

Refined holomorphic anomaly equation for local \mathbb{P}^2

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- Talk based on arXiv:2001.05347 "Holomorphic anomaly equation for (\mathbb{P}^2, E) and the Nekrasov-Shatashvili limit of local \mathbb{P}^2 ", joint with Honglu Fan, Shuai Guo, and Longting Wu.
- Relies on some of my previous works:
 - arXiv:1706.07762 "Tropical refined curve counting from higher genera and lambda classes"
 - arXiv:1806.11495 "The quantum tropical vertex"
 - arXiv:1909.02985 "Scattering diagrams, stability conditions, and coherent sheaves on \mathbb{P}^2 "
 - arXiv:1909.02992 "A proof of N.Takahashi's conjecture for (\mathbb{P}^2, E) and a refined sheaves/Gromov-Witten correspondence"
- Closely related:
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- Refined topological string on non-compact Calabi-Yau 3-folds is expected to satisfy the refined holomorphic anomaly equation.
- The case of the ordinary/unrefined topological string starts to be well-understood mathematically.
- Our result: first mathematical result going non-trivially in the “refined direction”.
- Focus on local \mathbb{P}^2 : total space of the canonical line bundle $\mathcal{O}_{\mathbb{P}^2}(-3)$ of \mathbb{P}^2 , and the Nekrasov-Shatashvili limit of the refined topological string.
- Purely mathematical result: construction of quasimodular forms from Betti numbers of moduli spaces of one-dimensional coherent sheaves on \mathbb{P}^2 .

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- Review: topological string and holomorphic anomaly equation.
- Review: refined topological string and refined holomorphic anomaly equation.
- Statement of the main result: mathematical derivation of the refined holomorphic anomaly equation for local \mathbb{P}^2 in the Nekrasov-Shatashvili limit.
- Main idea: reformulation of the NS limit of the refined (closed) topological string as an open topological string on a dual geometry.
- Proof of this reformulation: scattering diagram form of the Kontsevich-Soibelman wall-crossing formula and tropical geometry. Need to consider the full D4-D2-D0 BPS spectrum.
- Derivation of the holomorphic anomaly equation on the open string side by some degeneration from the (unrefined) closed topological string on local \mathbb{P}^2 . Corollary: a new formula relating the unrefined topological string and the NS limit of the refined topological string (proved for local \mathbb{P}^2 , conjectured to hold at least for all local del Pezzo surfaces).

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Physics overview

- Fix X a Calabi-Yau 3-fold.
- From coupling of appropriate twists of the two-dimensional supersymmetric sigma model of target X with two-dimensional topological gravity, define the A/B models topological string on X . [Witten, 1988-1990]
- Genus g amplitudes are obtained by integration over the moduli space of genus g Riemann surfaces.
- Moduli space M of parameters, complex manifold. For the B -model, M is the moduli space of complex structures on X . For the A -model, M is the "stringy complexified Kähler moduli space of X ".
- Genus g amplitudes of the topological string are functions $F_g(t, \bar{t})$ on M , where t and \bar{t} are holomorphic and antiholomorphic coordinates on M .

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Holomorphic anomaly equation

- Naive arguments suggest that $F_g(t, \bar{t})$ is an holomorphic function on M (independent of \bar{t}).
- It is not the case: subtlety comes from the fact that the moduli space of genus g Riemann surfaces is not compact, need to add (stable) nodal curves to get a compact moduli space. The holomorphic anomaly equation expresses the fact that the defect of holomorphy comes entirely from there. Schematically (without details of special geometry)

$$\bar{\partial}F_g = \frac{1}{2} \sum_{i=1}^{g-1} (DF_i)(DF_{g-i}) + \frac{1}{2} D^2 F_{g-1}$$

where D is a first order differential operator and D^2 is a second order differential operator. [Bershadsky-Cecotti-Ooguri-Vafa, 1993]

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Holomorphic anomaly equation

- The holomorphic anomaly equation is a very powerful tool to solve the topological string: it recursively computes F_g up to ambiguities which are holomorphic functions on \mathcal{M} . These holomorphic ambiguities can sometimes be fixed simply from their boundary behavior.
- The A -model of the quintic 3-fold can be solved for $g \geq 51$ using the holomorphic anomaly equation and known/expected boundary conditions (large volume leading behavior, orbifold regularity, conifold gap, Castelnuovo bound). [Huang-Klemm-Quackenbush, 2006]
- The A -model of some non-compact Calabi-Yau 3-folds, such as local \mathbb{P}^2 , is uniquely determined by the holomorphic anomaly equation and known/expected boundary conditions (large volume leading behavior, orbifold regularity, conifold gap). [Aganagic- Bouchard-Klemm, 2006][Haghighat-Klemm-Rauch, 2008]

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Focus on the A -model.

