Miura Operators, Degenerate Fields and the M2-M5 Intersection

in progress with Davide Gaiotto

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1. Introduction

1.1. $\mathfrak{gl}(1)$ current algebra

Let me start by recalling the simplest vertex operator algebra, the $gl(1)$ **current algebra (Heisenberg vertex operator algebra), generated by** $J(z)$ with operator product expansion

$$J(z)J(w) \sim -\frac{1}{\epsilon_1\epsilon_2}\frac{1}{(z-w)^2}.$$

Each vertex operator algebra leads to an associative algebra of its modes with commutation relations recovered from the operator product expansion by a simple contour-deformation argument. In our case,

$$J_m = \oint_0 \frac{dz}{2\pi i} z^m J(z)$$

with **commutation relations** given in terms of

$$[J_m, J_n] = -\frac{1}{\epsilon_1\epsilon_2}\oint_0 \frac{dz}{2\pi i}\oint_z \frac{dw}{2\pi i}\frac{z^m w^n}{(z-w)^2} = -\frac{1}{\epsilon_1\epsilon_2} m\delta_{m,-n}.$$

1. Introduction

1.2. Vertex operators

The $gl(1)$ current algebra admits a nice family of **highest-weight modules** induced from the highest-weight state $|u\rangle$ satisfying

$$J_0|u\rangle = -\frac{u}{\epsilon_1\epsilon_2}|u\rangle, \qquad J_m|u\rangle = 0, \qquad \text{for } m > 0,$$

with a tower of states generated by the action of negative modes

$$\begin{aligned}
&J_{-1}|u\rangle, \\
&J_{-2}|u\rangle, \qquad J_{-1}^2|u\rangle, \\
&J_{-3}|u\rangle, \qquad J_{-2}J_{-1}|u\rangle, \qquad J_{-1}^3|u\rangle.
\end{aligned}$$

1. Introduction

We can now define the **vertex operator**

$$\exp[a\phi(z)] =: \exp[a\sum_{n\neq 0}\frac{J_n}{n}z^{-n}]T_a :,$$

with $\phi(z)$ viewed as $J(z)=\partial\phi(z)$. We introduced T_a that commutes with J_n and shifts the highest weight $T_a|u\rangle = |u+a\rangle$ and :: is the normal ordering that orders J_n in increasing value of n. The vertex operator is an elementary object in the theory of vertex operator algebras and satisfies

$$\begin{aligned} J(z) : \exp[a\phi(z)] : \quad &\sim \quad -\frac{a}{\epsilon_1\epsilon_2}\frac{: \exp[a\phi(z)] :}{z-w}, \\ : \exp[a\phi(z)] :: \exp[b\phi(w)] : \quad &\sim \quad (z-w)^{-\frac{ab}{\epsilon_1\epsilon_2}} : \exp[a\phi(z)+b\phi(w)] : . \end{aligned}$$

Furthermore, we obviously have $: \exp[a\phi(0)] : |0\rangle = |a\rangle$.

1. Introduction

1.3. First trivially-sounding observation

Objects

$$\exp[-\epsilon_2\phi(z)]|0\rangle \quad \text{and} \quad \exp[\epsilon_2\phi(z)]|0\rangle,$$

thought of as formal power series in z with coefficients in the module induced from $|\epsilon_2\rangle$, generate a right module for the $\mathfrak{gl}(1)$ current algebra generated by the action of J_n but also a right/left module for the algebra generated by

$$t_{0,n} = \frac{z^n}{\epsilon_1}, \qquad t_{2,0} = \epsilon_1\partial^2.$$

Recall that differential operators act on functions from the right by

$$f(z) \circ \partial = -\partial f(z).$$

Note also that the general $t_{m,n} \sim z^m\partial^n$ can be obtained by commuting $t_{0,n}$ with $t_{2,0}$.

1. Introduction

1.3. The Miura operator

Let us now introduce another important object from the theory of vertex operator algebras, the **Miura operator**

$$\epsilon_3\partial - \epsilon_1\epsilon_2 J(z) = \epsilon_3\partial - \epsilon_1\epsilon_2 \sum_{m=-\infty}^{\infty} J_n z^{-n-1},$$

thought of as a differential operator in one variable with coefficients in the $\mathfrak{gl}(1)$ current algebra. Composition of such Miura operators is known to lead to other vertex operator algebras realized in terms of an embedding inside tensor products of $\mathfrak{gl}(1)$ current algebras as I will review later.

1. Introduction

1.4. Second trivially-sounding observation

The object

$$(\epsilon_3\partial - \epsilon_1\epsilon_2 J(z))|0\rangle,$$

thought of as a differential operator with coefficients in the module induced from $|0\rangle$ generates a module for J_n and a bimodule for the algebra generated by

$$t_{0,n} = \frac{z^n}{\epsilon_3}, \qquad t_{2,0} = \epsilon_3\partial^2.$$

since we can act from the right and from the left.

1. Introduction

1.5. Trivially-sounding observation becoming deep

The trivially-sounding observations lead to very deep statements if we realize that:

1. M2-brane on $\mathbb{R} \times \mathbb{C}_{\epsilon_3}$ leads to a topological quantum mechanics on $\mathbb{R}$ with the algebra of (Coulomb-branch) operators $\frac{z^n}{\epsilon_3}$ and $\epsilon_3 \partial^2$.
2. M5-brane on $\mathbb{C} \times \mathbb{C}_{\epsilon_1} \times \mathbb{C}_{\epsilon_2}$ leads to a holomorphic quantum field theory on $\mathbb{C}$ with an algebra of operators generated by J_m.

