Cluster structure on K-theoretic Coulomb branches

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Warm-up: Springer theory

G — Lie group, $\mathfrak{g} = Lie(G)$, $B \subset G$ — Borel subgroup,
$\mathcal{B} \simeq G/B$ — flag variety, $\mathcal{N} \subset \mathfrak{g}$ — nilpotent cone.

$$T^*\mathcal{B} \simeq \widetilde{\mathcal{N}} := \{(x, \mathfrak{b}) \,|\, x \in \mathcal{N},\ \mathfrak{b} \ni x\}$$

Steinberg variety:

$$Z := \widetilde{\mathcal{N}} \times_{\mathcal{N}} \widetilde{\mathcal{N}} = \left\{(x, \mathfrak{b}, \mathfrak{b}') \,|\, x \in \mathfrak{b} \cap \mathfrak{b}'\right\} \subset T^*\mathcal{B} \times T^*\mathcal{B}$$

It carries a diagonal action of $G \times \mathbb{C}^\times$, where $\mathbb{C}^\times$ acts by dilation along the fibers.

Let $K_\bullet^{G\times\mathbb{C}^\times}(Z)$ be the equivariant K-theory of the Steinberg variety. The convolution product endows it with an algebra structure, and

$$K_\bullet^{G\times\mathbb{C}^\times}(Z) \simeq \mathcal{H}_{\mathrm{aff}},$$

where $\mathcal{H}_{\mathrm{aff}}$ is the affine Hecke algebra.

Variety of triples

Braverman, Finkelberg, and Nakajima define K-theoretic Coulomb branches (of $4d$ $\mathcal{N}=2$ SUSY gauge theories compactified on a circle) in the spirit of "generalized affine Springer theory".

G — complex reductive group, N — its complex representation
Set $\mathcal{K}=\mathbb{C}((z))$, $\mathcal{O}=\mathbb{C}[[z]]$
$\mathrm{Gr}_G = G(\mathcal{K})/G(\mathcal{O})$ — affine Grassmannian.

Variety of triples:

$$\mathcal{R}_{G,N} = \left\{([g],s)\,|\,[g]\in\mathrm{Gr}_G,\, s\in N[[z]]\cap gN[[z]]\right\}$$

Similarly to the Steinberg variety, $\mathcal{R}_{G,N}$ has a convolution, and admits a $G(\mathcal{O})\rtimes\mathbb{C}^\times$-action, where $\mathbb{C}^\times$ acts via loop rotation.

K-theoretic Coulomb branches

Theorem (Braverman – Finkelberg – Nakajima)

$$\mathcal{A}^q_{G,N} := K_\bullet^{G(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathcal{R}_{G,N})$$

is an associative algebra, commutative at $q = 1$.

Definition

The K-theoretic Coulomb branch $\mathcal{M}_{G,N} = \mathrm{Spec}(\mathcal{A}^{q=1}_{G,N})$.

One can use the equivariant localization to obtain an embedding

$$\mathcal{A}^q_{G,N} \hookrightarrow (\mathcal{A}^q_{T,0})^{loc},$$

where $T \subset G$ is the maximal torus, and $(\mathcal{A}^q_{T,0})^{loc}$ is isomorphic to an algebra of q-difference operators in variables Λ_i, localized at root hyperplanes $\Lambda_i - \Lambda_j$.

Minuscule monopole operators

How does one describe $\mathcal{A}^q_{G,N}$ explicitly?

First, there is a commutative subalgebra

$$K_\bullet^{G(\mathcal{O})\rtimes\mathbb{C}^\times}(pt) \subset \mathcal{A}^q_{G,N}$$

generated by symmetric functions in Λ_j.