- There does not exist yet a general mathematical definition of the “stringy complexified Kähler moduli space”. When we know that X is mirror to Y , it is the moduli space of complex structures on Y .
- In particular, there does not exist yet a general mathematical definition of the genus g amplitudes $F_g(t, \bar{t})$.
- What exists mathematically is the power expansion around the large volume point $t = +\infty$ of the holomorphic limit $F_g(t) := \lim_{\bar{t} \rightarrow 0} F_g(t, \bar{t})$ where t is a linear coordinate on $H^2(X, \mathbb{C})$: up to known perturbative terms,

$$F_g(t) = \sum_{\beta \in H_2(X, \mathbb{Z})} N_{g, \beta} e^{-\int_{\beta} t}$$

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- The physics prediction of the existence of $F_g(t, \bar{t})$ implies that the mathematically defined series $F_g(t)$ should have non-zero radius of convergence and analytic continuation to the universal cover of \mathcal{M} , with quasi-modular-like transformations with respect to the action of the fundamental group.
- The holomorphic anomaly equation for $F_g(t, \bar{t})$ translates into an equation describing $F_g(t)$ up to strictly modular ambiguities.
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Theorem (Conjecture of (Aganagic-Bouchard-Klemm, 2006), proofs: (Fang-Liu-Zong, 2016), (Lho-Pandharipande, 2017), (Coates-Iritani, 2018))

For $X = \mathcal{O}_{\mathbb{P}^2}(-3)$, the generating series $F_g(t)$ of Gromov-Witten invariants are (after the mirror map change of variables) quasi-modular forms for the group $\Gamma_1(3)$ and satisfy the holomorphic anomaly equation.

The topological vertex ([Aganagic-Klemm-Marino-Vafa, 2003], [Li-Liu-Liu-Zhou, 2004]) gives a way to compute A -model/Gromov-Witten invariants of toric Calabi-Yau 3-folds. However, it is not known how to deduce quasimodularity and holomorphic anomaly equation from the topological vertex.

The genus g A -model amplitudes of a Calabi-Yau 3-fold compute F -terms in the low-energy effective action of Type IIA string theory on $\mathbb{R}^{1,3} \times X$. Using that the strong coupling limit of Type IIA string theory is M -theory, one obtains a M -theory description of the topological string [Gopakumar-Vafa, 1998]

$$\sum_{g \geq 0} F_g(t) g_s^{2g-2} = - \sum_{\beta \in H_2(X, \mathbb{Z})} \sum_{k \geq 1} \frac{1}{k} \frac{\text{Tr}_{\mathcal{H}_\beta} [(-1)^{2(J_L + J_R)} q^{2kJ_L}]}{(q^{\frac{k}{2}} - q^{-\frac{k}{2}})^2} e^{-k \int_\beta t}$$

where \mathcal{H}_β is the spaces of BPS states of M2 branes of charge β , representation of the 5d little group $SO(4) \sim SU(2)_L \times SU(2)_R$, and $q = e^{ig_s}$.

Let X be a Calabi-Yau 3-fold with a $U(1)_T$ symmetry scaling non-trivially the holomorphic volume form, e.g. $X = \mathcal{O}_{\mathbb{P}^2}(-3)$ with $U(1)_T$ rotating the fibers of the natural projection $\mathcal{O}_{\mathbb{P}^2}(-3) \rightarrow \mathbb{P}^2$. Define the refined topological string amplitudes $F_{g_1, g_2}(t)$ by

$$\begin{aligned} & \sum_{g_1, g_2 \geq 0} F_{g_1, g_2}(t) (\epsilon_1 + \epsilon_2)^{2g_1} (-\epsilon_1 \epsilon_2)^{g_2 - 1} \\ := & \sum_{\beta \in H_2(X, \mathbb{Z})} \sum_{k \geq 1} \frac{1}{k} \frac{\text{Tr}_{\mathcal{H}_\beta} \left((-1)^{2(J_L + J_R)} q_L^{2kJ_L} q_R^{2k(J_R + I_T)} \right)}{(q_1^{\frac{k}{2}} - q_1^{-\frac{k}{2}})(q_2^{\frac{k}{2}} - q_2^{-\frac{k}{2}})} e^{-k \int_\beta t} \end{aligned}$$

where

$$\begin{aligned} q_1 &= e^{\epsilon_1}, q_2 = e^{\epsilon_2}, q_L = e^{\epsilon_L}, q_R = e^{\epsilon_R}, \\ \epsilon_R &= \frac{\epsilon_1 + \epsilon_2}{2}, \epsilon_L = \frac{\epsilon_1 - \epsilon_2}{2}. \end{aligned}$$