Vertex operators $\exp[\epsilon_1 \phi(z)]|0\rangle$ and $\exp[-\epsilon_1 \phi(z)]|0\rangle$ and the Miura operator $(\epsilon_3 \partial - \epsilon_1 \epsilon_2 J(z))|0\rangle$ generate modules for both. The purpose of this talk is to convince you that they can be interpreted as elementary building blocks of "gauge invariant" operators describing M2-M5 brane intersections. This leads to deep connections between $\mathcal{W}$-algebras associated to M5-branes and Coulomb-branch algebras associated to M2-branes with consequences such as categorification of the relation between Donaldson-Thomas and Pandharipande-Thomas topological vertices and more.

2. M-theory expectations

2.1. M2-brane and M5-brane theories

Let us introduce the following physical system:

M-theory	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{R}$	$\mathbb{C}_{\epsilon_1}$	$\mathbb{C}_{\epsilon_2}$	$\mathbb{C}_{\epsilon_3}$
N_1 M5	$\times$				$\times$	$\times$
N_2 M5	$\times$			$\times$		$\times$
N_3 M5	$\times$			$\times$	$\times$	
n_1 M2			$\times$	$\times$		
n_2 M2			$\times$		$\times$	
n_3 M2			$\times$			$\times$

We turn on Ω-background along three complex directions with parameters ϵ_i satisfying $\epsilon_1+\epsilon_2+\epsilon_3=0$. We place M2 branes along $\mathbb{R}\times\mathbb{C}_{\epsilon_i}$ and M5 branes along $\mathbb{C}\times\mathbb{C}_{\epsilon_i}\times\mathbb{C}_{\epsilon_j}$. From the perspective of the directions $\mathbb{C}^2\times\mathbb{R}$, M5-branes lead to surface operators S_{N_1,N_2,N_3} living on $\mathbb{C}$ and M2-branes lead to line operators L_{n_1,n_2,n_3} living on $\mathbb{R}$.

2. M-theory expectations

- The world-volume theory on N_3 M5-branes in Ω-background leads to vertex operator algebras $Y_{0,0,N_3}$ on $\mathbb{C}$ [Alday-Gaiotto-Tachikawa (2009), Wyllard (2009)]. There exists a three-parameter generalization Y_{N_1,N_2,N_3} associated to the general configuration of M5-branes leading to S_{N_1,N_2,N_3} form our previous work [Gaiotto-MR (2017)].
- A 3d supersymmetric gauge theory in Ω background leads to a Coulomb branch algebra of local operators on $\mathbb{R}$ [Oh-Yagi (2019), Jeong (2019), Dedushenko (2019)]. The theory associated to n_3 M2-branes on $\mathbb{C}_{\epsilon_3} \times \mathbb{R}$ has a UV completion in terms of an ADHM quiver gauge theory with the Coulomb-branch algebra $A_{0,0,n_3}$ studied by [Kodera-Nakajima (2016), Costello (2016)]. One of the results of our today's discussion is a three-parameter generalization of these algebras A_{n_1,n_2,n_3} associated to the general configurations of M2-branes L_{n_1,n_2,n_3} in the M-theory picture above [Gaiotto-MR (2020)].

2. M-theory expectations

2.2. Coupling to the twisted M-theory

According to the proposal of [Costello (2017)], twisted M-theory on $\mathbb{C}^2 \times \mathbb{R} \times \mathbb{C}_{\epsilon_1} \times \mathbb{C}_{\epsilon_2} \times \mathbb{C}_{\epsilon_3}$ is described in terms of a $\mathfrak{gl}(1)$ **5d non-commutative Chern-Simons-like theory** with the action

$$\frac{1}{\epsilon_1} \int (A *_{\epsilon_3} dA + A *_{\epsilon_3} A *_{\epsilon_3} A) dz_1 dz_2$$

living on $\mathbb{C}^2 \times \mathbb{R}$, where $*_{\epsilon_3}$ is a non-commutative deformation of the standard wedge product. The 5d Chern-Simons theory (before the ϵ_3 deformation) has a large gauge symmetry

$$A \to A + d\phi, \quad \text{for } \phi \in \mathbb{C}[z_1, z_2].$$

The non-commutative deformation leads to an obvious deformation of the **algebra of gauge transformations**

$$[z_1, z_2] = \epsilon_3.$$

S_{N_1,N_2,N_3} and L_{n_1,n_2,n_3} must admit a consistent coupling to this theory!