Second, affine Grassmannian admits stratification

$$\mathrm{Gr}_G = \bigsqcup_\lambda \mathrm{Gr}^\lambda_G, \qquad \mathrm{Gr}^\lambda_G = G(\mathcal{O})[z^\lambda], \qquad \overline{\mathrm{Gr}^\lambda_G} = \sqcup_{\mu\le\lambda} \mathrm{Gr}^\mu_G,$$

where

$$\overline{\mathrm{Gr}^\lambda_G} \text{ is smooth } \iff \overline{\mathrm{Gr}^\lambda_G} = \mathrm{Gr}^\lambda_G \iff \lambda \text{ is minuscule.}$$

Using the natural projection

$$\pi\colon \mathcal{R}_{G,N} \longrightarrow \mathcal{R}_{G,0} = \mathrm{Gr}_G,$$

for λ minuscule one defines the **minuscule monopole operator**

$$[\mathcal{O}_{\mathcal{R}^\lambda}] \in \mathcal{A}^q_{G,N} \qquad \text{where} \qquad \mathcal{R}^\lambda = \pi^{-1}(\overline{\mathrm{Gr}^\lambda_G}).$$

Quiver gauge theories

Let $\Gamma = (\Gamma_0, \Gamma_1)$ be a quiver with the set of vertices Γ_0, and the set of arrows Γ_1, and V be a Γ_0-graded vector space. (For simplicity, until the very end of the talk we will consider quivers without framing.)

Let us set

$$G = \prod_{i \in \Gamma_0} GL(V_i), \qquad N = \prod_{i \to j} \mathrm{Hom}(V_i, V_j).$$

Let $\dim(V_i) = d_i$ for each node $i \in \Gamma_0$. We denote the n-th fundamental coweight of $GL(V_i)$ by $\varpi_{i,n}$, and define

$$\varpi^*_{i,n} = \varpi_{i,n} - \varpi_{i,d_i}.$$

Let λ be a general minuscule G-coweight, then its restriction to $GL(V_i)$ is either $\varpi_{i,n}$ or $\varpi^*_{i,n}$ for some $1 \le n \le d_i$.

Generators in quiver gauge theories

Consider the minuscule monopole operators

$$E_{i,n} = [\mathcal{O}_{\mathcal{R}^{\varpi_{i,n}}}] \qquad \text{and} \qquad F_{i,n} = [\mathcal{O}_{\mathcal{R}^{\varpi^*_{i,n}}}].$$

Theorem (Weekes'19)

Quantized Coulomb branch $\mathcal{A}^q_\Gamma = \mathcal{A}^q_{G,N}$ of a quiver gauge theory is generated by

- *all (dressed) minuscule monopole operators over $K_\bullet^{G(\mathcal{O})\rtimes\mathbb{C}^\times}(pt)$;*
- *(dressed) monopole operators $E_{i,1}$, $F_{i,1}$, where $i \in \Gamma_0$, over $K_\bullet^{G(\mathcal{O})\rtimes\mathbb{C}^\times}(pt) \otimes_{\mathbb{C}[q,q^{-1}]} \mathbb{C}(q)$.*

"Dressed" here means that the bundles $\mathcal{O}_{\mathcal{R}^\lambda}$ are twisted by a wedge power of a tautological bundle.

Under localization, the operator $E_{i,n}$ reads

$$E_{i,n} \longmapsto \sum_{\substack{J\subset\{1,\dots,d_i\}\\ |J|=n}} \prod_{j\leftarrow i} \left(\prod_{r\in J} \prod_{s=1}^{d_j} \left(1 + q\Lambda_{i,r}\Lambda_{j,s}^{-1}\right) \right) \cdot \mathfrak{D}_{i,J}$$

where

$$\mathfrak{D}_{i,J} = \prod_{r\in J} \prod_{s\notin J} (1 - \Lambda_{i,s}\Lambda_{i,r}^{-1})^{-1} D_{i,r}.$$

Note that $E_{i,n}$ takes especially simple form if $i \in \Gamma_0$ is a sink:

$$E_{i,n} \longmapsto \sum_{\substack{J\subset\{1,\dots,d_i\}\\ |J|=n}} \mathfrak{D}_{i,J}, \qquad E_{i,1} \longmapsto \sum_{r=1}^{n} \prod_{s\neq r} (1 - \Lambda_{i,s}/\Lambda_{i,r})^{-1} D_{i,r}.$$

Same goes for $F_{i,n}$, but it is the simplest when $i \in \Gamma_0$ is a source.