[Nekrasov, 2002][Hollowood-Iqbal-Vafa, 2003]

Refined topological string

- Unrefined limit: $\epsilon_1 = -\epsilon_2 = ig_s$, so $\epsilon_L = ig_s, \epsilon_R = 0, q_L = e^{ig_s}, q_R = 1$, and so $F_{0,g}(t) = F_g(t)$.
- From the M -theory definition, one expects $F_{g_1,g_2}(t)$, a priori a formal power series in $Q = e^{-t}$, to be convergent and to come from a globally defined function $F_{g_1,g_2}(t, \bar{t})$ on the space M of parameters.
- No obvious worldsheet definition of $F_{g_1,g_2}(t, \bar{t})$, so no BCOV derivation for an holomorphic anomaly equation.
- Nevertheless, a refined holomorphic anomaly equation was guessed [Krefl-Walcher, 2010][Huang-Klemm, 2010]

$$\bar{\partial} F_{g_1,g_2} = \frac{1}{2} \sum_{\substack{0 < j_1 \leq g_1 \\ 0 < j_2 \leq g_2 \\ (j_1, j_2) \neq (0,0) \\ (j_1, j_2) \neq (g_1, g_2)}}^{g-1} (DF_{j_1, j_2}) (DF_{(g_1-j_1, g_2-j_2)}) + \frac{1}{2} D^2 F_{g_1, g_2-1}.$$

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- No obvious worldsheet definition of $F_{g_1,g_2}(t, \bar{t})$, so no BCOV derivation for an holomorphic anomaly equation.
- Nevertheless, a refined holomorphic anomaly equation was guessed [Krefl-Walcher, 2010][Huang-Klemm, 2010]

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Refined topological string

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- The refined topological string of some non-compact Calabi-Yau 3-folds, such as local \mathbb{P}^2 , is uniquely determined by the refined holomorphic anomaly equation and known/expected boundary conditions (large volume leading behavior, orbifold regularity, conifold gap). [Huang-Klemm, 2010]
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The spaces of $M2$ -branes \mathcal{H}_β can be defined as cohomology of moduli spaces of $D2 - D0$ branes, i.e. moduli spaces of coherent sheaves on X supported on curves, and the $SU(2)_L \times SU(2)_R$ action can be understood geometrically ([Gopakumar-Vafa, 1998], subtleties coming from the fact that these moduli spaces are singular [Hosono-Saito-Takahashi, 2001],[Kiem-Li, 2016],[Maulik-Toda, 2016]).

$$\begin{aligned}
 & \sum_{g_1, g_2 \geq 0} F_{g_1, g_2}(t) (\epsilon_1 + \epsilon_2)^{2g_1} (-\epsilon_1 \epsilon_2)^{g_2 - 1} \\
 := & \sum_{\beta \in H_2(X, \mathbb{Z})} \sum_{k \geq 1} \frac{1}{k} \frac{\text{Tr}_{\mathcal{H}_\beta} \left((-1)^{2(J_L + J_R)} q_L^{2kJ_L} q_R^{2k(J_R + I_T)} \right)}{\left(q_1^{\frac{k}{2}} - q_1^{-\frac{k}{2}} \right) \left(q_2^{\frac{k}{2}} - q_2^{-\frac{k}{2}} \right)} e^{-k \int_\beta t}
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Main difficulty: the refined holomorphic anomaly equation is a recursion on (g_1, g_2) , but $F_{g_1, g_2}(t)$ is defined through the non-trivial change of variables $q_1 = e^{\epsilon_1}$ and $q_2 = e^{\epsilon_2}$. Without some geometric/worldsheet-like interpretation of the parameters (g_1, g_2) , it seems very difficult to prove anything. It is not known how to deduce quasimodularity and holomorphic anomaly from the refined topological vertex for example.