2. M-theory expectations

Classically, one expects that the line L_{n_1,n_2,n_3} coming from M2-branes can couple to $\partial_{\bar{z}_1}^m \partial_{\bar{z}_2}^n A_t$ by $t_{m,n}$ satisfying relations of the gauge algebra. As argued by [Costello (2018)], this expectation receives quantum corrections and vanishing of gauge anomalies actually requires $t_{m,n}$ to satisfy relations of an algebra $\mathcal{A}$ deforming the classical gauge algebra above. Algebra $\mathcal{A}$ is an associative algebra generated by $t_{2,0}$ and $t_{0,n}$ with other generators $t_{m,n}$ defined in terms of

$$[t_{2,0}, t_{m,n}] = 2n t_{m+1,n-1}$$

and relations determined using

$$\begin{aligned}[t_{3,0}, t_{0,d}] &= 3d\, t_{2,d-1} + \sigma_2 \frac{d(d-1)(d-2)}{4} t_{0,d-3} \\ &\quad + \frac{3}{2}\sigma_3 \sum_{m=0}^{d-3} (m+1)(d-m-2) t_{0,m} t_{0,d-3-m},\end{aligned}$$

introducing an explicit dependence on $\sigma_3 = \epsilon_1\epsilon_2\epsilon_3, \sigma_2 = \epsilon_1\epsilon_2 + \epsilon_1\epsilon_3 + \epsilon_2\epsilon_3$.

2. M-theory expectations

Note that $\mathcal{A}$ admits a truncation map to the Coulomb-branch algebra $A_{0,0,1}$ associated to a single M2-brane

$$I_{0,0,1} : \mathcal{A} \to A_{0,0,1}$$

given by

$$I_{0,0,1} : t_{0,n} \to \frac{1}{\epsilon_3} z^n \quad \text{and} \quad I_{0,0,1} : t_{2,0} \to \epsilon_3 \partial^2 .$$

Analogously, permuting parameters $\epsilon_1, \epsilon_2, \epsilon_3$ gives rise to three elementary truncation maps $I_{1,0,0}, I_{0,1,0}, I_{0,0,1}$.

2. M-theory expectations

An analogous argument from [Costello (2017)] leads to a condition on local operators of surface defects S_{N_1,N_2,N_3} that should admit a consistent coupling to the 5d Chern-Simons theory. In particular, vanishing of gauge anomalies requires surface defects S_{N_1,N_2,N_3} to couple to $\partial_{z_1}^n A$ via operators $W_n(z)$ that satisfy the algebra $\mathcal{W}_\infty$ generated by infinitely many fields $W_1, W_2, W_3, \ldots$ studied by many people [Shiffman-Vasserot (2012), Gaberdiel-Gopakumar (2012), Procházka (2015),...] and satisfying a system of operator product expansions

$$\begin{aligned}
W_1(z)W_1(w) &\sim \frac{\psi_0}{(z-w)^2}, \\
W_2(z)W_1(w) &\sim \frac{W_1(w)}{(z-w)^2} + \frac{\partial W_1(w)}{z-w}, \\
W_2(z)W_2(w) &\sim -\frac{\sigma_2\psi_0 + \sigma_3^2\psi_0^3}{2}\frac{1}{(z-w)^4} + \frac{2W_2(w)}{(z-w)^2} + \frac{\partial W_2(w)}{z-w}
\end{aligned}$$

depending on $\sigma_3 = \epsilon_1\epsilon_2\epsilon_3, \sigma_2 = \epsilon_1\epsilon_2 + \epsilon_1\epsilon_3 + \epsilon_2\epsilon_3$ and parameter ψ_0.

2. M-theory expectations

Note that the $\mathfrak{gl}(1)$ current algebra associated to $S_{0,0,1}$ satisfy relations of $\mathcal{W}_\infty$ since there exists a truncation map

$$S_{0,0,1} : \mathcal{W}_\infty \to Y_{0,0,1}$$

given by

$$\begin{aligned} S_{0,0,1} &: \quad W_1(z) \to J(z), \\ S_{0,0,1} &: \quad W_2(z) \to -\frac{\epsilon_1\epsilon_2}{2} : JJ : (z), \\ S_{0,0,1} &: \quad W_n(z) \to 0, \qquad \text{for } n > 2, \end{aligned}$$

with normalization

$$J(z)J(w) \sim -\frac{1}{\epsilon_1\epsilon_2}\frac{1}{(z-w)^2}.$$

Similarly, there exist elementary truncation maps $S_{1,0,0}$ and $S_{0,1,0}$ obtained by permutation of ϵ_i and leading to $Y_{1,0,0}$ and $Y_{0,1,0}$.

2. M-theory expectations

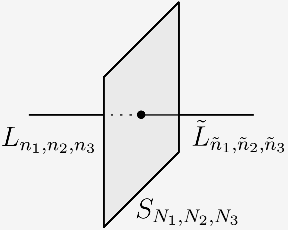
As we will see in later sections, there also exist truncation maps S_{N_1,N_2,N_3} and L_{n_1,n_2,n_3} from $\mathcal{W}_\infty$ and $\mathcal{A}$ to the algebra of operators Y_{N_1,N_2,N_3} on surface S_{N_1,N_2,N_3} and the algebra of operators A_{n_1,n_2,n_3} on line L_{n_1,n_2,n_3}:

$$
\begin{aligned}
S_{N_1,N_2,N_3} &: \mathcal{W}_\infty \to Y_{N_1,N_2,N_3}, \\
L_{n_1,n_2,n_3} &: \mathcal{A} \to A_{n_1,n_2,n_3}.
\end{aligned}
$$

The elementary truncations are sufficient for the discussion of elementary building blocks of gauge-invariant junctions.