Example: sDAHA

Let Γ be a quiver with one vertex and one loop, $V = \mathbb{C}^d$. Then $G = GL(d)$ and $N = \mathrm{End}(\mathbb{C}^d)$ is the adjoint representation.

$$\mathcal{A}^q_{T,0} \simeq \frac{\mathbb{C}[q^{\pm 1}] \left\langle \Lambda_i, D_i \right\rangle_{i=1}^d}{D_i \Lambda_j = q^{\delta_{ij}} \Lambda_j D_i}$$

$$K^T_\bullet(pt) \simeq \mathbb{C} \left\langle \Lambda_i \right\rangle_{i=1}^d \qquad \text{and} \qquad K^G_\bullet(pt) \simeq \Big(K^T_\bullet(pt) \Big)^{S_d}$$

$$[\mathcal{O}_{\mathcal{R}^{\omega_n}}] = \sum_{\substack{J \subseteq \{1,\dots,d\} \\ |J| = n}} \prod_{r \in J} \prod_{s \notin J} \frac{t\Lambda_r - \Lambda_s}{\Lambda_r - \Lambda_s} D_r$$

Note that

$$\mathcal{A}^q_\Gamma \simeq sDAHA(GL_n).$$

Coulomb branches and clusters

Conjecture (Gaiotto)

K-theoretic Coulomb branches are cluster varieties.

In physics, the Coulomb branch $\mathcal{M}_{\mathcal{C}} = \mathrm{Spec}(\mathcal{A}_{G,N})$ is the Coulomb branch of moduli of vacua in a $4d$ $\mathcal{N} = 2$ G-gauge theory on $\mathbb{R}^3 \times S^1$.

BPS quiver of the theory $\longleftrightarrow$ quiver of the cluster variety.

Theorem (Schrader-S. 'in progress)

For each quiver Γ, there is a quantum cluster variety $\mathbb{L}^q_{\mathcal{Q}_\Gamma}$ and an embedding

$$\iota\colon \mathcal{A}^q_\Gamma \hookrightarrow \mathbb{L}^q_{\mathcal{Q}_\Gamma}.$$

Under ι, operators $E_{i,n}$ and $F_{i,n}$ become cluster monomials, provided that $i \in \Gamma_0$ has no adjacent loops in Γ_1.

Cluster varieties

A **cluster variety** is an affine Poisson variety with an (in general infinite) collection of charts such that

- each chart is a torus $(\mathbb{C}^\times)^d$
- the Poisson brackets between toric coordinates are log-canonical:

$$\{y_i, y_j\} = \varepsilon_{ij} y_i y_j$$

- gluing data is given by subtraction-free rational expressions, the **cluster mutations.** A cluster mutation **in direction k** only affects y_k itself, and coordinates that have nontrivial Poisson brackets with y_k.

It is convenient to encode cluster charts by quivers with vertices corresponding to coordinates y_j, and ε being the adjacency matrix.

Quantum cluster charts

Let $\Lambda \simeq \mathbb{Z}^d$ be a lattice with a skew form $\langle \cdot, \cdot \rangle$. Define a quantum torus

$$\mathcal{T}^q = \mathbb{Z}[q^{\pm 1}] \left\langle Y_\lambda \right\rangle_{\lambda \in \Lambda}, \qquad q^{\langle \lambda, \mu \rangle} Y_\lambda Y_\mu = Y_{\lambda + \mu}.$$

A choice of basis $\{e_j\} \in \Lambda$ defines a quiver

$$\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1), \qquad \mathcal{Q}_0 = \{1, \dots, d\}, \qquad \#(i \to j) = \langle e_i, e_j \rangle$$

and the corresponding **quantum cluster chart**

$$\mathcal{T}^q_{\mathcal{Q}} = \mathbb{Z}[q^{\pm 1}] \left\langle Y_{e_i} \right\rangle_{i=1}^d, \qquad q^{\left\langle e_i, e_j \right\rangle} Y_{e_i} Y_{e_j} = Y_{e_i + e_j} = q^{\left\langle e_j, e_i \right\rangle} Y_{e_j} Y_{e_i}.$$

A **mutation** μ'_k in direction $k \in \mathcal{Q}_0$ is the change of basis

$$e'_i = \begin{cases} -e_k, & i = k \\ e_i + [\varepsilon_{ik}]_+ e_k, & i \neq k, \end{cases}$$

where $[a]_+ = \max(a, 0)$.

