Towards geometric construction of quantum difference equations

$\mathcal{N}=4$ 3d theories : 3d mirror symmetry

$$\mathcal{T} \longleftrightarrow \mathcal{T}^!$$

theory mirror dual

correlation functions are the same : $\mathcal{T}$ and $\mathcal{T}^!$ are two "languages" to describe the same "physics"

In low energy limit: moduli spaces of vacua

$$X \rightleftarrows X^!$$

enumerative invariants are connected in a non-perturbative way.

Vertex functions - certain equivariant counts of rational curves. It satisfies difference equations in equivariant and Kähler variables.

3d mirror symmetry.

$X, X^!$ - symplectic resolutions of singularities

X $\qquad\qquad$ $X^!$

$\theta \in H^2(X, \mathbb{R})$ $\qquad\qquad$ $\theta^! \in H^2(X^!, \mathbb{R})$

$\hbar$ $\qquad\qquad$ $\hbar^!$

$A \subset \mathrm{Aut}(X, \omega_X)$ $\qquad\qquad$ $A^! \subset \mathrm{Aut}(X^!, \omega_{X^!})$

$K = \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{C}^*$ $\qquad\qquad$ $K^! = \mathrm{Pic}(X^!) \otimes_{\mathbb{Z}} \mathbb{C}^*$

$T = A \times \mathbb{C}^*_\hbar$ $\qquad\qquad$ $T^! = A^! \times \mathbb{C}^*_{\hbar^!}$

Assumption: X^{A}, $(X^!)^{A^!}$ are finite.

Main examples include Nakajima quiver varieties of finite and affine type, resolutions of slices in affine Grassmannians of A_n-type, bow varieties, ...

We say that X and $X^!$ are 3d-dual if

① $\exists$ isomorphism κ: $A \to K^!$, $K \to A^!$, $\mathbb{C}^\times_\hbar \to \mathbb{C}^\times_{\hbar^!}$

(3d symmetry exchanges equivariant and Kähler variables)

Denote $\sigma = dk^{-1}(\theta^!)$ $\quad \sigma^! \in dk(\theta)$

$A \xrightarrow{k} K^! \ni e^{\theta^!}$ $\quad \cap \qquad \cap$

$e^{\theta} \in K \xrightarrow{k} A^!$ $\quad \mathrm{Lie}_{\mathbb{R}}(A) \qquad \mathrm{Lie}_{\mathbb{R}}(A^!)$

σ and $\sigma^!$ define decompositions of TX and $TX^!$ to attracting and repelling directions.

$$\mathrm{Attr}_\sigma(p) = \{ x \in X \mid \lim_{t \to 0} \sigma(t) \cdot x = p \};$$

$p \in \mathrm{Attr}^f_\sigma(p)$ - smallest closed subset closed under Attr. $\Rightarrow$

$\Rightarrow$ there is partial ordering on fixed points

$$p_1 \geqslant p_2 \iff p_2 \in \mathrm{Attr}^f_\sigma(p_1)$$

② $\exists$ a bijection $X^A \xrightarrow{\sim} (X^!)^{A^!}$

$p \mapsto p^!$

inversing the partial order.

$E = \mathbb{C}^*/q^{\mathbb{Z}}$ - elliptic curve. ;

$\mathrm{Ell}_T(X)$ - equivariant elliptic cohomology.

$$\mathrm{Ell}_T(pt) = T/q^{\mathrm{Hom}(\mathbb{C}^* \to T)} \cong E^{\dim T}.$$

extended $$E_T(X) = \mathrm{Ell}_T(X) \times \underset{\displaystyle K/q^{\mathrm{Hom}(\mathbb{C}^* \to K)}.}{\underset{\parallel}{\mathrm{Ell}_{T^!}(pt)}}$$