2. M-theory expectations

As pointed out by [Gaiotto-Oh (2019)], one should be able to associate gauge-invariant operators $\mathcal{O}$ also to intersections of L_{n_1,n_2,n_3}, $\tilde{L}_{\tilde{n}_1,\tilde{n}_2,\tilde{n}_3}$ and S_{N_1,N_2,N_3} defects, corresponding to the following configuration:



One can expect that the gauge-invariance condition receives contributions from all the operators involved and deforms the naive constraint

$$\mathcal{O}\tilde{t}_{a,b} = (t_{a,b} + W_{a,b})\mathcal{O}.$$

The main objective of our work was to identify the precise condition and explore its mathematical implications.

3. Gauge-invariant junctions

3.1. The $\mathcal{A} \to \mathcal{A} \otimes \mathcal{W}_\infty$ coproduct

It is straightforward to check that formulas

$$\begin{aligned} \tilde{t}_{0,n} &\to t_{0,n} + W_{1,n} \\ \tilde{t}_{2,0} &\to t_{2,0} + V_{-2} + \sigma_3 \sum_{n=1}^{\infty} n W_{1,-n-1} W_{1,n-1} + 2\sigma_3 \sum_{n=1}^{\infty} n W_{1,-n-1} t_{0,n-1}, \end{aligned}$$

where

$$V(z) = W_3 + \frac{2}{\psi_0} : W_1 W_2 : -\frac{2}{3}\frac{1}{\psi_0^2} : W_1 W_1 W_1 :,$$

define a coproduct

$$\Delta_{A,\mathcal{W}_\infty} : \mathcal{A} \to \mathcal{A} \otimes \mathcal{W}_\infty .$$

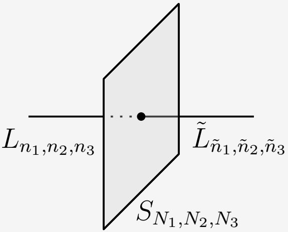
3. Gauge-invariant junctions

3.2. The gauge invariant junctions

We have now all the ingredients to state the gauge-invariance condition deforming the naive

$$\mathcal{O}\tilde{t}_{a,b} = (t_{a,b} + W_{a,b})\mathcal{O},$$

expected to be satisfied by the operator $\mathcal{O}$ describing the junction:



We propose it to have the following form:

$$\boxed{\mathcal{O} \circ L_{\tilde{n}_1,\tilde{n}_2,\tilde{n}_3}(t_{m,n}) = (L_{n_1,n_2,n_3} \otimes S_{N_1,N_2,N_3})(\Delta_{A,\mathcal{W}_\infty}(t_{m,n})) \circ \mathcal{O}.}$$

3. Gauge-invariant junctions

Let me decodify the condition

$$\boxed{\mathcal{O}\circ L_{\tilde{n}_1,\tilde{n}_2,\tilde{n}_3}(t_{m,n}) = (L_{n_1,n_2,n_3}\otimes S_{N_1,N_2,N_3})(\Delta_{\mathcal{A},\mathcal{W}_\infty}(t_{m,n}))\circ\mathcal{O}}$$

diagrammatically. We start with a generator $t_{m,n}$ and map it to A_{n_1,n_2,n_3} and $A_{\tilde{n}_1,\tilde{n}_2,\tilde{n}_3}\otimes Y_{N_1,N_2,N_3}$ via

$$\begin{array}{ccccc}
\mathcal{A} & \xrightarrow{\Delta_{\mathcal{A},\mathcal{W}_\infty}} & \mathcal{A} & \otimes & \mathcal{W}_\infty \\
\big\downarrow L_{n_1,n_2,n_3} & & \big\downarrow L_{\tilde{n}_1,\tilde{n}_2,\tilde{n}_3} & & \big\downarrow S_{N_1,N_2,N_3} \\
A_{n_1,n_2,n_3} & & A_{\tilde{n}_1,\tilde{n}_2,\tilde{n}_3} & \otimes & Y_{N_1,N_2,N_3}
\end{array}$$

The gauge-invariance condition is a statement that the operator $\mathcal{O}$ intertwines the right action of $t_{m,n}$ mapped by the arrow on the left of the diagram with the left action of $t_{m,n}$ mapped by the sequence of arrows on the right of the diagram.

4. Building blocks of gauge-invariant junctions

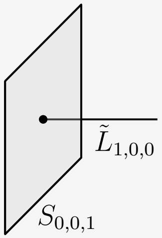
4.1. Elementary junctions

Many key objects in the theory of VOAs can be now given a new interpretation in terms of a gauge-invariant junction! In this section, we are going to provide such a re-interpretation for elements $\exp[-\epsilon_2\phi(z)]|0\rangle, \exp[\epsilon_2\phi(z)]|0\rangle$ and $(\epsilon_3\partial - \epsilon_1\epsilon_2 J(z))|0\rangle$ introduced in the introduction. As we will see later (and as obvious from the M-theory picture), these serve as elementary building blocks of more complicated junctions.