The Nekrasov-Shatashvili limit

- NS limit: $\epsilon_1 = \hbar$, $\epsilon_2 = 0$, and so $\epsilon_L = \epsilon_R = \frac{\hbar}{2}$.
- Denote $F_g^{NS}(t) := F_{g,0}(t)$ and $y = e^{\hbar}$

$$\sum_{g \geq 0} F_g^{NS}(t) \hbar^{2g-1} = \sum_{\beta \in H_2(X, \mathbb{Z})} \sum_{k \geq 1} \frac{1}{k^2} \frac{\Omega_\beta(y^k)}{y^{\frac{k}{2}} - y^{-\frac{k}{2}}} e^{-k \int_\beta t}$$

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$$\Omega_\beta(y) := \text{Tr}_{\mathcal{H}_\beta}((-1)^{2(J_L + J_R)} y^{J_L + J_R + I_T}).$$

Refined 4d BPS index for D2-D0 branes ($J_L + J_R$ is the 4d diagonal $SU(2)$ in the 5d $SU(2)_L \times SU(2)_R$), (roughly) the Hirzebruch χ_y -genus of the moduli space of D2-D0 branes.

- Refined holomorphic equation in the NS limit (note: no loop term)

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Theorem (Conjecture of (Huang-Klemm, 2010), proof: (B-Fan-Guo-Wu, 2020))

For $X = \mathcal{O}_{\mathbb{P}^2}(-3)$, the series $F_g^{NS}(t)$, defined in terms of Betti numbers of one-dimensional coherent sheaves on \mathbb{P}^2 , are (after the mirror map change of variables) quasi-modular forms for the group $\Gamma_1(3)$ and satisfy the NS limit of the refined holomorphic anomaly equation.

$$X = \mathcal{O}_{\mathbb{P}^2}(-3), H_2(X, \mathbb{Z}) = H_2(\mathbb{P}^2, \mathbb{Z}) = \mathbb{Z}$$

$$\sum_{g \geq 0} F_g^{NS}(t) \hbar^{2g-1} = \sum_{d \geq 1} \sum_{k \geq 1} \frac{1}{k^2} \frac{\Omega_d(y^k)}{y^{\frac{k}{2}} - y^{-\frac{k}{2}}} e^{-kdt}$$

- Precise mathematical definition of the Laurent polynomials $\Omega_d(y)$?
- Moduli space $M_{d,\chi}$ of D2-D0 branes: (coarse) moduli space of semistable coherent sheaves F on \mathbb{P}^2 supported on curves of degree d and with $\chi(F) = \chi$. Very natural in algebraic geometry (Abel-Jacobi in family)[Simpson, 1990], [Le Potier, 1993].
- Define $\Omega_{d,\chi}(y)$ as the Poincaré polynomial of Betti numbers of $M_{d,\chi}$ for the intersection cohomology (physics: think about L^2 -cohomology of the stable (smooth) locus. $M_{d,\chi}$ is smooth if $\gcd(d, \chi) = 1$, singular in general.

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Conjecture

$\Omega_{d,\chi}$ only depends on d and not on χ .

We have $M_{d,\chi} \simeq M_{d,\chi+d}$ ($F \mapsto F \otimes \mathcal{O}_{\mathbb{P}^2}(1)$) and $M_{d,\chi} \simeq M_{d,-\chi}$ ($F \mapsto F^\vee$) but $M_{d,\chi} \neq M_{d,\chi'}$ if $d \geq 3$ and $\chi \not\equiv \pm\chi' \pmod{d}$.

Theorem (B, 2019)

- $\Omega_{d,\chi}$ only depends on d and on $\gcd(d, \chi)$.
- The full conjecture is true for $d \leq 4$.

Proof using Gromov-Witten theory of a basic result on topology of moduli space of sheaves.

Without knowing the general conjecture, we define $\Omega_d(y)$ by average:

$$\Omega_d(y) := \frac{1}{d} \sum_{\chi \pmod{d}} \Omega_{d,\chi}(y).$$

$$y = e^{\hbar}$$

$$\sum_{g \geq 0} F_g^{NS}(t) \hbar^{2g-1} = \sum_{d \geq 1} \sum_{k \geq 1} \frac{1}{k^2} \frac{\Omega_d(y^k)}{y^{\frac{k}{2}} - y^{-\frac{k}{2}}} e^{-kdt}$$

Theorem (B-Fan-Guo-Wu, 2020)

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We would like a genus g worldsheet definition of $F_g^{NS}(t)$. This cannot be the unrefined topological string on $\mathcal{O}_{\mathbb{P}^2}(-3)$: $F_g^{NS}(t) \neq F_g(t)$ for $g \geq 1$.