Mina Aganagic and Andrei Okounkov constructed

$\mathrm{Stab}_{\sigma}^{X,\,ell}(p)$ - a section of certain line bundle over $E_T(X)$

twisted version $\underline{\mathrm{Stab}}_{\sigma}^{X,\,ell}(p) = \Theta(N_{p^!}^{-}) \cdot \mathrm{Stab}_{\sigma}^{X,\,ell}(p)$

$$\underline{\mathrm{Stab}}_{\sigma^!}^{X^!,\,ell}(p) = \Theta(N_p^{-}) \cdot \mathrm{Stab}_{\sigma^!}^{X^!,\,ell}(p^!)$$

$$\underline{\mathrm{Stab}}(p)\big|_p = \underline{\mathrm{Stab}}(p^!)\big|_{p^!} = \Theta(N_p^{-}) \cdot \Theta(N_{p^!}^{-}).$$

③ There is a bundle $\begin{array}{c}\mathcal{M}\\ \downarrow\\ \mathrm{Ell}_{T\times T^!}(X\times X^!)\end{array}$

and a section m such that

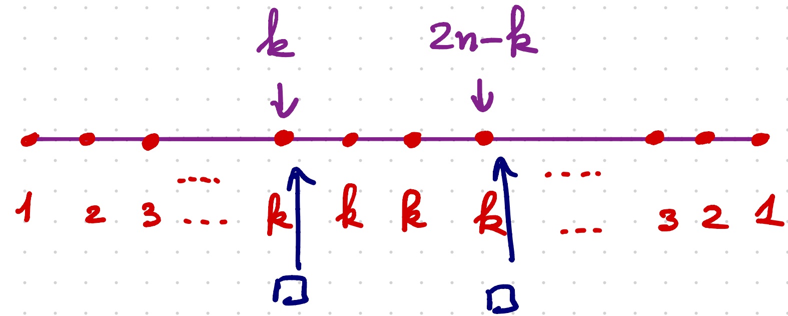
$$\begin{tikzcd}
 & \mathrm{Ell}_{T\times T^!}(X\times X^!) & \\
E_T(X) \arrow[ur, "i_{p^!}"] & & E_{T^!}(X^!) \arrow[ul, "i_p"']
\end{tikzcd}$$

$$i_{p^!}^* m = \underline{\mathrm{Stab}}_{G}^{X,\mathrm{ell}}(p) \qquad i_p^*(m) = \mathrm{Stab}_{G}^{X^!,\mathrm{ell}}(p^!)$$

Main examples:

$T^*Gr(k,n)$
$k \le \frac{n}{2}$

and



$$T^*(G/B) \longleftrightarrow T^*(G^L/B^L)$$

$$\bigsqcup_{n=0}^{\infty} \mathrm{Hilb}(\mathbb{C}^2, n)$$ - self - dual

Renormalize $\tilde{T}_{p,r}(z,a) = \dfrac{\underline{\mathrm{Stab}}_{6}^{X,\mathrm{ell}}(p)\big|_r}{\Theta(N_r^-)\,\Theta(N_{p^!}^-)}$

$$\begin{pmatrix} 1 & & * \\ & 1 & \\ & & \ddots \\ 0 & & & 1 \end{pmatrix}$$

$$\tilde{T}^{X}_{p,r}(z,a) = \kappa^*\left(\tilde{T}^{X^!}_{r^!,p^!}(z,a)\right)$$

matrix of elliptic functions

$$f(z,a) = \frac{\theta(a,z)}{\theta(a)\,\theta(z)}$$ Poincare bundle

$$\theta(x) = \left(x^{1/2} - x^{-\frac{1}{2}}\right)\prod_{j=1}^{\infty}(1-q^j x)(1-q^j/x)$$

Observe:

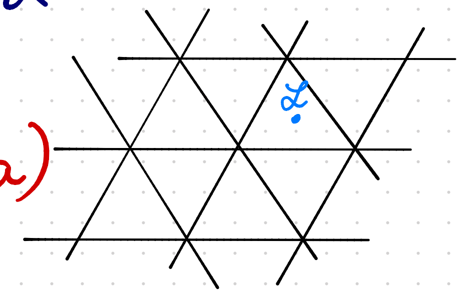
$$\lim_{q\to 0} f(zq^s, a) = \begin{cases} -\dfrac{a^{-\lfloor s\rfloor}}{1-a}, & s \notin \mathbb{Z} \\[2ex] -\dfrac{1-za}{(1-z)(1-a)}\, a^{-s}, & s \in \mathbb{Z}. \end{cases}$$