4. Building blocks of gauge-invariant junctions

4.2. The vertex operator $\exp[-\epsilon_2\phi(z)]$

Let us start with the example of $\mathcal{O} = \exp[-\epsilon_2\phi(z)]$ that we associate to:



Let us test its gauge invariance. Composing the coproduct $\Delta_{\mathcal{A},\mathcal{W}_\infty}$ with $I_{0,0,0}$ produces an embedding $\Delta_{\mathcal{W}_\infty} : \mathcal{A} \to \mathcal{W}_\infty$ and the gauge-invariance condition simplifies to

$$\mathcal{O}|0\rangle \circ L_{1,0,0}(t_{a,b}) = S_{0,0,1}(\Delta_{\mathcal{W}_\infty}(t_{a,b})) \circ \mathcal{O}|0\rangle.$$

The operator $\mathcal{O}|0\rangle$ should intertwine the right action of $A_{1,0,0}$ and the action of $A_{1,0,0}$ via its image in $Y_{0,0,1}$.

4. Building blocks of gauge-invariant junctions

It is straightforward to check that indeed

$$\begin{aligned} \mathcal{O}|0\rangle \frac{1}{\epsilon_1} z^n &= J_n \mathcal{O}|0\rangle, \\ \mathcal{O}|0\rangle \epsilon_1 \partial^2 &= \left(V_{-2} + \frac{\sigma_3}{2} \sum_{n=-\infty}^{\infty} |n| : J_{-n-1} J_{n-1} : \right) \mathcal{O}|0\rangle \end{aligned}$$

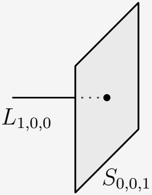
with relevant truncation maps given by

$$\begin{aligned} l_{1,0,0}(t_{0,n}) &= \frac{1}{\epsilon_1} z^n, \\ l_{1,0,0}(t_{2,0}) &= \epsilon_1 \partial^2, \\ l_{0,0,0} \otimes S_{0,0,1}(\Delta_{\mathcal{A},\mathcal{W}_\infty} t_{0,n}) &= J_n, \\ l_{0,0,0} \otimes S_{0,0,1}(\Delta_{\mathcal{A},\mathcal{W}_\infty}(t_{2,0})) &= V_{-2} + \frac{\sigma_3}{2} \sum_{n=-\infty}^{\infty} |n| : J_{-n-1} J_{n-1} : . \end{aligned}$$

4. Building blocks of gauge-invariant junctions

4.3. The vertex operator $\exp[\epsilon_2\phi(z)]$

Let us now look at the example of $\mathcal{O} = \exp[\epsilon_2\phi(z)]$ that we associate to the junction



The generator $t_{m,n}$ now acts trivially from the right but the action from the left becomes more complicated.

4. Building blocks of gauge-invariant junctions

It is straightforward to check that indeed

$$
\begin{aligned}
0 &= \left(\frac{z^n}{\epsilon_1} + J_n\right)\mathcal{O}|0\rangle, \\
0 &= \left(\epsilon_1\partial^2 + V_{-2} + \sigma_3\sum_{m=0}^{\infty} mJ_{-n-1}J_{n-1} + 2\epsilon_2\epsilon_3\sum_{m=0}^{\infty} mz^{m-1}J_{-n-1}\right)\mathcal{O}|0\rangle
\end{aligned}
$$

with relevant truncation maps given by

$$
L_{0,0,0}(t_{0,n}) = 0, \qquad L_{0,0,0}(t_{2,0}) = 0,
$$

$$
\begin{aligned}
L_{1,0,0}\otimes S_{0,0,1}(\Delta_{\mathcal{A},\mathcal{W}_\infty} t_{0,n}) &= \frac{1}{\epsilon_1}z^n + J_n, \\
L_{1,0,0}\otimes S_{0,0,1}(\Delta_{\mathcal{A},\mathcal{W}_\infty}(t_{2,0})) &= \epsilon_1\partial^2 + V_{-2} + \sigma_3\sum_{m=0}^{\infty} mJ_{-n-1}J_{n-1} \\
&\quad + 2\epsilon_2\epsilon_3\sum_{m=0}^{\infty} mz^{m-1}J_{-n-1}.
\end{aligned}
$$

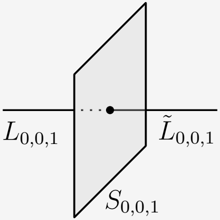
4. Building blocks of gauge-invariant junctions

4.4. The Miura operator $\epsilon_3\partial - \epsilon_1\epsilon_2 J(z)$

The last elementary junction can be identified with the Miura operator

$$\mathcal{O} = \epsilon_3\partial - \epsilon_1\epsilon_2 J(z)$$

that we associate to the intersection



Note also that intersection of $S_{0,0,1}$ with lines of different orientation, such as $L_{1,0,0}$, can be obtained by colliding the two endpoints discussed above. This observation turns out to lead to a new relation between Miura operators and degenerate field as we will see later.

4. Building blocks of gauge-invariant junctions

In the present case, both the left and the right line operators are non-trivial, leading to a condition combining the previous two cases

$$\begin{aligned} \mathcal{O}|0\rangle \frac{1}{\epsilon_3} z^n &= \left(\frac{1}{\epsilon_3} z^n + J_n \right) \mathcal{O}|0\rangle, \\ \mathcal{O}|0\rangle \epsilon_3 \partial^2 &= \left(\epsilon_3 \partial^2 + V_{-2} + \epsilon_1 \epsilon_2 \sum_{n=1}^{\infty} |n| J_{-n-1} z^{n-1} \right) \mathcal{O}|0\rangle \end{aligned}$$

that can be checked by a straightforward calculation.

