^1_R
- $R \to \infty$ limit is large complex structure / semi-flat limit studied by [Greene-Shapere-Vafa-Yau '90] and familiar from F-theory on K3



Finite R

- Corrections away from this limit are determined by instantons in this theory.
- These instantons are obtained by taking the worldlines of 4d BPS particles and wrapping them around S^1_R
- Exponentially small away from singular fibers: $e^{-2\pi RM}$

$$\mathcal{X}_\gamma(\zeta) = \mathcal{X}_\gamma^{\text{sf}}(\zeta) \exp\left[-\frac{1}{4\pi i}\sum_{\gamma'\in\hat{\Gamma}'_a}\Omega(\gamma';a)\langle\gamma,\gamma'\rangle \times \int_{\ell_{\gamma'}(a)}\frac{d\zeta'}{\zeta'}\frac{\zeta'+\zeta}{\zeta'-\zeta}\log(1-\mathcal{X}_{\gamma'}(\zeta'))\right]$$

$$\mathcal{Y}_\gamma(\zeta) = \log\mathcal{X}_\gamma(\zeta)\,, \quad \mathcal{Y}_\gamma^{\text{sf}}(\zeta) = \frac{\pi R}{\zeta}Z_\gamma + i\theta_\gamma + \pi R\zeta\overline{Z_\gamma}$$

$$\varpi(\zeta) = \frac{1}{4\pi^2 R}d\mathcal{Y}_m(\zeta)\wedge d\mathcal{Y}_e(\zeta) = -\frac{i}{2\zeta}\omega_+ + \omega_K - \frac{i\zeta}{2}\omega_-$$

$$\omega_\pm = \omega_I \pm i\omega_J$$

$$g = -\omega_I\omega_J^{-1}\omega_K$$

[Gaiotto-Moore-Neitzke '08]

Instanton corrections

- At large R, these $\mathcal{X}_\gamma$ take a universal form, up to exponentially-suppressed corrections that result from 4d BPS states running around this circle.
- We have thus reduced the determination of a K3 metric to the simpler problem of counting BPS states in a little string theory on T^2. Specifically, need the BPS index (second helicity supertrace) $\Omega(\gamma; a)$ that counts 4d BPS states at a point in (4d) moduli space a.
- Thanks to wall crossing formula, in principle only need to determine BPS state counts at one point in parameter and moduli space

Approximation

Iterate integral equation once: $\varpi^{\text{inst}}(\zeta) = \sum_\gamma \Omega(\gamma)\varpi_\gamma^{\text{inst}}$

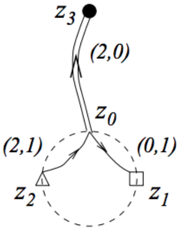
$$\varpi_\gamma^{\text{inst}}(\zeta) = -\frac{i}{8\pi^2} d\mathcal{Y}_\gamma^{\text{sf}}(\zeta) \wedge \left[-A^{\text{inst}} d\log\left(Z_\gamma/\overline{Z_\gamma}\right) + V^{\text{inst}} \left(\frac{1}{\zeta} dZ_\gamma - \zeta d\overline{Z_\gamma} \right) \right]$$

$$A^{\text{inst}} = \sum_{n>0} e^{in\theta_\gamma} |Z_\gamma| K_1(2\pi R n |Z_\gamma|)$$

$$V^{\text{inst}} = \sum_{n>0} e^{in\theta_\gamma} K_0(2\pi R n |Z_\gamma|)$$

[Ooguri-Vafa '96, Seiberg-Shenker '96, GMN '08]

String webs



- Particularly nice at points in moduli space with constant τ – flat base, so combinatorial flat surface problem.

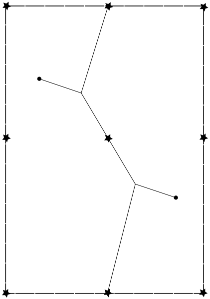
T^4/Z_q orbifold limits

- $T^4/Z_q = (T_F^2 \times T_B^2)/Z_q$, T_F^2 fibration over T_B^2/Z_q [Sen '96, Dasgupta-Mukhi '96].
- Non-abelian global symmetry from coincident 7-branes. Moving D3-brane probe near one of these 7-brane stacks and taking low energy limit yields either $SU(2)$ $N_f = 4$ SCFT or E_6, E_7, or E_8 Minahan-Nemeschansky (MN) SCFT

q	4d global symmetry	τ_F	τ_B
2	$\mathrm{Spin}(8)^4 \times U(1)^4$		
3	$E_6^3 \times U(1)^2$	κ_3	κ_3
4	$E_7^2 \times \mathrm{Spin}(8) \times U(1)^2$	i	i
6	$E_6 \times E_8 \times \mathrm{Spin}(8) \times U(1)^2$	κ_3	κ_3