Quantum cluster varieties

To each mutation one associates a birational automorphism $\mu_k^\sharp = \mathrm{Ad}_{\Psi^q(Y_k)}$ of $\mathcal{T}^q$, where

$$\Psi^q(z) = \prod_{n=1}^{\infty}(1+q^{2n+1}z)^{-1}, \qquad \Psi^q(q^2z) = (1+qz)\Psi^q(z)$$

The **quantum cluster mutations** μ_k is the change of basis μ_k' followed by $\mu_k^\sharp$.

Example: for a quiver $\mathcal{Q} = \{1 \to 2\}$, we have $Y_2Y_1 = q^2Y_1Y_2$, $\mu_2(Y_2) = Y_2^{-1}$, and

$$\mu_2(Y_1) = \Psi^q(Y_2)Y_1\Psi^q(Y_2)^{-1} = Y_1(1+qY_2).$$

Definition

The **quantum cluster variety** $\mathbb{L}^q = \mathbb{L}^q_{\mathcal{Q}}$ is a subalgebra of $\mathcal{T}^q$, consisting of elements $a \in \mathcal{T}^q$, which stay in $\mathcal{T}^q$ under any finite sequence of cluster mutations.

Positive representations

Under parametrization

$$q = e^{\pi i b^2}, \qquad b^2 \in \mathbb{R}_{>0} \setminus \mathbb{Q}$$

there is a homomorphism

$$\mathcal{T}_{\mathcal{Q}}^q \to \mathcal{H} \qquad Y_j \mapsto e^{2\pi b \hat{y}_j}.$$

where $\mathcal{H} = \mathbb{C}\langle \hat{y}_1, \ldots, \hat{y}_d \rangle$ is the Heisenberg algebra with relations

$$[\hat{y}_j, \hat{y}_k] = (2\pi i)^{-1} \varepsilon_{jk}.$$

The Heisenberg algebra $\mathcal{H}$ has irreducible Hilbert space representation in which the generators Y_j act by (unbounded) positive self-adjoint operators. Half of them act by multiplication operators, another half as shifts.

Fock–Goncharov: if $|q| = 1$, pull-backs of these representations to $\mathbb{L}^q$ are unitary equivalent. This defines a canonical **positive** representation of $\mathbb{L}^q$.

Non-compact quantum dilogarithm

Problem: $\Psi^q(z)$ diverges at $|q| = 1$.

Replace $\Psi^q(z)$ by the **non-compact quantum dilogarithm** $\varphi(z)$. It is the unique solution to the **pair** of difference equations

$$\varphi(z - ib^{\pm 1}/2) = (1 + e^{2\pi b^{\pm 1} z})\varphi(z + ib^{\pm 1}/2).$$

Each $\hat{y}_k$ acts by a self-adjoint operator and

$$z \in \mathbb{R} \implies |\varphi(z)| = 1,$$

hence mutation in direction k gives rise to a **unitary** operator:

$$\text{quantum mutation in direction } k \longrightarrow \varphi(-\hat{y}_k)^{-1}$$

Note: Quantum dilogarithm satisfies the **pentagon identity**:

$$[\hat{p}, \hat{x}] = \frac{1}{2\pi i} \implies \varphi(\hat{p})\varphi(\hat{x}) = \varphi(\hat{x})\varphi(\hat{p} + \hat{x})\varphi(\hat{p})$$

Coxeter–Toda phase space

Back to Coulomb branches: how does one construct $\mathcal{Q}_\Gamma$ out of Γ?

Let us consider the simplest example: $G = GL_n$, $N = 0$.

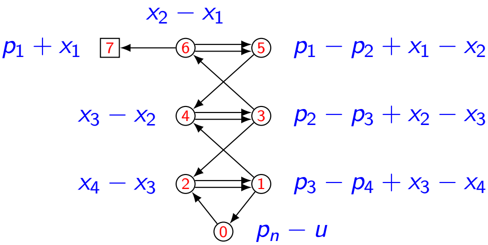
Theorem (Bezrukavnikov – Finkelberg – Mirković, '05)

Algebra $\mathcal{A}^q_{GL_n,0}$ is isomorphic to the quantized phase space of the GL_n Coxeter–Toda integrable system (a.k.a. quantum open relativistic Toda).