5. Composing gauge-invariant junctions

5.1. Fusion in topological direction

Let us now discuss various examples of composing elementary junctions. The Miura operator serves in the literature as an object that produces algebras $Y_{0,0,N_3}$ by so-called Miura transformation. Let us first look at the Miura transformation for $N_3 = 2$. Taking a product of two Miura operators

$$\begin{aligned}&(\epsilon_3\partial - \epsilon_1\epsilon_2 J(z))(\epsilon_3\partial - \epsilon_1\epsilon_2\tilde{J}(z))\\ &= (\epsilon_3\partial)^2 \underbrace{-\epsilon_1\epsilon_2(J(z) + J(z))}_{U_1(z)}\epsilon_3\partial \underbrace{+(\epsilon_1\epsilon_2)^2 J_1 J_2(z) - \epsilon_1\epsilon_2\epsilon_3\partial\tilde{J}}_{U_2(z)},\end{aligned}$$

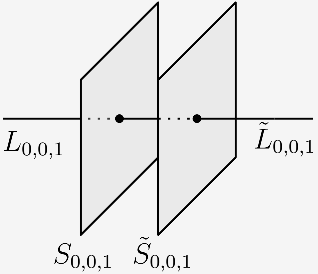
one can show that fields U_1, U_2 generate $Y_{0,0,2}$, i.e. the Virasoro algebra tensored with the $\mathfrak{gl}(1)$ current algebra. The Miura transformation above gives a coproduct $Y_{0,0,2} \to Y_{0,0,1} \otimes Y_{0,0,1}$.

5. Composing gauge-invariant junctions

From our new perspective, it is natural to expect that

$$\mathcal{O} = (\epsilon_3 \partial - \epsilon_1 \epsilon_2 J(z))(\epsilon_3 \partial - \epsilon_1 \epsilon_2 \tilde{J}(z))$$

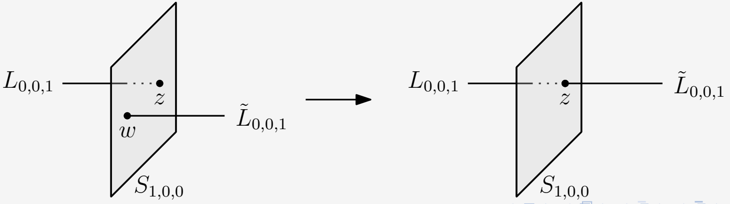
describes a junction between $S_{0,0,2}$ coming from the fusion of $S_{0,0,1}$ and $\tilde{S}_{0,0,1}$ with the line $L_{0,0,1}$ as in the figure.



Analogously to the simple example of a single Miura operator, $\mathcal{O}|0\rangle$ indeed satisfies the gauge-invariance condition with $Y_{0,0,2}$ generators acting via $Y_{0,0,2} \to Y_{0,0,1} \otimes Y_{0,0,1}$ from the Miura transformation.

5. Composing gauge invariant junctions

We have just seen that composition of two elementary Miura operators produces a gauge-invariant junction between $S_{0,0,2}$ and $L_{0,0,1}$. Analogously, composing N_3 elementary Miura operators produces a gauge-invariant junction between $S_{0,0,N_3}$ and $L_{0,0,1}$ on which the $Y_{0,0,N_3}$ algebra acts via $Y_{0,0,N_3} \to (Y_{0,0,1})^{N_3}$ coming from the Miura transformation involving N_3 elementary Miura operators. To obtain an operator associated to the general intersection of S_{N_1,N_2,N_3} with $L_{0,0,1}$, we need to work a bit harder. Let us start with $S_{1,0,0}$, where the line $L_{0,0,1}$ can actually split into $\exp[-\epsilon_2\phi(z)]$ associated to $L_{0,0,1}$ ending from the right and $\exp[\epsilon_2\phi(w)]$ associated to the line $L_{0,0,1}$ ending from the left.



5. Composing gauge-invariant junctions

By taking a contour integral of the product of the two vertex operators

$$\oint dw \exp[\epsilon_2\phi(z)]\exp[-\epsilon_2\phi(w)] = \oint \frac{dw}{(w-z)^{\frac{\epsilon_1}{\epsilon_3}+1}}\exp[\epsilon_2\phi(z)-\epsilon_2\phi(w)],$$

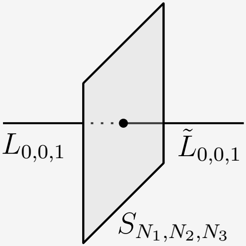
we can be re-written as

$$\begin{aligned} &\propto\ :\exp[\epsilon_2\phi(z)](\epsilon_3\partial)^{\frac{\epsilon_1}{\epsilon_3}}\exp[-\epsilon_2\phi(z)]: \\ &\propto\ (\epsilon_3\partial)^{\frac{\epsilon_1}{\epsilon_3}} - \epsilon_1\epsilon_2 J(z)(\epsilon_3\partial)^{\frac{\epsilon_3}{\epsilon_1}-1} \\ &\quad + \frac{\epsilon_1(\epsilon_1-\epsilon_3)}{2}\left(\epsilon_2^2 J^2(z) - \epsilon_2\partial J(z)\right)(\epsilon_3\partial)^{\frac{\epsilon_3}{\epsilon_1}-2} + \dots, \end{aligned}$$

and surprisingly recover the generalized Miura operator form [Procházka-MR (2018), Procházka (2018)] with fields U_i multiplying $(\epsilon_1\partial)^{\frac{\epsilon_3}{\epsilon_1}-i}$ satisfying OPEs of $\mathcal{W}_\infty$ and producing the truncation map $S_{1,0,0}:\mathcal{W}_\infty\to Y_{1,0,0}$. An analogous calculation produces the Miura operator associated to the gauge-invariant junction of $S_{0,1,0}$ with $L_{0,0,1}$.