$\kappa_q = e^{2\pi i/q}$

LST vs SCFTs



$\mathcal{N} = 2$ SUSY: $M = |Z_\gamma|$. So, abelian global symmetries must be associated to F1 and D1 winding about the two 1-cycles of T^2_B. For $q \neq 2$, only two linear combinations of these four charges are conserved

LST BPS spectra encoded in K3 metrics

- Turn on arbitrary Wilson lines for the 4d global symmetry as we reduce on S^1_R in order to smooth out the orbifold. (Correspond to extra moduli for heterotic on T^3 vs. T^2, in addition to $M_s R$.)
- Contributions to $\varpi^{\mathrm{inst}}(\zeta)$ from the BPS states of the LST with gauge and global charges of the form $\gamma = m\gamma_g + \gamma_f$:

$$\varpi^{\mathrm{eff}}_{\gamma_g} = -\frac{i}{8\pi^2} d\mathcal{Y}^{\mathrm{sf}}_{\gamma_g}(\zeta) \wedge \sum_{n>0} e^{in\theta_{\gamma_g}} \sum_{m|n} m^2 \sum_{\gamma_f} \Omega(m\gamma_g + \gamma_f) e^{in\theta_{\gamma_f}/m} \times \Big(-|Z_\gamma/m| K_1(2\pi Rn|Z_\gamma/m|) d\log(Z_\gamma/\bar{Z}_\gamma) + K_0(2\pi Rn|Z_\gamma/m|) \left(\frac{1}{\zeta} dZ_{\gamma_g} - \zeta d\bar{Z}_{\gamma_g} \right) \Big)$$

CFT BPS spectra encoded in K3 metrics

- At orbifold point, all flavor contributions to central charge are from winding, and for simplest string webs winding part of γ_f is also divisible by m: $\gamma_f = m\gamma_w + \tilde{\gamma}_f$. Letting $Z_{\gamma''} = Z_{\gamma_g+\gamma_w} = Z_\gamma / m$ gives

$$\varpi^{\mathrm{eff,CFT}}_{\gamma_g} = -\frac{i}{8\pi^2} d\mathcal{Y}^{\mathrm{sf}}_{\gamma_g}(\zeta) \wedge \sum_{n>0} e^{in\theta_{\gamma_g}} \times$$
$$\sum_{\gamma_w} e^{in\theta_{\gamma_w}} \left(-|Z_{\gamma''}| K_1(2\pi Rn|Z_{\gamma''}|) d\log(Z_{\gamma''}/\bar{Z}_{\gamma''}) \right.$$
$$\left. + K_0(2\pi Rn|Z_{\gamma''}|) \left(\frac{1}{\zeta} dZ_{\gamma''} - \zeta d\bar{Z}_{\gamma''} \right) \right) \times$$
$$\sum_{m|n} m^2 \sum_{\tilde{\gamma}_f} \Omega(m\gamma_g + \gamma_f) e^{in\theta_{\tilde{\gamma}_f}/m}$$

CFT BPS spectra encoded in K3 metrics, continued

- So, CFT BPS spectra are encoded in K3 metrics in the form of functions

$$F_{n,p,q}(\theta) = \sum_{m|n} m^2 \sum_{\tilde{\gamma}_f} \Omega(\gamma) e^{in\theta_{\tilde{\gamma}_f}/m}$$
$$= \sum_{m|n} m^2 \sum_{\mathcal{R}} \Omega(m,p,q,\mathcal{R}) \phi_{\mathcal{R}}(n\theta/m)$$

- (Dropped dependence on γ_w, since BPS spectrum only depends on which singular fiber strings are ending on, not number of times they wound around before terminating.)
- In contrast with LST spectrum, these CFT spectra don't wall cross, thanks to scale invariance plus R-symmetry

K3 as a Higgs branch

- D2-brane probing T^4/Z_q orbifold: K3 is *Higgs branch*. No quantum corrections!
- Perturbative type IIA string vacuum: no non-Abelian gauge symmetry. So, not just S^1-reduction of earlier M-theory frame on K3. B-field [Aspinwall '95]. From D2-brane point of view, this B-field breaks global symmetries.
- Non-renormalization theorem: g_s is in background vector multiplet, B-field dilutes away in $g_s \to \infty$ limit. So, moduli space is same as that of the M2-brane.
- Reminiscent of 3d mirror symmetry; not an accident! As discussed in [Porrati-Zaffaroni '96], this picture yields the simplest mirror pairs studied in [Intriligator-Seiberg '96]