Theorem (Berenstein – Zelevinsky, '03)

The quantized phase space of the GL_n Coxeter–Toda integrable system is isomorphic to the quantum cluster algebra $\mathbb{L}^q_{\mathcal{Q}}$ with the quiver shown on the next slide.

The Coxeter–Toda quiver:



Heisenberg algebra: $\mathcal{H}_n = \mathbb{C}[q^{\pm 1}] \langle x_j, p_j \rangle_{j=1}^n$, $[p_j, x_k] = (2\pi i)^{-1}\delta_{jk}$, p_j acts on $L^2(\mathbb{R}^n)$ via

$$p_j \mapsto \frac{1}{2\pi i}\frac{\partial}{\partial x_j}.$$

For example,

$$Y_6 \mapsto e^{2\pi b(x_2 - x_1)} \qquad \text{and} \qquad Y_7 \mapsto e^{2\pi b(p_1 + x_1)}$$

Baxter operator

Theorem (Schrader–S.)

Consider the **Baxter operator**

$$Q_n(u) = \varphi(u - p_n)\varphi(u - p_{n-1} + x_n - x_{n-1})\varphi(u - p_{n-1}) \dots \varphi(u - p_1)$$

obtained by mutating consecutively at $0, 1, 2, \dots, 2n-2$. *Then*

1. *Unitary operators* $Q_n(u)$ *satisfy*

$$[Q_n(u), Q_n(v)] = 0,$$

2. *If* $A_n(u) = Q_n(u - ib/2)Q_n(u + ib/2)^{-1}$, *then one can expand*

$$A_n(u) = \sum_{k=0}^{n} H_k U^k, \quad U := e^{2\pi bu}$$

and the commuting operators $H_1, \dots, H_n$ *quantize the* GL_n *Coxeter–Toda Hamiltonians.*

Dehn twist

Additionally, there is a **Dehn twist operator** realized as mutations at all even nodes postcomposed with $e^{\pi i(p_1^2+\cdots+p_n^2)}$:

$$\tau_n = e^{\pi i(p_1^2+\cdots+p_n^2)}\varphi(x_2 - x_1)\ldots\varphi(x_n - x_{n-1})$$

which commutes with the Baxter operator

$$[\tau_n, Q_n(u)] = 0$$

Problem: Construct complete set of joint eigenfunctions (a.k.a. the b-Whittaker functions) for operators $Q_n(u), \tau_n$.

b-Whittaker transform

Set $\mathcal{R}_n(u)$ to be the same as the Baxter operator $Q_n(u)$ but without the last mutation. We then define

$$\Psi_{\boldsymbol{\lambda}}(\boldsymbol{x}) := \mathcal{R}_n(c_b - \lambda_n) \dots \mathcal{R}_2(c_b - \lambda_2) \cdot e^{2\pi b(\boldsymbol{\lambda} \cdot \boldsymbol{x})},$$

where

$$\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n), \qquad \boldsymbol{x} = (x_1, \dots, x_n),$$

and $c_b = i(b + b^{-1})/2$.

Define the b-Whittaker transform as follows:

$$\mathcal{W}\colon L^2(\mathbb{R}^n) \longrightarrow L^2_{sym}(\mathbb{R}^n, m(\boldsymbol{\lambda})d\boldsymbol{\lambda}),$$
$$(\mathcal{W}[f])(\boldsymbol{\lambda}) = \int_{\mathbb{R}^n} \overline{\Psi^{(n)}_{\boldsymbol{\lambda}}(\boldsymbol{x})} f(\boldsymbol{x}) d\boldsymbol{x}$$

Intertwining properties

Theorem (Schrader–S.)