$$\theta(x) = \sum_{n\in\mathbb{Z}} x^{n+\frac{1}{2}}\, q^{\frac{n(n+1)}{2}} (-1)^n$$

leading term: minimize $s\left(n+\frac{1}{2}\right) + \frac{n(n+1)}{2}$ for $n \in \mathbb{Z}$

limit

- piecewise constant function of $s \in \mathbb{R}$
- changes when s crosses walls at $\mathbb{Z} \subset \mathbb{R}$
- for regular s the limit does not depend on s



Theorem (Mina Aganagic, Andrei Okounkov):

If $\mathcal{L} \in \mathrm{Pic}(X) \otimes \mathbb{R}$, then $\lim\limits_{q \to 0} \tilde{T}_{p,r}(z q^{\mathcal{L}}, a)$

1) piecewise constant function of $\mathcal{L}$
2) changes when $\mathcal{L}$ crosses $\mathrm{Walls}(X) \subset H^2(X, \mathbb{R})$
3) for regular $\mathcal{L} \in H^2(X, \mathbb{R}) \setminus \mathrm{Walls}(X)$ the limit does not depend on Kähler parameters:

$$\lim T_{p,r}(z q^{\mathcal{L}}, a) = \tilde{A}^{\mathcal{L}, X}_{p,r}, \quad \tilde{A}^{\mathcal{L}, X}_{p,r} = \frac{\mathrm{Stab}_{\mathfrak{C}}^{\mathcal{L}, X, k\text{-th}}(p)\big|_r}{\mathrm{Stab}_{\mathfrak{C}}^{\mathcal{L}, X, k}(r)\big|_r}$$

For $w \in \mathrm{Lie}_{\mathbb{R}} A$ define $\gamma_w = \langle e^{2\pi i w} \rangle \subset A$;

$$\mathrm{Res}(X) = \{ w \mid X^{\gamma_w} \neq X^A \} \subset \mathrm{Lie}_{\mathbb{R}}(A)$$

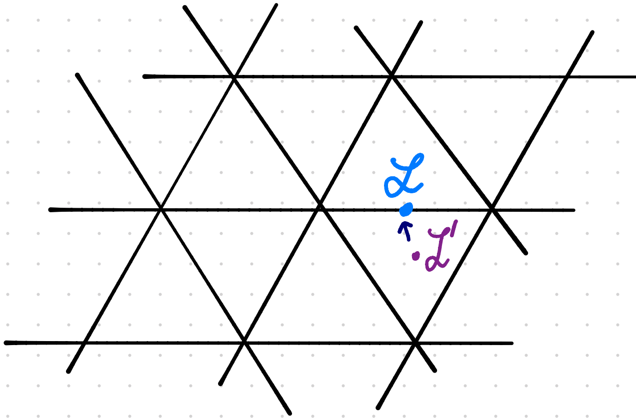
Theorem (Andrey Smirnov, Ya.K.): 3d-mirror symmetry switches walls with resonances

$$\mathrm{Res}(X) = \mathrm{Walls}(X^!) , \quad \mathrm{Res}(X^!) = \mathrm{Walls}(X) ;$$

$\lim\limits_{q \to 0} \tilde{T}_{p,r}(z, aq^w)$ – piecewise constant function, changing on $\mathrm{Res}(X)$.

for generic w: $\lim\limits_{q \to 0} \tilde{T}_{p,r}(z, aq^w) = \tilde{Z}^{w, X^!, k}_{r^!, p^!} = \dfrac{\mathrm{Stab}^{w, X^!, k}_{\mathfrak{S}^!}(p^!)|_{r^!}}{\mathrm{Stab}^{w, X^!, k}_{\mathfrak{S}^!}(r^!)|_{r^!}}$

What about a limit to a wall?



<u>Theorem</u> (Andrey Smirnov, Ya.K.)