Hyper-Kähler quotient

- Superpotential takes form $\operatorname{Tr}\Phi\mu_+$, where Φ is chiral multiplet in $\mathcal{N}=4$ vector multiplet whose vev vanishes on Higgs branch and μ_+ is function of hypermultiplet fields. F-term equation is then $\mu_+=0$.
- D-terms analogously take form $\mu_{\mathbb{R}}=0$, where $\mu_{\mathbb{R}}$ is a Hermitian function of the hypermultiplet fields.
- Higgs branch is the quotient of the space $\mu_{\mathbb{R}}=\mu_+=0$ by the gauge group.

$\mathrm{Sym}^N \mathbb{C}^2$

- Higgs branch of N parallel D2-branes. 3d $\mathcal{N} = 8$ $U(N)$ gauge theory; from $\mathcal{N} = 4$ point of view, adjoint hyper consisting of chiral multiplets U, V.
- $\mu_+ = -2[U, V]$, $\mu_{\mathbb{R}} = [U, U^\dagger] + [V, V^\dagger]$
- $\mu_+ = 0$ implies U and V can be simultaneously unitarily upper triangulized, $\mu_{\mathbb{R}} = 0$ implies that these upper triangular matrices are actually diagonal. Can then fix most of gauge group by demanding U and V be diagonal. Remaining gauge symmetry is S_N Weyl group.

$\mathbb{C}^2/Z_2$

- D2-brane probing $\mathbb{C}^2/Z_2$. Worldvolume is obtained by starting on $\mathbb{C}^2$ covering space with D2-brane and its image and the imposing orbifold projections. [Douglas-Moore '96]
- So, starting point is the $N = 2$ theory from last slide. We then require

$$U = -\sigma_z U \sigma_z \ , \quad V = -\sigma_z U \sigma_z \ , \quad g = \sigma_z g \sigma_z$$

$$U = \begin{pmatrix} & u_+ \\ u_- & \end{pmatrix} \ , \quad V = \begin{pmatrix} & v_+ \\ v_- & \end{pmatrix} \ , \quad g = e^{i\theta} \begin{pmatrix} e^{i\alpha/2} & \\ & e^{-i\alpha/2} \end{pmatrix}$$

- $\mu_+ = 0 \Rightarrow \begin{pmatrix} u_+ \\ v_+ \end{pmatrix} = \lambda \begin{pmatrix} u_- \\ v_- \end{pmatrix} \ , \quad \mu_{\mathbb{R}} = 0 \Rightarrow |\lambda| = 1.$
- $U(1)$: $\lambda = 1$; $\alpha = \pi$: $(u, v) \sim (-u, -v)$

$T^4 = \mathbb{C}^2/\mathbb{Z}^4$

- Same idea, but now we have an infinite-dimensional gauge group. [Taylor '96]
- Start with $U(\infty^4)$ and impose $\mathbb{Z}^4$ orbifold projection: $(u, v) \mapsto (u, v) + (n^u, n^v)$, $n \in \Lambda$
- Result is $\widehat{U(1)} = \mathrm{Maps}(\hat{T}^4 \to U(1))$, $\hat{T}^4 = \mathbb{C}^2/\hat{\Lambda}$, $\hat{\Lambda} = \mathrm{Hom}(\Lambda, 2\pi\mathbb{Z})$.
- T-duality: D2 probing T^4 becomes D6 wrapping $\hat{T}^4$
- U and V now define a $U(1)$ connection on $\hat{T}^4$:

$$B = \sum_n (U_n d\psi_1 + V_n d\psi_2) e(n) + \mathrm{h.c.}$$

$$e(n) = e^{i(n^u\psi_1 + n^v\psi_2 + \mathrm{c.c.})} = e^{in\cdot y},\ \psi_1 = \frac{y_1 - iy_2}{2},\ \psi_2 = \frac{y_3 - iy_4}{2}$$

T^4 continued

- The moment map equations, taken together, are equivalent to

$$F = - * F \ .$$

So, just looking at moduli space of ASD connections, mod gauge equivalence.

$$\|F\|^2 \equiv \int F \wedge * F = - \int F \wedge F = - \int dCS_3 = 0$$

So, moduli space of flat $U(1)$ connections / Wilson lines on $\hat{T}^4$, which is indeed T^4.

- Physically sensible that we reduce to constant gauge fields: Kaluza-Klein masses. Moduli space is compact because of large gauge transformations.