The b-Whittaker transform is a unitary equivalence. Moreover

$$\mathcal{W} \circ \tau = e^{\pi i(\lambda_1^2+\dots+\lambda_n^2)} \circ \mathcal{W},$$

$$\mathcal{W} \circ Q_n(u) = \prod_{j=1}^{n} \varphi(u-\lambda_j) \circ \mathcal{W},$$

$$\mathcal{W} \circ H_k^{(n)} = e_k(\Lambda^{-1}) \circ \mathcal{W},$$

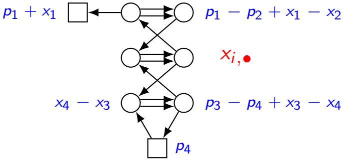
$$\mathcal{W} \circ Y_{2n-1} = \sum_{j=1}^{n} \prod_{k \neq j} \frac{1}{1-\Lambda_k/\Lambda_j} D_j \circ \mathcal{W},$$

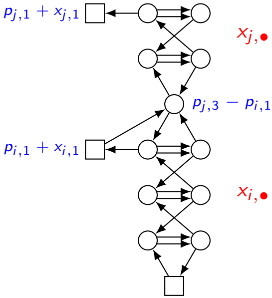
where e_k is the elementary symmetric function and $\Lambda = e^{2\pi b\lambda}$.

Converting quivers $\Gamma \rightsquigarrow \mathcal{Q}_\Gamma$

Let Γ be a gauge theory quiver. We associate to it a cluster quiver $\mathcal{Q}_\Gamma$ by the following rule.

- To each node $i \in \Gamma$ with label n_i, we associate a GL_{n_i} Coxeter–Toda quiver $\mathcal{Q}_i$;
- For each directed edge $e\colon i \to j$ in Γ_1, we **glue** the top of $\mathcal{Q}_i$ to the bottom of $\mathcal{Q}_j$ as shown.





From Coulomb branches to clusters

- Localization embeds $\mathcal{A}_\Gamma^q$ into rational q-difference operators in $\Lambda_{i,j}$, with $i \in \Gamma_0$ and $1 \leq j \leq d_i$;
- Applying inverse b- Whittaker transform $\mathcal{W}^{-1}$ at each node we embed $\mathcal{A}_\Gamma^q$ into polynomial q-difference operators;
- Need to construct a map $\mathcal{A}_\Gamma^q \to \mathcal{T}_{\mathcal{Q}_\Gamma}^q$ which makes the diagram commutative with respect to positive representation of $\mathcal{T}_{\mathcal{Q}_\Gamma}^q$;
- Need to show that images of minuscule monopole operators land in the universally Laurent algebra $\mathbb{L}_{\mathcal{Q}_\Gamma}^q$.

Then at each node we send

- $e_k(\Lambda^{-1}) \longmapsto H_k^{(d_i)}$ (in cluster terms, $H_1^{(n)} = \sum_{j=0}^{2n-2} Y_{e_0+\cdots+e_j}$);
- $E_{i,1}$ to Y_{2n-1} if $i \in \Gamma_0$ is a sink (and similar formula for $F_{i,1}$);
- "dressing" is achieved by applying Dehn twists τ_{d_i}.

It is easy to express $H_k^{(d_i)}$ and Y_{2n-1} in the so-called cluster $\mathcal{A}$-variables, certain elements in $\mathbb{L}_{\mathcal{Q}_\Gamma}^q$.

Orientation of quivers

Note: at this point we'd be (almost) done if we knew how to swap orientation of arrows in Γ.

On the Coulomb side, changing the arrow $i \to j$ to $j \to i$ corresponds to conjugating monopole operators by

$$\prod_{r=1}^{n_i} \prod_{s=1}^{n_j} \varphi(\lambda_{j,s} - \lambda_{i,r}). \qquad (*)$$

Need to find a sequence of mutations that acts on the product

$$\overline{\Psi}^{(n_i)}_{\boldsymbol{\lambda}_{i,\bullet}}(\boldsymbol{x}_{i,\bullet}) \overline{\Psi}^{(n_j)}_{\boldsymbol{\lambda}_{j,\bullet}}(\boldsymbol{x}_{j,\bullet})$$

with the eigenvalue $(*)$.