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Take the derivative d/dt

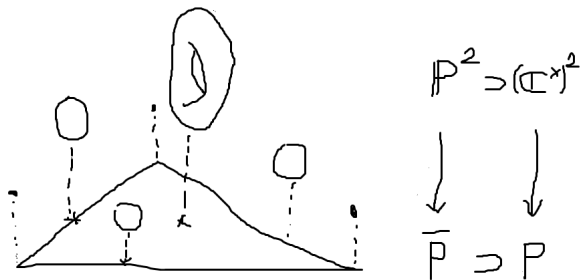
$$\sum_{g \geq 0} \frac{d}{dt} F_g^{NS} \hbar^{2g-1} = - \sum_{d \geq 1} \sum_{k \geq 1} \frac{1}{k} \frac{d\Omega_d(y^k)}{y^{\frac{k}{2}} - y^{-\frac{k}{2}}} e^{-kdt}$$

The derivative of the NS limit of the refined topological string looks exactly like an open topological string! [Ooguri-Vafa, 1999]

Main claim: for $X = \mathcal{O}_{\mathbb{P}^2}(-3)$ (and more generally local del Pezzo surfaces), we can find such open topological string.

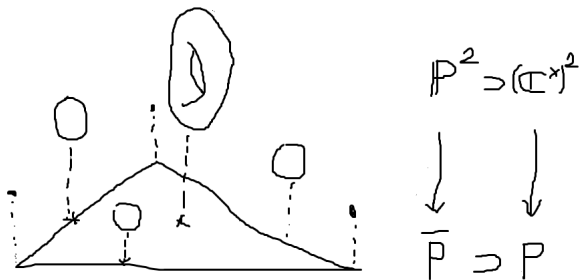
Towards the open string geometry

- \mathbb{P}^2 : complex projective plane, complex dimension 2, real dimension 4. Standard symplectic structure, T^2 -Hamiltonian action leading to a toric description.
- Moment map: $\mu : \mathbb{P}^2 \rightarrow \bar{P}$. Over the interior P of \bar{P} , $(\mathbb{C}^*)^2 \rightarrow P$, Lagrangian T^2 -fibration. $(\mathbb{C}^*)^2$: complement of a triangle of lines in \mathbb{P}^2 .



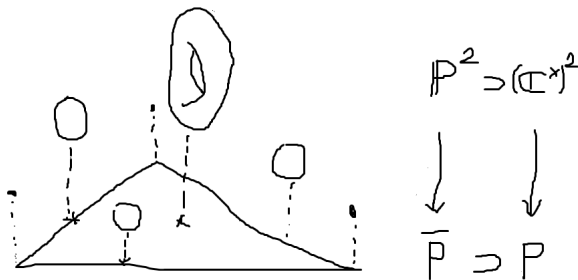
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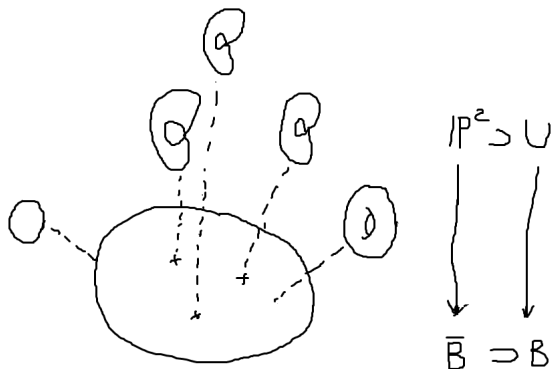
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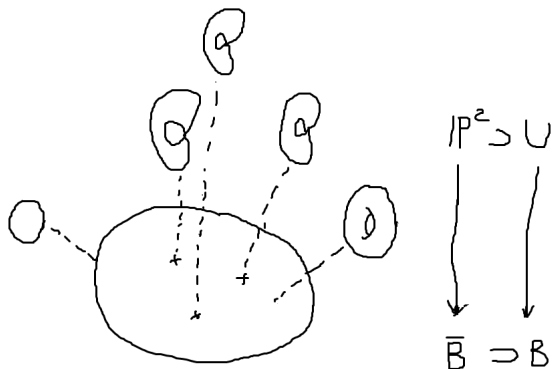
Towards the opens string geometry: (\mathbb{P}^2, E)