$K3 = T^4/Z_q = \mathbb{C}^2/\mathbb{Z}^4 \rtimes Z_q$

- Now, realize K3 as resolution of T^4/Z_q; i.e., orbifold $\mathbb{C}^2$ by Λ, and then by Z_q, or equivalently by $\mathbb{Z}^4 \rtimes Z_q$. [$q = 2$ case studied in Ramgoolam-Waldram '98, Greene-Lazaroiu-Yi '98. Similar constructions exist for all torus orbifold limits of K3]
- Start with $U(q)$ gauge theory on $\hat{T}^4$ and then impose Z_q projections:

$$\iota^* B = \sigma_q B \sigma_q^\dagger \,, \quad g \circ \iota = \sigma_q g \sigma_q^\dagger$$

$$\sigma_q = \begin{pmatrix} 1 & & & \\ & \kappa_q & & \\ & & \ddots & \\ & & & \kappa_q^{q-1} \end{pmatrix}$$

K3: blow-up parameters

$$F = - * F + \sum_{y'} \sum_{i=1}^{q-1} \eta_{y',i} \sigma_q^i \delta^4(y - y')$$

- So, K3 is hyper-Kähler quotient of infinite-dimensional flat space of Z_q-equivariant $SU(q)$ connections on $\hat{T}^4$ with prescribed (singular, for generic FI parameters) boundary conditions by group of equivariant $SU(q)$ gauge transformations (that preserve the boundary conditions).
- $q = 2$: 16 triples of FI parameters plus 10 T^4 moduli $= 58$ moduli
- $q \neq 2$: 18 triples of FI parameters plus 4 T^4 moduli $= 58$ moduli

K3: moduli space with vanishing FI parameters

- Can restrict to zero-modes, thanks to Kaluza-Klein masses and gauge transformations.
- Zero-mode moment maps and gauge transformations allow us to set $U_0 = us_q$, $V_0 = vs_q^\dagger$, where

$$s_q = \begin{pmatrix} & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ 1 & & & & \end{pmatrix},$$

  and $(u, v) \sim (\kappa_q u, \kappa_q^* v)$.
- 'Quasi-large' gauge transformations preserve this gauge and implement $(u, v) \sim (u + n^u, v + n^v)$.

Perturbation theory

- Parametrize general zero modes as $U_0 = U_0^{\text{orb}} + \Delta U_0$, $V_0 = V_0^{\text{orb}} + \Delta V_0$, where

$$\text{Tr}\,(U_0^{\text{orb}})^\dagger \Delta U_0 = \text{Tr}\,(V_0^{\text{orb}})^\dagger \Delta V_0 = 0\ .$$

- Goal: solve for $U_n(u,v)$, $V_n(u,v)$ (in a particular gauge) – carve K3 out of infinite-dimensional flat space

Perturbation theory, continued

- Suppose, inductively, that one knows $(\nu-1)$-th order approximations $U_n^{(\nu-1)}(u,v)$, $V_n^{(\nu-1)}(u,v)$. Then, write $U_n^{(\nu)} = U_n^{(\nu-1)} + \delta U_n^{(\nu)}$, and similarly for V.
- Writing the moment map equations and keeping only order ν terms, we find that they are linear in $\delta U_n^{(\nu)}$ and $\delta V_n^{(\nu)}$ and decouple into infinitely many equations, each involving only finitely many variables.
- Furthermore, there is a natural gauge choice,

$$d_{B^{\text{orb}}} * B = 0 \ ,$$

which shares these features.

Perturbation theory, continued

Explicitly, for each n we solve the linear equations

$$\xi^{(\nu)}_{n,+} = \delta U^{(\nu)}_n n^v - \delta V^{(\nu)}_n n^u + [U^{\rm orb}_0, \delta V^{(\nu)}_n] + [\delta U^{(\nu)}_n, V^{\rm orb}_0]$$

$$\xi^{(\nu)}_{n,\mathbb{R}} = -n^u (\delta U^{(\nu)}_{-n})^\dagger + n^{\bar u} \delta U^{(\nu)}_n + [U^{\rm orb}_0, (\delta U^{(\nu)}_{-n})^\dagger] + [\delta U^{(\nu)}_n, (U^{\rm orb}_0)^\dagger] + (U \mapsto V)$$

$$0 = -n^u (\delta U^{(\nu)}_{-n})^\dagger - n^{\bar u} \delta U^{(\nu)}_n + [U^{\rm orb}_0, (\delta U^{(\nu)}_{-n})^\dagger] + [(U^{\rm orb}_0)^\dagger, \delta U^{(\nu)}_n] + (U \mapsto V) ,$$

where $\xi^{(\nu)}_{n,+/\mathbb{R}}$ are constructed out of $\delta U^{(\nu')}_n, \delta V^{(\nu')}_n$ with $\nu' < \nu$ and $\xi^{(1)}_{n,+/\mathbb{R}}$ are the FI parameters. Note: coefficients on right side of equation are identical for all ν!