Let $\mathcal{L} \in \mathrm{Pic}(X)\otimes\mathbb{R}$ and $\mathcal{L}'$ – arbitrary slope in infinitezimal neighborhood of $\mathcal{L}$. Then

$$\lim_{q\to 0} \tilde{f}(zq^{\mathcal{L}}, a) = \tilde{Z}'' \cdot \tilde{A}^{\mathcal{L}', X}$$

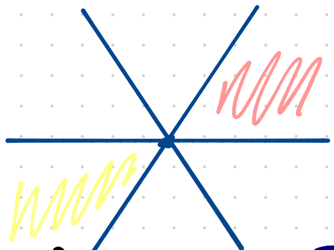
K-theoretic stable envelope of X with slope $\mathcal{L}'$

$$\tilde{Z}'' = \mathcal{L}\cdot Z' \cdot \mathcal{L}^{-1}$$

depends only on z.

How to describe $\tilde{Z}^4$ geometrically?

$\mathcal{U}_0 \subset H^2(X,\mathbb{R}) \supset$ Walls.



$D_+(X)$ ample slopes

$D_-(X)$ anti-ample slopes

$$D_\pm(X^!) = d_\kappa(\pm \mathcal{C}) \cap \mathcal{U}_0 \subset H^2(X^!,\mathbb{R})$$

If $\mathcal{L} \in \mathrm{Walls}(X) \Rightarrow \kappa(\mathcal{L}) \in \mathrm{Res}(X) \Rightarrow$

$\Rightarrow$ it corresponds to a nontrivial subvariety $Y_{\mathcal{L}} \subset X^!$

<u>Theorem</u> (Andrey Smirnov, Ya.K.): If $\mathcal{L}' = \mathcal{L} + \varepsilon$, $\varepsilon \in D_+(X)$, then $Z'(z)$ is conjugate to K-theoretic stable envelopes of $Y_{\mathcal{L}}$ with small ample slope.

$$\tilde{Z}' = H \tilde{Z} H^{-1};$$

$$\tilde{Z} = \frac{\mathrm{Stab}_{\sigma^!}^{D_+(Y_{\mathcal{L}}),\, Y_{\mathcal{L}},\, k}(p^!)\Big|_{r^!}}{\mathrm{Stab}_{\sigma^!}^{D_+(Y_{\mathcal{L}}),\, Y_{\mathcal{L}},\, k}(p^!)\Big|_{p^!}};$$

H = diagonal matrix of $\hbar$

Equivariant limit: X, $w \in A$

from $\lim_{q \to 0} \mathrm{Stab}^{\mathrm{ell}}(aq^{w}, z)$ one

can extract K-theoretic stable envelopes of

X^{A_w} of slope 0.

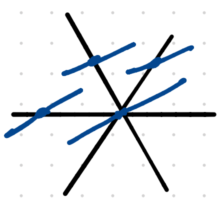
Wall R-matrices

$\mathcal{L} \in H^2(X,\mathbb{R})$; $\varepsilon \in D_+(X)$

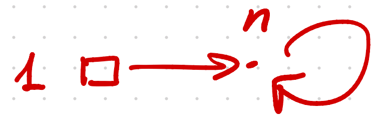
$$R^X(\mathcal{L},\sigma) = \left(\mathrm{Stab}_\sigma^{\mathcal{L}-\varepsilon, X, k}\right)^{-1} \cdot \left(\mathrm{Stab}_\sigma^{\mathcal{L}+\varepsilon, X, k}\right)$$

Theorem (Andrey Smirnov, Ya.K.):

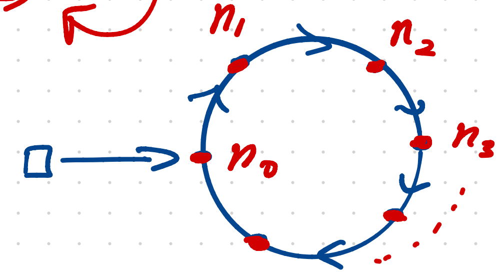
$$R^X(s,\sigma) \sim R^{Y_s}(0,-\sigma^!)^{-1}$$