5. Composing gauge-invariant junctions

Composition of the elementary Miura operator describing the gauge-invariant junction of $L_{0,0,1}$ with $S_{0,0,1}$ and the generalized Miura operators describing the gauge-invariant junction of $L_{0,0,1}$ with $S_{1,0,0}$ and $S_{0,1,0}$ produces a (conjecturally) gauge-invariant intersection of $L_{0,0,1}$ with general S_{N_1,N_2,N_3}!

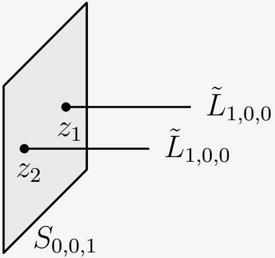


The standard Miura transformation can be thought of as probing a system of M5-brane algebras by a an $L_{0,0,1}$ probe. This explains why the description of $\mathcal{W}_\infty$ in terms of the U_i basis from Miura operators breaks the explicit triality symmetry of $\mathcal{W}_\infty$.

5. Composing gauge-invariant junctions

5.2. Fusion in holomorphic direction

I would like to now briefly discuss fusion of line operators in the holomorphic direction. The simplest example would be the one in the figure



with the gauge-invariant operator

$$\mathcal{O} = \exp[-\epsilon_2 \phi(z_1)] \exp[-\epsilon_2 \phi(z_2)].$$

To probe the gauge-invariance, we need to identify algebras A_{n_1,n_2,n_3}.

5. Composing gauge-invariant junctions

Analogously to the coproduct producing Y_{N_1,N_2,N_3}, there exists a coproduct that can be used to find an explicit realization of A_{n_1,n_2,n_3} starting from $A_{1,0,0}$, $A_{0,1,0}$ and $A_{0,0,1}$. Let me skip the details and write directly:

$$
\begin{aligned}
t_{0,n} &= \epsilon_1^{-1}\sum_{i=1}^{n_1} z_i^d + \epsilon_2^{-1}\sum_{i=1}^{n_2}(z_i')^d + \epsilon_3^{-1}\sum_{i=1}^{n_3}(z_i'')^d, \\
t_{2,0} &= \epsilon_1\sum_{i=1}^{n_1}\partial_{z_i}^2 + \frac{\epsilon_2\epsilon_3}{\epsilon_1}\sum_{i<j}\frac{2}{(z_i-z_j)^2} + \epsilon_1\sum_{i,j}\frac{2}{(z_i'-z_j'')^2} + \\
&\quad \epsilon_2\sum_{i=1}^{n_2}\partial_{z_i'}^2 + \frac{\epsilon_1\epsilon_3}{\epsilon_2}\sum_{i<j}\frac{2}{(z_i'-z_j')^2} + \epsilon_2\sum_{i,j}\frac{2}{(z_i-z_j'')^2} + \\
&\quad \epsilon_3\sum_{i=1}^{n_3}\partial_{z_i''}^2 + \frac{\epsilon_1\epsilon_2}{\epsilon_3}\sum_{i<j}\frac{2}{(z_i''-z_j'')^2} + \epsilon_3\sum_{i,j}\frac{2}{(z_i-z_j')^2}.
\end{aligned}
$$

Note that $t_{n,0}$ form a system of mutually commuting differential operators and generalizes the well-known Calogero-Moser integrable system!

5. Composing gauge-invariant junctions

Going back to our figure, we can compose two vertex operators at a given location and show that

$$\mathcal{O} = \exp[-\epsilon_1 \phi(z_1)] \exp[-\epsilon_1 \phi(z_2)]$$

satisfies

$$J_n \mathcal{O}|0\rangle = \mathcal{O}|0\rangle \frac{z_1^n + z_2^n}{\epsilon_1}$$

and

$$\left(\frac{\epsilon_1^2 \epsilon_2^2}{2} (J^3)_{-2} + \frac{\sigma_3}{2} \sum_{n=-\infty}^{\infty} |n| : J_{-n-1} J_{n-1} : \right) \mathcal{O}|0\rangle$$
$$= \mathcal{O}|0\rangle \left(\epsilon_1 \partial_1^2 + \epsilon_1 \partial_1^2 + \frac{\epsilon_2 \epsilon_3}{\epsilon_1} \frac{2}{(z_1 - z_2)^2} \right)$$

as expected from the gauge-invariance condition.