Perturbation theory, continued

For $\nu \geq 2$,

$$\xi_{n,+}^{(\nu)} = -\sum_{m} \sum_{\nu'=1}^{\nu-1} [\delta U_{n-m}^{(\nu')}, \delta V_{m}^{(\nu-\nu')}]$$

$$\xi_{n,\mathbb{R}}^{(\nu)} = -\sum_{m} \sum_{\nu'=1}^{\nu-1} [\delta U_{n+m}^{(\nu')}, (\delta U_{m}^{(\nu-\nu')})^{\dagger}] + (U \mapsto V)$$

Solution

$$N^u_{i,j} = n^u + (1-\kappa_q^i)\kappa_q^j u\,,\ N^v_{i,j} = n^v + (1-\kappa_q^{-i})\kappa_q^{-j} v\,,\ D_{i,j} = |N^u_{i,j}|^2 + |N^v_{i,j}|^2$$

$$\tilde{\xi}^{(\nu)}_{n,i,j,+} = \frac{1}{q}\,\mathrm{Tr}\, S_j S^\dagger_{i+j} \xi^{(\nu)}_{n,+}\,,\ \tilde{\xi}^{(\nu)}_{n,i,j,\mathbb{R}} = \frac{1}{q}\,\mathrm{Tr}\, S_j S^\dagger_{i+j} \xi^{(\nu)}_{n,\mathbb{R}}\,,\ S_j = \begin{pmatrix} 1 \\ \kappa_q^j \\ \vdots \\ \kappa_q^{(q-1)j} \end{pmatrix}$$

$$\delta U^{(\nu)}_n = \frac{1}{2q} \sum_{j=0}^{q-1} \sum_{i=\delta_{n,0}}^{q-1} \frac{2\tilde{\xi}^{(\nu)}_{n,i,j,+} \bar{N}^v_{i,j} + \tilde{\xi}^{(\nu)}_{n,i,j,\mathbb{R}} N^u_{i,j}}{D_{i,j}} S_{i+j} S^\dagger_j$$

$$\delta V^{(\nu)}_n = \frac{1}{2q} \sum_{j=0}^{q-1} \sum_{i=\delta_{n,0}}^{q-1} \frac{-2\tilde{\xi}^{(\nu)}_{n,i,j,+} \bar{N}^u_{i,j} + \tilde{\xi}^{(\nu)}_{n,i,j,\mathbb{R}} N^v_{i,j}}{D_{i,j}} S_{i+j} S^\dagger_j$$

Integral equation

Summing up the contribution from each ν and writing $e(n) = e^{in\cdot y}$, $U = U^{\rm orb} + \Delta U$, and $V = V^{\rm orb} + \Delta V$, we find

$$\Delta U = \frac{1}{2q}\sum_n \sum_{j=0}^{q-1}\sum_{i=\delta_{n,0}}^{q-1} \frac{S_{i+j}S_j^\dagger}{D_{i,j}}\Bigg[\left(2\xi_{n,i,+}e(n)\bar{N}^v_{i,j} + \xi_{n,i,\mathbb{R}}e(n)N^u_{i,j}\right) - \frac{1}{q}\sum_m \mathrm{Tr}\Big[S_jS^\dagger_{i+j}\Big(2[\Delta U_{n-m}e(n-m),\Delta V_m e(m)]\bar{N}^v_{i,j} + \Big([\Delta U_{n+m}e(n+m),\Delta U^\dagger_m e(-m)] + [\Delta V_{n+m}e(n+m),\Delta V^\dagger_m e(-m)]\Big)N^u_{i,j}\Big)\Big]\Bigg] .$$

Similarly for ΔV. Coupled integral equations on $\hat{T}^4$!