- Replace the triangle of lines by a smooth cubic E (genus one curve) and $(\mathbb{C}^*)^2$ by $U := \mathbb{P}^2 - E$. U is a non-compact Calabi-Yau surface.
- Over the interior B of \bar{B} , $U \rightarrow B$, Lagrangian T^2 -fibration with 3 nodal singular fibers. Topological check:
 $\chi_{top}(U) = \chi_{top}(\mathbb{P}^2) - \chi_{top}(E) = 0$. In fact, there exists such special Lagrangian fibration. [Collins, Jacob, Lin, 2019]



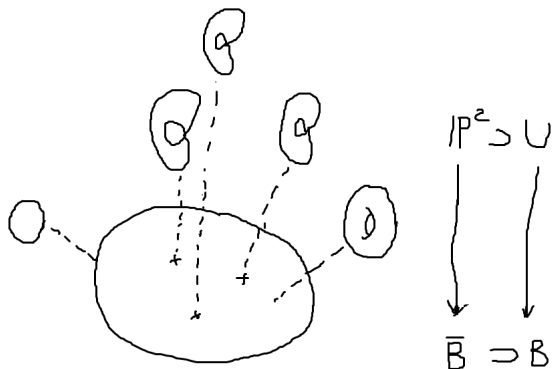
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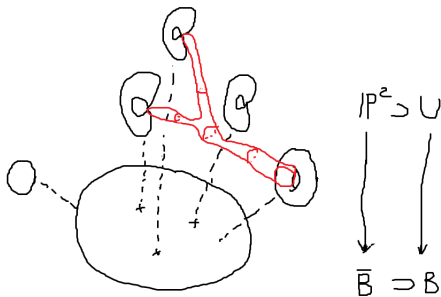
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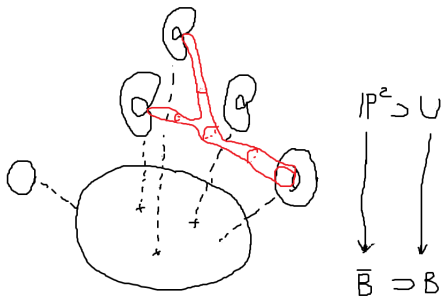
Open topological string: Holomorphic curves in (\mathbb{P}^2, E)

- Study holomorphic curves in U with one boundary on a Lagrangian T^2 -fiber.
- In fact U is hyperkähler. Study the open topological string on the Calabi-Yau 3-fold Z obtained as the \mathbb{C}^* -twistor family of U , with brane $T_b^2 \times \mathbb{R}$: $N_{g,v}^{open}(b)$ “counts” of genus g Riemann surfaces in Z with one boundary of class $v \in H_1(T_b^2 \times \mathbb{R}, \mathbb{Z}) \simeq \mathbb{Z}^2$.



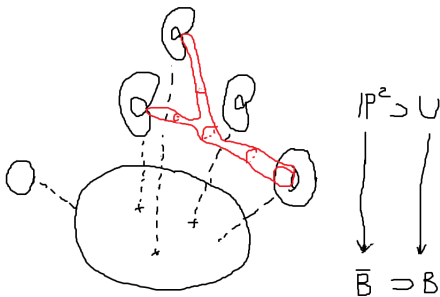
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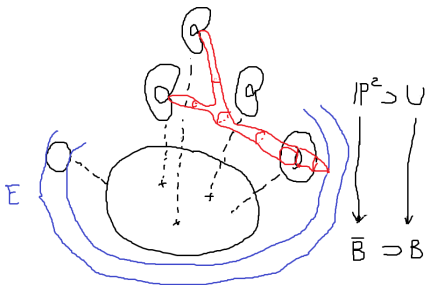
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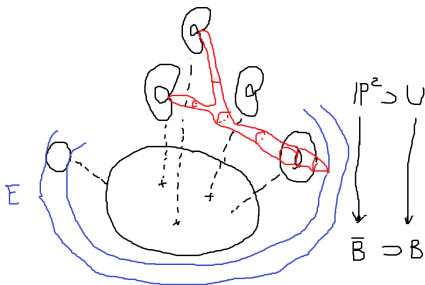
Open topological string: Holomorphic curves in (\mathbb{P}^2, E)

- The open invariants $N_{g,v}^{open}(b)$ jump as a function of b (wall-crossing) and are difficult to define rigorously (e.g. we are not in a toric situation).
- We provide a mathematically precise definition when b is close to E and $v \in H_1(T_b^2, \mathbb{Z})$ is a multiple of the cycle $(1, 0)$ collapsing on E . For open curves wrapping a cycle collapsing on E , close them by gluing a disc: get a closed holomorphic curve in \mathbb{P}^2 meeting E in one point. Use relative Gromov-Witten theory to define $N_{g,(d,0)}^{open} \in \mathbb{Q}$.