Baxter operators revisited

Recall, that for $d_j = 1$ we have already seen such a sequence, namely **the Baxter operator**

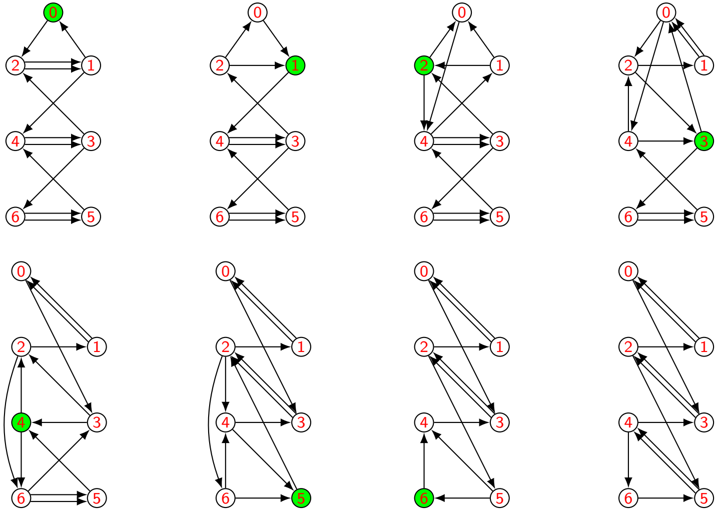
$$Q_n(\boldsymbol{x}, \boldsymbol{p}_{\boldsymbol{x}}; \mu)\Psi^{(n)}_{\boldsymbol{\lambda}}(\boldsymbol{x}) = \prod_{j=1}^{n} \varphi(\mu - \lambda_j)\Psi^{(n)}_{\boldsymbol{\lambda}}(\boldsymbol{x}).$$

Since $\Psi^{(1)}_{\lambda}(x) = e^{2\pi i \lambda x}$, we can also write

$$Q_n(\boldsymbol{x}, \boldsymbol{p}_{\boldsymbol{x}}; p_y)\Psi^{(n)}_{\boldsymbol{\lambda}}(\boldsymbol{x})\Psi^{(1)}_{\mu}(y) = \prod_{j=1}^{n} \varphi(\mu - \lambda_j)\Psi^{(n)}_{\boldsymbol{\lambda}}(\boldsymbol{x})\Psi^{(1)}_{\mu}(y).$$

Moreover, Baxter operator is a sequence of mutations.

Example: $G = PGL_4$



Bi-fundamental Baxter operator

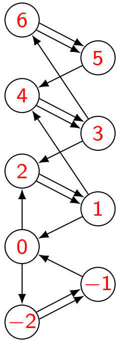


Figure: $\mathfrak{gl}_4 \times \mathfrak{gl}_2$ Toda

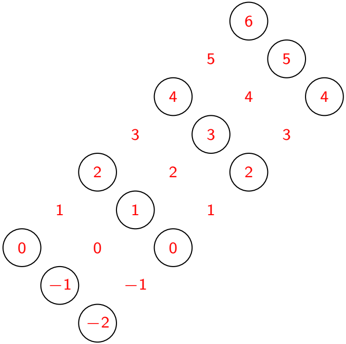


Figure: Exchange sequence.

Mutate column-by-column, reading left to right. In each column mutate at circled vertices first bottom to top, and then in the rest top to bottom.

Bi-fundamental Baxter operator

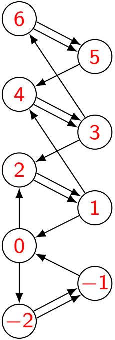


Figure: $\mathfrak{gl}_4 \times \mathfrak{gl}_2$ Toda

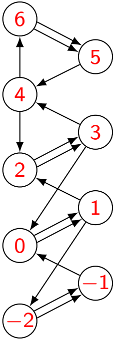


Figure: $\mathfrak{gl}_2 \times \mathfrak{gl}_4$ Toda

Result of applying the bi-fundamental Baxter operator.

- Given a gauge quiver Γ together with an orientation $\mathfrak{o}$, we have a recipe to construct the corresponding cluster quiver $\mathcal{Q}_\Gamma$.
- We have an injective homomorphism $\iota_{\mathfrak{o}} \colon \mathcal{A}_\Gamma^q \longrightarrow \mathrm{Frac}\left(\mathbb{L}^q_{\mathcal{Q}_\Gamma}\right)$.
- $\iota_{\mathfrak{o}}$ and $\iota_{\mathfrak{o}'}$ are related by applying sequences of bi-fundamental Baxter operators.
- If Γ does not have loops, we show that $\iota_{\mathfrak{o}}$ lands in $\mathbb{L}^q_{\mathcal{Q}_\Gamma}$ by replacing it with different $\iota_{\mathfrak{o}'}$ for different generators of $\mathcal{A}_\Gamma^q$.