Kähler forms

$$\omega_I = \frac{i}{2q}\sum_n \mathrm{Tr}\Big(-dU_n \wedge dV_{-n} + dU_n^\dagger \wedge dV_{-n}^\dagger\Big)$$

$$\omega_J = -\frac{1}{2q}\sum_n \mathrm{Tr}\Big(dU_n \wedge dV_{-n} + dU_n^\dagger \wedge dV_{-n}^\dagger\Big)$$

$$\omega_K = \frac{i}{2q}\sum_n \mathrm{Tr}\Big(dU_n \wedge dU_n^\dagger + dV_n \wedge dV_n^\dagger\Big)$$

$$dU_n = \frac{\partial U_n}{\partial u}du + \frac{\partial U_n}{\partial u^*}du^* + \frac{\partial U_n}{\partial v}dv + \frac{\partial U_n}{\partial v^*}dv^*$$

Kähler forms – first order corrections

$$\omega_+^{\text{orb}} = -i\, du \wedge dv \,, \quad \omega_K^{\text{orb}} = \frac{i}{2}(du \wedge du^* + dv \wedge dv^*)$$

$$\varpi(\zeta) = \varpi^{\text{orb}}(\zeta) + \varpi^{\text{pert}}(\zeta)$$

$$\varpi^{\text{pert}}(\zeta) = -\frac{i}{2\zeta}\omega_+^{\text{pert}} + \omega_K^{\text{pert}} - \frac{i\zeta}{2}\omega_-^{\text{pert}}$$

$$= \sum_n \sum_{i=1}^{\lfloor q/2 \rfloor} f_i \sum_{t=\pm 1} \left(-\frac{i}{2\zeta}\omega_{nti+} + \omega_{ntiK} - \frac{i\zeta}{2}\omega_{nti-} \right)$$

$$f_i = \begin{cases} \frac{1}{2} & : i = q/2 \\ 1 & : \text{else} \end{cases}$$

$N_i^u = N_{i,0}^u$, etc.

$$\omega_{nti+\,u\bar{u}} = \frac{i|1-\kappa_q^i|^2}{4}\frac{(2\xi_{nti+}\bar{N}_i^v + \xi_{nti\mathbb{R}}N_i^u)(2\xi_{n(-t)i+}\bar{N}_i^u - \xi_{nti\mathbb{R}}^* N_i^v)}{D_i^3}$$

$$\omega_{nti+\,uv} = 0$$

$$\omega_{nti+\,u\bar{v}} = -\frac{i(1-\kappa_q^i)^2}{4}\frac{(2\xi_{nti+}\bar{N}_i^u - \xi_{nti\mathbb{R}}N_i^v)(2\xi_{n(-t)i+}\bar{N}_i^u - \xi_{nti\mathbb{R}}^* N_i^v)}{D_i^3}$$

$$\omega_{nti+\,\bar{u}v} = -\frac{i(1-\kappa_q^{-i})^2}{4}\frac{(2\xi_{nti+}\bar{N}_i^v + \xi_{nti\mathbb{R}}N_i^u)(2\xi_{n(-t)i+}\bar{N}_i^v + \xi_{nti\mathbb{R}}^* N_i^u)}{D_i^3}$$

$$\omega_{nti+\,\bar{u}\bar{v}} = 0$$

$$\omega_{nti+\,v\bar{v}} = -\omega_{nti+\,u\bar{u}}$$

Similar expressions for ω_K

$$g = -\omega_I \omega_J^{-1} \omega_K = g^{\text{orb}} + \sum_n g_n$$

$$J_I = -\omega_J^{-1} \omega_K = J_I^{\text{orb}} + \sum_n J_{nI} \;, \quad \ldots$$

$$R_{km} = R^{\ell}{}_{k\ell m} \approx (g^{\text{orb}})^{\ell i} R_{ik\ell m}$$

$$\approx \frac{1}{2} \sum_n (g^{\text{orb}})^{\ell i} \left(g_{n\,im,k\ell} + g_{n\,k\ell,im} - g_{n\,i\ell,km} - g_{n\,km,i\ell} \right) = 0$$

$$J_\sigma^2 \approx (J_\sigma^{\text{orb}})^2 + \sum_n \{ J_\sigma^{\text{orb}}, J_{n\sigma} \} = -1$$

$$\sum_n \delta(x-n) = \sum_k e^{2\pi i k x}$$

$$\sum_n \lim_{x\to n} f(x) = \sum_k \mathcal{F}[f](k)$$

- We now perform a *2-dimensional* Poisson resummation over lattice parametrized by n^v. Motivated by geometric picture we're trying to make contact with – corrections to semi-flat geometry.
- Set $\xi_+ = 0$ for simplicity – focus on BPS spectrum of 4d theory at orbifold point

$$\varpi^{\text{inst}}(\zeta) = \sum_{\gamma_g} \varpi^{\text{eff}}_{\gamma_g}$$

$$\varpi^{\text{eff}}_{\gamma_g} = -\frac{i}{8\pi^2} d\mathcal{Y}^{\text{sf}}_{\gamma_g}(\zeta) \wedge \sum_{n>0} e^{in\theta_{\gamma_g}} \times$$
$$\sum_{\gamma_w} e^{in\theta_{\gamma_w}} \left(-|Z_{\gamma''}| K_1(2\pi Rn|Z_{\gamma''}|) d\log(Z_{\gamma''}/\bar{Z}_{\gamma''})\right.$$
$$\left. + K_0(2\pi Rn|Z_{\gamma''}|)\left(\frac{1}{\zeta} dZ_{\gamma''} - \zeta d\bar{Z}_{\gamma''}\right)\right) \times$$
$$F_{n,p,q,\gamma_w}$$