Main application: $X = \mathrm{Hilb}(\mathbb{C}^2, n)$



$$X \simeq X^!$$



$\mathrm{Pic}(X) \simeq \mathbb{Z}$ generated by $\mathcal{O}(1)$

$$\mathrm{Walls}(X) = \left\{ \frac{a}{b} \in \mathbb{Q},\ |b| \le n \right\}$$

$$s = \frac{a}{b} \Rightarrow Y_s = \bigsqcup_{n_0 + \dots + n_{b-1} = n} X(n_0, n_1, \dots, n_{b-1})$$

$$\mathrm{Fock} = \bigoplus_{n=0}^{\infty} K\left((X^!)^{\mathbb{Z}_n} \right) \circlearrowleft U_\hbar \widehat{\mathfrak{gl}}_\ell$$

Stable envelopes with small $\frac{\text{ample}}{\text{anti-ample}}$ slopes

$\updownarrow$

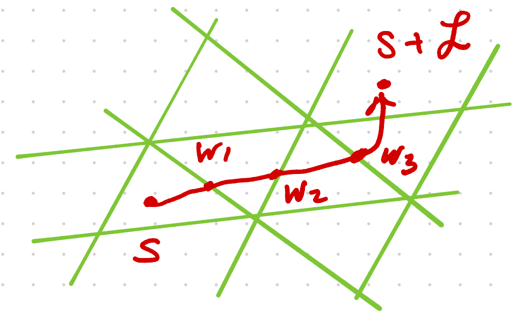
$\frac{\text{standard}}{\text{costandard}}$ basis in Fock.

Corollary (Eugene Gorsky, Andrei Neguț conjecture)

Wall R-matrix for Hilb for slope $\frac{a}{b}$ is conjugate to the transition matrix between standard and costandard bases for $U_\hbar \widehat{\mathfrak{gl}}_\ell$

Quantum difference equation

$$\Psi(q^{\mathscr{L}} z)\,\mathscr{L} = M_{\mathscr{L}}(z)\,\Psi(z)$$



$$M_{\mathscr{L}} = B_{w_1}(z)\,B_{w_2}(z)\cdots$$

$$\mathrm{Mon}(z) = \overleftrightarrow{\prod_{w}} B_w(z)$$

Idea: invent a limit

$$\mathrm{Mon}(z) \xrightarrow[q\to 0]{} B_w(z)$$

Theorem (Andrey Smirnov, Ya. K.): The wall-crossing operator $B_{a/b}(z)$ for $\mathrm{Hilb}(\mathbb{C}^2, n)$ in the stable basis for slope a/b coincides with the R-matrix for $U_\hbar(\widehat{\mathfrak{gl}}_b)$ in the basis of fixed points.

$$\mathrm{Mon} = \left(\mathrm{Stab}^{ell}_{flop}\right)^{-1} \cdot \mathrm{Stab}^{ell}$$

$$B_w = \lim_{q\to 0} \mathrm{Mon}(z q^w) = \lim \left({\mathrm{Stab}^{ell}_{flop}}^{-1} \right) \cdot \lim \left(\mathrm{Stab}^{ell} \right)$$

$$\lim \left({\mathrm{Stab}^{ell}_{flop}}^{-1} \right) = \left(\mathrm{Stab}^{X,\,k\text{-}th}_{\mathcal{L}-\varepsilon}\right)^{-1} \cdot \left(\mathrm{Stab}^{(X^!)^\vee,\,K}_{\mathfrak{C}}\right)^{-1}$$

$$\lim \left(\mathrm{Stab}^{ell} \right) = \left(\mathrm{Stab}^{(X^!)^\vee,\,K}_{-\mathfrak{C}}\right) \cdot \left(\mathrm{Stab}^{X,\,k\text{-}th}_{\mathcal{L}+\varepsilon}\right)$$

R-matrix for $(X^!)^\vee$

Similarly one can construct shift operators

by considering $\mathrm{Stab}_{-e}^{\mathfrak{ll}-1} \circ \mathrm{Stab}_{e}$.