Work in progress, joint with Di Francesco, Kedem, Schrader:

One has $\mathcal{A}_\Gamma^q \longrightarrow \mathbb{L}^q_{\mathcal{Q}_\Gamma}$, when Γ consists of one node and one loop. Moreover, the natural $GL_2(\mathbb{Z})$-action on $\mathcal{A}_\Gamma^q$ is realized via cluster transformations. In particular, Toda Hamiltonians and monopole operators are mutation equivalent, which settles the case of Γ having loops.

Donaldson–Thomas invariants

Each cluster variety has an associated 3-CY category given by the quiver $\mathcal{Q}$ with generic potential. Its DT-invariants can be collected into a generating function $\mathbb{E}_{\mathcal{Q}}$. If there exists a sequence of cluster mutations μ^q, which at the end changes all logarithmic labels of cluster variables to their negatives, one has $\mu_q = \mathrm{Ad}_{\mathbb{E}_{\mathcal{Q}}}$.

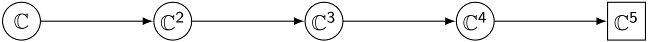
For quivers $\mathcal{Q}_\Gamma$ constructed from gauge quivers Γ without loops, such a sequence can be obtained by

1. changing orientation of every arrow in Γ;
2. applying certain amount of Dehn twists at every node.

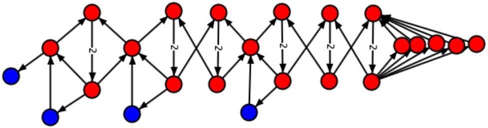
Presumably, $\mathbb{E}_{\mathcal{Q}}$ counts the BPS states of the corresponding quiver gauge theory.

Quantum groups as Coulomb branches

Another example of a quantum K-theoretic Coulomb branch is the quantum group $U_q(\mathfrak{sl}_n)$. The corresponding gauge quiver is as follows (node with $\mathbb{C}^5$ is the framing node: G does not involve GL_5, but additional equivariance is taken with respect to the maximal torus $T \in GL(5)$).



The corresponding cluster quiver $\mathcal{Q}_\Gamma$ is



Gelfand–Tsetlin subalgebra

The chain of embeddings

$$\mathfrak{gl}_1 \subset \mathfrak{gl}_2 \subset \cdots \subset \mathfrak{gl}_{n-1} \subset \mathfrak{gl}_n,$$

where $\mathfrak{gl}_k$ sits in the top-left corner, induces embeddings

$$U_q(\mathfrak{gl}_1) \subset U_q(\mathfrak{gl}_2) \subset \cdots \subset U_q(\mathfrak{gl}_n).$$

Therefore, if Z_k is the center of $U(\mathfrak{gl}_k)$, then

$$[Z_j, Z_k] = 0 \qquad \text{for all} \qquad 1 \leq j, k \leq n.$$

The **Gelfand–Tsetlin subalgebra** $\mathrm{GZ}_n \subset U_q(\mathfrak{gl}_n)$ is a commutative subalgebra generated by $Z_1, \ldots, Z_n$.

Any finite dimensional irreducible representation of $U_q(\mathfrak{gl}_n)$ breaks up into 1-dimensional weight spaces for GZ_n, where each weight space has multiplicity ≤ 1.

Positive representations of quantum groups

Theorem (Schrader–S.)

- *Representations of $U_q(\mathfrak{sl}_n)$ coming from its cluster structure coincide with its positive representations studied by Frenkel–Ip and Ponsot–Teschner.*
- *Positive representations of $U_q(\mathfrak{sl}_n)$ are equivalent to its* **Gelfand–Tsetlin representations** *studied by Gerasimov–Kharchev–Lebedev–Oblezin via applying Whittaker transform at every node.*
- *Toda Hamiltonians at the k-th node generate the subalgebra $Z_k \subset U_q(\mathfrak{gl}_k)$, and embedding $U_q(\mathfrak{gl}_k) \hookrightarrow U_q(\mathfrak{gl}_n)$ is realized via applying $\mathcal{W}$ to Z_k.*

Corollary

Positive representations of $U_q(\mathfrak{sl}_n)$ decompose with multiplicity one with respect to its Gelfand–Tsetlin subalgebra.

Thank you!