Geometry of string webs is encoded in lattice of winding charges and the flavor central charges Z_{γ_w}:
$Z_{\gamma''} = (p\tau_F + q)(a - a_0)$

F_{n,p,q,γ_w} depends very weakly on γ_w: only depends on subgroup of Z_q that stabilizes fixed point a_0 – i.e., type of singular fiber

$$F_{n,p,q,Z_2} = n^2(-1)^n \sum_{\lambda \in Z_2^2} \left(-\frac{1}{2}\pi^4 R^2 \xi^2_{\lambda 1 \mathbb{R}} \right) (-1)^{n(\lambda^3 p + \lambda^4 q)}$$

$$F_{n,p,q,Z_3} = n^2(-1)^n \sum_{\lambda \in Z_3} \left(-\frac{4}{3}\pi^4 R^2 |\xi_{\lambda 2 \mathbb{R}}|^2 \right) \kappa_3^{n\lambda(p+q)}$$

$$F_{n,p,q,Z_4} = n^2(-1)^n \sum_{\lambda \in Z_2} \left(-2\pi^4 R^2 |\xi_{\lambda 3\mathbb{R}}|^2\right)(-1)^{n\lambda(p+q)}$$
$$+ F_{n,p,q,Z_2}(\xi_{(1,0)1\mathbb{R}} = \xi_{(0,1)1\mathbb{R}})$$
$$F_{n,p,q,Z_6} = n^2(-1)^n \left(-4\pi^4 R^2 |\xi_{4\mathbb{R}}|^2\right)$$
$$+ F_{n,p,q,Z_2}(\xi_{(1,0)1\mathbb{R}} = \xi_{(0,1)1\mathbb{R}} = \xi_{(1,1)1\mathbb{R}})$$
$$+ F_{n,p,q,Z_3}(\xi_{12\mathbb{R}} = \xi_{22\mathbb{R}})$$

Conjectural exact relationships

$$F_{n,p,q,Z_4}(\xi_{\lambda 3\mathbb{R}} = 0) = F_{n,p,q,Z_2}(\xi_{(1,0)1\mathbb{R}} = \xi_{(0,1)1\mathbb{R}})$$

$$F_{n,p,q,Z_6}(\xi_{4\mathbb{R}} = \xi_{\lambda 1\mathbb{R}} = 0) = F_{n,p,q,Z_3}(\xi_{12\mathbb{R}} = \xi_{22\mathbb{R}})$$

$$F_{n,p,q,Z_6}(\xi_{4\mathbb{R}} = \xi_{\lambda 2\mathbb{R}} = 0) = F_{n,p,q,Z_2}(\xi_{(1,0)1\mathbb{R}} = \xi_{(0,1)1\mathbb{R}} = \xi_{(1,1)1\mathbb{R}})$$

Z_2: $SU(2)$ $N_f = 4$

$$F_{n,p,q}(\theta) = \begin{cases} \phi_{\mathcal{R}_{p,q}}(n\theta) - 8 & : 2|n \\ \phi_{\mathcal{R}_{p,q}}(n\theta) & : 2 \nmid n \end{cases} \quad , \quad \mathcal{R}_{p,q} = \begin{cases} \mathbf{8_v} & : 2|p \wedge 2 \nmid q \\ \mathbf{8_s} & : 2 \nmid p \wedge 2 \nmid q \\ \mathbf{8_c} & : 2 \nmid p \wedge 2|q \end{cases}$$

- Half-hyper ($\Omega = 1$) with gauge charge (p, q) in one of the 3 8-dimensional reps of Spin(8), depending on whether p, q, or both are odd.
- Vector ($\Omega = -2$) with gauge charge $(2p, 2q)$ in singlet of Spin(8)

Agrees with result from hyper-Kähler quotient after a simple linear change of variables from θ to ξ