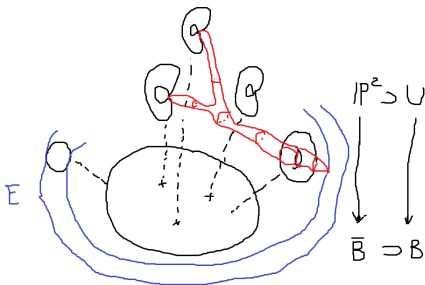
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Theorem (B, 2019)

$$\sum_{g \geq 0} \frac{d}{dt} F_g^{NS} \hbar^{2g-1} = -\frac{1}{3} \sum_{g \geq 0} \sum_{d \geq 1} (-1)^{d-1} N_{g,(d,0)}^{open} \hbar^{2g-1} e^{-dt}$$

In other words, the refined BPS counts $\Omega_d(y)$ of (closed) D2-D0 branes on $X = \mathcal{O}_{\mathbb{P}^2}(-3)$ are equal (up to simple factor) to Ooguri-Vafa open BPS counts on $Z = U \times \mathbb{C}^ = (\mathbb{P}^2 - E) \times \mathbb{C}^*$.*

Open string interpretation of the BPS spectrum of $D4 - D2 - D0$ branes of $X = \mathcal{O}_{\mathbb{P}^2}(-3)$

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Almost theorem (for any reasonable mathematical definition of the open invariants)(B.2019)

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Heuristic chain of geometric connections:

- Start with open holomorphic curves in U .
- U is hyperkähler: do some hyperkähler rotation. In some different complex structure, get V an elliptic fibration $W: V \rightarrow B$. Open holomorphic curves in U becomes open special Lagrangians.
- Suspension construction: S Calabi-Yau 3-fold $uv = W$, fibration in affine conics over V , degenerate over T^2 . Open special Lagrangians in V lift to closed special Lagrangians in S .
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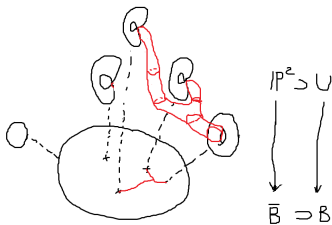
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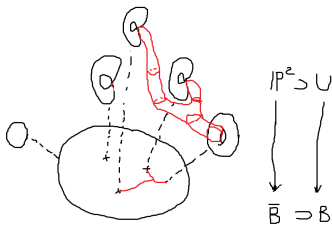
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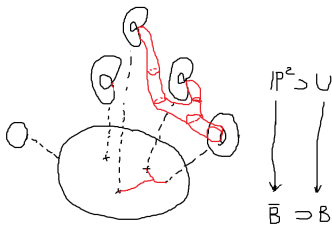
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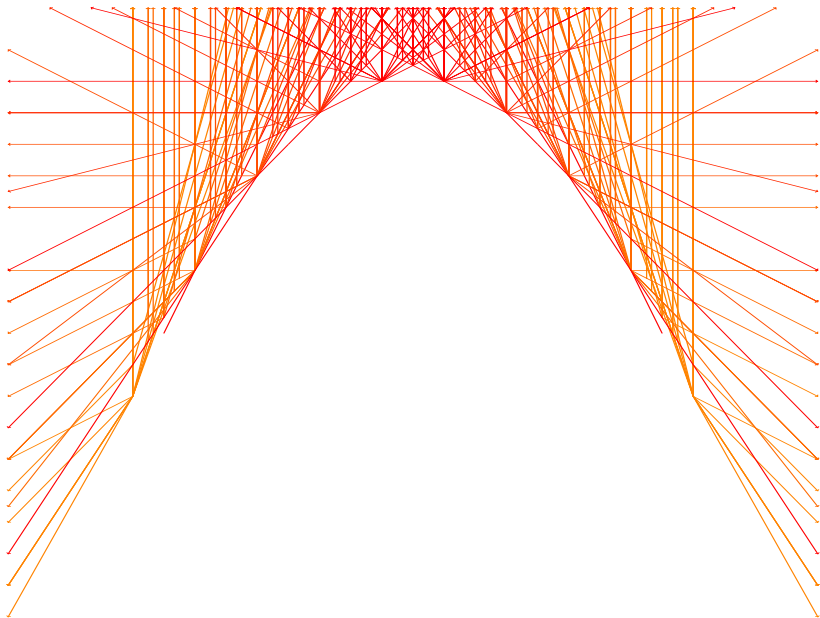
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Scattering diagram