5. Composing gauge-invariant junctions

Analogously, we can fuse n_1 lines with orientation along $\mathbb{C}_{\epsilon_1}$ and n_2 lines with orientation along $\mathbb{C}_{\epsilon_2}$ ending from the right to produce an operator intertwining the action of $t_{m,n}$ embedded inside $Y_{0,0,1}$ with the action of the algebra $A_{n_1,n_2,0}$ realized in terms of the differential operators above. Generally, we expect that this leads to a gauge-invariant endpoint of $L_{n_1,n_2,0}$ on $S_{0,0,1}$. To identify the more complicated algebras A_{n_1,n_2,n_3}, one would need to consider more complicated surface operators, such as $S_{0,1,1}$. With the elements above, one can study more complicated junctions by combining Miura operators and vertex operators fused along topological or holomorphic direction and recover a very rich story that I do not have time to discuss in its full glory.

5. Composing gauge-invariant junctions

5.3. Connection to PT modules

Let me at least briefly mention a connection to the Pandaripande-Thomas box counting [Pandharipande-Thomas (2009)] of the topological vertex $C_{\lambda,\mu,\nu}$ [Aganagic-Klemm-Marino-Vafa (2003)] labelled by a triple of partitions λ,μ,ν. We have identified the gauge-invariant operator associated to the configuration of n lines with orientation along $\mathbb{C}_{\epsilon_1}$ and m lines with orientation along $\mathbb{C}_{\epsilon_2}$ ending from the right with

$$\mathcal{O}|0\rangle = \exp[-\epsilon_1\phi(z_1)]\ldots\exp[-\epsilon_1\phi(z_n)]\exp[-\epsilon_2\phi(\tilde{z}_1)]\ldots\exp[-\epsilon_2\phi(\tilde{z}_m)]|0\rangle.$$

The right action of $A_{n,m,0}$ and the action of modes of the $\mathfrak{gl}(1)$ current algebra generates a rather complicated module for both algebras. One can project to an $A_{n,m,0}$-module by taking an overlap the highest-weight state

$$\langle n\epsilon_1 + m\epsilon_2|\mathcal{O}|0\rangle = \prod_{m<n}(z_i - z_j)^{-\frac{\epsilon_1}{\epsilon_2}}\prod_{m,n}(z_i - \tilde{z}_j)^{-1}\prod_{m<n}(\tilde{z}_i - \tilde{z}_j)^{-\frac{\epsilon_2}{\epsilon_1}}.$$

5. Composing gauge-invariant junctions

One can check that the action of $A_{m,n,0}$ on $\langle n\epsilon_1 + m\epsilon_2|\mathcal{O}|0\rangle$ generates a nice highest-weight module with a basis labelled by Pandharipande-Thomas box configurations associated to the topological vertex $C_{(m),(n),0}$. A trivial example is $A_{1,0,0}$ acting on $\langle\epsilon_1|\mathcal{O}|0\rangle = 1$ and producing the module $\mathbb{C}[z]$ with the obvious action of $\frac{z^n}{\epsilon_1}$ and $\epsilon_1\partial^2$ and with character

$$C_{(1),0,0} = \mathrm{Tr}_{\mathbb{C}[z]} q^{z\partial} = \frac{1}{1-q} = 1 + q + q^2 + q^3 + \dots .$$

It turns out that one can generally define $\mathcal{A}$-modules labelled by a triple of partitions λ, μ, ν with a basis labelled by Pandharipande-Thomas box configurations. The gauge-invariance condition relating the action of $\mathcal{A}$ and $\mathcal{W}_\infty$ then leads to a categorification of the relation between PT and DT topological vertices due to Pandharipande and Thomas

$$\chi_{\lambda,\mu,\nu} = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n} C_{\lambda,\mu,\nu}.$$

6. Summary and outlook

To summarize:

- We identified the gauge-invariance condition satisfied by operators describing M2-M5 brane junctions in twisted M-theory.
- We re-interpret basic objects from the theory of VOAs (the Miura operator, degenerate fields) as building blocks of gauge-invariant junctions. This leads to a new perspective on these objects and their connection with Coulomb-branch algebras and integrable systems.
- Composition of junctions is consistent with the fusion of line and surface operators. This allows us to built more complicated junctions starting from the elementary ones.
- Endpoints of M2-branes ending on M5-branes lead to $\mathcal{W}_\infty$-modules labelled by a triple of partitions when viewed from the point of view of M5-branes and $\mathcal{A}$-modules labelled by a triple of partitions when viewed from the point of view of M2-branes. The connection between them categorifies the Pandharipande-Thomas conjectural relation between the DT and the PT topological vertex.

6. Summary

There are many possible directions one could take from here:

- Some of our claims remain somewhat conjectural and deserve further study. I expect our definition of PT modules to admit a simplification.
- A natural question is a generalization of our setup to configurations with M2 and M5 branes compactified on other toric three-folds.
- Our gauge-invariance condition (together with other results) should be derivable directly from a perturbative analysis in the 5d Chern-Simons theory.
- Some of the modules we construct should admit a geometric construction via correspondences acting on moduli spaces associated to our geometry.
- Our story should admit a generalization replacing $\mathbb{C}^2 \times \mathbb{R}$ by $\mathbb{C} \times \mathbb{C}^* \times \mathbb{R}$ or $\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{R}$ leading to trigonometric Calogero-Moser systems and more.
- M5 branes can be supported on either of the two $\mathbb{C}$'s inside $\mathbb{C}^2 \times \mathbb{R}$, a possibility that we have not investigated.