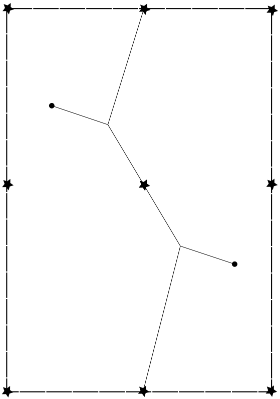
Z_3: E_6 MN

n	$\Omega_{\text{red}}(n\gamma_1)$
1	$\overline{\mathbf{27}}$
2	$\mathbf{27}$
3	$\mathbf{78} + 2 \times \mathbf{1}$
4	$\overline{\mathbf{351}} + 2 \times \overline{\mathbf{27}}$
5	$\mathbf{1728} + 2 \times \mathbf{351} + 6 \times \mathbf{27}$
6	$\mathbf{5824} + \mathbf{2430} + 2 \times \mathbf{2925} + 6 \times \mathbf{650} + 13 \times \mathbf{78} + 16 \times \mathbf{1}$
7	$\overline{\mathbf{19305}} + 3 \times \overline{\mathbf{17550}} + 6 \times \overline{\mathbf{7371}} + 13 \times \overline{\mathbf{1728}} + 12 \times \overline{\mathbf{351'}} + 29 \times \overline{\mathbf{351}} + 44 \times \overline{\mathbf{27}}$

[Hollands-Neitzke ’16]. We also compared with data on E_6 and E_7 theories from [Hao-Hollands-Neitzke ’19]

- We have derived constraints on the spectra of these field theories for arbitrarily large imprimitivity!
- At leading order in the FI parameters, they are fairly weak, but we have obtained some new BPS state counts.
- Proceeding to higher orders will yield the entire spectra.
- Motivated by the leading order expressions produced by the hyper-Kähler quotient, we have conjectured strong all-orders relationships between the BPS spectra of the various field theories that coexist within the same F-theory compactifications (which are satisfied by all existing data)

Missing BPS states of LSTs

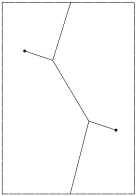


A_1 $\mathcal{N} = (1, 1)$ LST

- Considerations from before show that moduli space of LST on T^3 is $\mathrm{Sym}^2(T^4)$
- However, can turn on holonomy of background R-symmetry gauge field which preserves 3d $\mathcal{N} = 4$, and resulting moduli space is essentially $T^4 \times K3$ [Cheung-Ganor-Krogh '98]
- Mathematically, this is related to construction of K3 as a generalized Kummer variety
- So, can read off K3 metric from metric on this moduli space
- Only get special K3 surfaces from this construction: always have Z_2^4 symmetries.

BPS state counting

- One 1-real-dimensional family is particularly nice: if holonomy is only on the third circle, then the BPS state counting problem is simply that of the $(1, 1)$ (or $(2, 0)$) LST on T^2, with no R-symmetry holonomies
- String web formulation: type IIB on T^2 with two transverse D3-branes



- Geometric engineering: type IIA on affine A_1 singularity, i.e. total space of I_2 singular fiber, times T^2

Conclusion

- A hyper-Kähler quotient yields computationally useful, explicit, analytic expressions for K3 metrics.
- They secretly encode the solution to a little string theory BPS state counting problem. In particular, there are piecewise constant lists of integers hiding inside of K3 metrics! Similarly, we find characters of Spin(8) and E_n representations. We also find an interesting dependence on the geometry of string webs.
- Via string dualities, we can recast this BPS state counting problem in terms of open string reduced Gromov-Witten theory of K3. Aligns with the Strominger-Yau-Zaslow construction of mirror manifolds.

Coulomb branch construction

- By finding the full BPS spectrum of the little string theory, we will complete the specification of a second, equivalent construction of K3 metrics. We intend to do so by Poisson resumming the Higgs branch result at all orders.
- Other approaches: geometric engineering, holography, DLCQ, deconstruction. Neat connections with $\mathcal{N} = (1,1)$ A_1 little string theory and open topological string theory.
- Even without most counts, Coulomb branch construction gives some very accurate approximations, similar to (and generalizing) [Gross-Wilson '00]

Generalizations

- Adding D6-branes wrapping T^4 or an orbifold thereof to the hyper-Kähler quotient construction will allow us to obtain nearly all (hopefully all) known compact hyper-Kähler manifolds. 3d mirror symmetry again relates these configurations to little string theories
- Poisson resumming 1, 3, or 4 times is also possible. Do these yield other interesting expansions with corresponding counting problems?
- Although we've focused in this talk on K3 and little string theories, analogous stories hold for moduli spaces of various field theories whose Coulomb branches are non-compact 4-dimensional hyper-Kähler manifolds.