- Main result (B. 2019): the scattering diagram can be embedded in the space of Bridgeland stability conditions on the derived category $D_C^b \text{Coh}(K_{\mathbb{P}^2})$, such that the rays correspond to stability conditions for which there exists stable objects of phase $\pi/2$. Use coordinates on the stringy Kähler moduli space given by the real part of the central charges. Similar to attractor flows in supergravity [Denef,2001][Denef, Moore,2007].
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- (Prince) Triangles in the scattering diagrams are indexed by Markov triples (integer solutions of $x^2 + y^2 + z^2 = 3xyz$). Discs potential associated to Vianna's monotone tori can be extracted from the scattering diagram.
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I have sketched above the proof of:

Theorem (B, 2019)

$$\sum_{g \geq 0} \frac{d}{dt} F_g^{NS} \hbar^{2g-1} = -\frac{1}{3} \sum_{g \geq 0} \sum_{d \geq 1} (-1)^{d-1} N_{g,(d,0)}^{open} \hbar^{2g-1} e^{-dt}$$

In other words, the refined BPS counts $\Omega_d(y)$ of (closed) D2-D0 branes on $X = \mathcal{O}_{\mathbb{P}^2}(-3)$ are equal (up to simple factor) to Ooguri-Vafa open BPS counts on $Z = U \times \mathbb{C}^ = (\mathbb{P}^2 - E) \times \mathbb{C}^*$.*

To prove the quasimodularity and refined holomorphic equation for F_g^{NS} , it remains to work on the open topological string side. There is some hope because the \hbar -expansion is now the geometric genus expansion of the topological string.

Modularity from the Gromov-Witten side (with Fan, Guo, Wu, 2020)

- Degeneration argument. Degeneration of \mathbb{P}^2 to the normal cone of E . Line bundle defined by the family of divisors E . General fiber: $X = \mathcal{O}_{\mathbb{P}^2}(-3) = \mathcal{O}(-E)$. Special fiber: $\mathbb{P}^2 \times \mathbb{A}^1$, glued along $E \times \mathbb{C}^1$ to a non-trivial line bundle over $\mathbb{P}(N_{E|\mathbb{P}^2} \oplus \mathcal{O})$.
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- Degeneration argument. Degeneration of \mathbb{P}^2 to the normal cone of E . Line bundle defined by the family of divisors E . General fiber: $X = \mathcal{O}_{\mathbb{P}^2}(-3) = \mathcal{O}(-E)$. Special fiber: $\mathbb{P}^2 \times \mathbb{A}^1$, glued along $E \times \mathbb{C}^1$ to a non-trivial line bundle over $\mathbb{P}(N_{E|\mathbb{P}^2} \oplus \mathcal{O})$.
- Localization on the bubble $\mathbb{P}(N_{E|\mathbb{P}^2} \oplus \mathcal{O})$: reduction to equivariant Gromov-Witten theory of $N_{E|\mathbb{P}^2} \oplus N_{E|\mathbb{P}^2}^\vee \rightarrow E$ with stationary descendent insertions.
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- Upshot: formula computing open Gromov-Witten invariants $N_{g,(d,0)}^{open}$ of (\mathbb{P}^2, E) in terms of Gromov-Witten invariants of $X = \mathcal{O}_{\mathbb{P}^2}(-3)$ and the elliptic curve E .

$$F_g = (-1)^g F_g^{NS} + \sum_{n \geq 0} \sum_{\substack{g=h+g_1+\dots+g_n, \\ \mathbf{a}=(a_1,\dots,a_n) \in \mathbb{Z}_{\geq 0}^n \\ (a_j, g_j) \neq (0,0), \sum_{j=1}^n a_j = 2h-2}} \frac{(-1)^{h-1} F_{h,\mathbf{a}}^E}{|\text{Aut}(\mathbf{a}, \mathbf{g})|} \prod_{j=1}^n (-1)^{g_j-1} D^{a_j+2} F_{g_j}^{NS}.$$

- $F_{h,\mathbf{a}}^E$: Gromov-Witten theory of E with stationary descendent insertions, known in closed form by [Okounkov-Pandharipande, 2002]
- New formula relating the unrefined topological string F_g and the NS limit F_g^{NS} of the refined topological string (conjecturally valid at least for all local del Pezzo surfaces).

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Thank you for your attention!