Higher representations and cornered Heegaard Floer homology

Andrew Manion (joint with Raphaël Rouquier)

Outline

Background: 4d SW theory

Background: Extended TQFT

Heegaard Floer homology

Tensor products and cornered HF

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2 Background: Extended TQFT

3 Heegaard Floer homology

4 Tensor products and cornered Heegaard Floer homology

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Important for physics: metric-independent QFTs interesting for quantum gravity?

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Similar power to Donaldson invariants but easier to work with

Modern perspective; categorified Chern–Simons theory

Gukov–Putrov–Vafa '16: Hilbert space of the 4d Seiberg–Witten TQFT on M^3 can be obtained from the 6d $(2,0)$ superconformal field theory on $\mathbb{R} \times M^3 \times \Sigma_{SW}$ after reducing on Σ_{SW}

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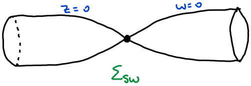

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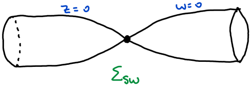

6-manifold $\mathbb{R} \times M^3 \times \Sigma_{SW}$ is an $M5$ brane in 11-dimensional spacetime $\mathbb{R} \times T^*M^3 \times \mathbb{C}^2$; 6d theory is worldvolume theory of M-theory on this brane

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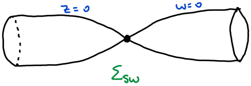

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Replace Σ_{SW} with $\Sigma_n := \{w^n = 0\} \subset \mathbb{C}^2$ ("n branes stacked on $\{w = 0\}$"): get theories related to categorified $U(n)$ Chern–Simons (e.g. Khovanov homology for $n = 2$)

The $U(1|1)$ case

Can also consider "n branes and m antibranes" stacked on $\{w = 0\}$ (curve $\Sigma_{n|m}$); related to categorified $U(n|m)$ Chern–Simons theory

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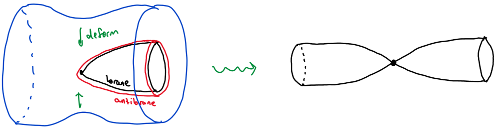

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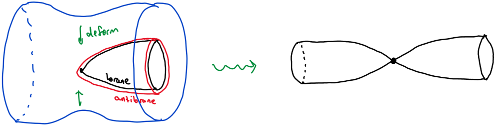

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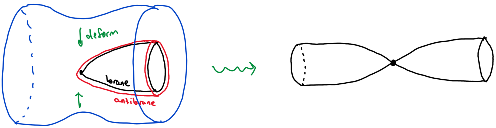

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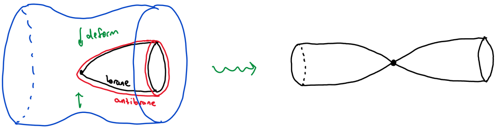

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Mathematically: some known relationships especially involving quantum representations of $\mathfrak{gl}(1|1)$, would like more

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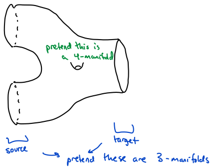

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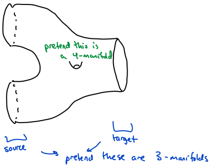

Cases of interest e.g. Donaldson, SW: not defined on all 4-manifolds (b_2^+ restriction), doesn't satisfy all the axioms, ...

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Lower-dimensional manifolds get higher categories; in the best cases this goes all the way down to a point

Extending down (continued)

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Lurie '09: TQFTs that extend to 0-manifolds, satisfying strong axioms, are completely determined by what they assign to a point

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Forgetting about the axioms, maybe even the basic idea of assigning higher-categorical data to higher morphisms in cobordism categories (without extra decorations) is insufficient for the examples of interest?

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This is most naturally explained in terms of Heegaard Floer homology, which we'll discuss next

The Atiyah–Floer conjecture

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- Surface $F \mapsto$ Fukaya category of moduli space of flat $SU(2)$ connections on F
- 3-manifold M with boundary $F \mapsto$ flat connections extending over M^3 (a Lagrangian submanifold of the moduli space over F)

The Atiyah–Floer conjecture (continued)

So: if M is glued from two pieces M_1, M_2 along F: Hilbert space of theory on M (instanton Floer homology) is the Lagrangian intersection Floer group of the Lagrangians from M_i in the moduli space over F

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Lagrangian Floer group is the homology of a complex whose generators are intersection points between Lagrangians (here: flat connections on all of M), differential counts *pseudoholomorphic curves* rather than instantons

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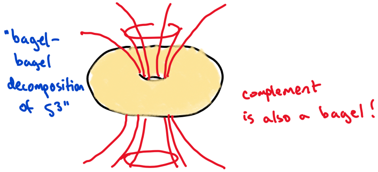
Example:



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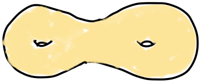

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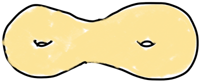

Could try to give an alternate definition of instanton Floer homology of M by: pick a Heegaard splitting for M, use it to compute a Lagrangian Floer group as above, show the result is independent of splitting

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Looking at solutions to SW equations on $\mathbb{R}^2 \times F$ for a surface F: moduli space of flat connections should be replaced by moduli space of solutions to "vortex equations" (the symmetric power $\mathrm{Sym}^k(F)$ for vortex number k)

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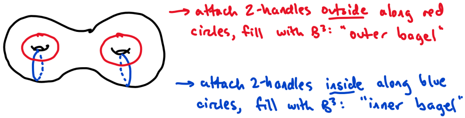
A diagram for the genus-2 splitting of S^3 above:



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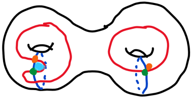

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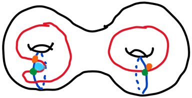

Computations can be manageable: understand some basic patterns for the moduli spaces, hope they suffice to constrain the answer uniquely

Bordered Floer homology

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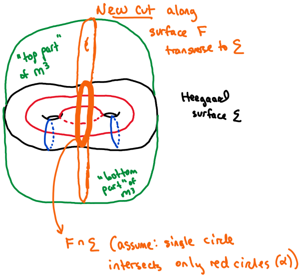
- DOESN'T: generalize the Heegaard splittings used in HF homology to allow more general splittings
- DOES: introduce a *new cut* on M^3 along a surface F, *transverse to* the Heegaard surface Σ, and analyze everything in terms of its intersection with Σ

Bordered Floer homology (continued)

Picture:



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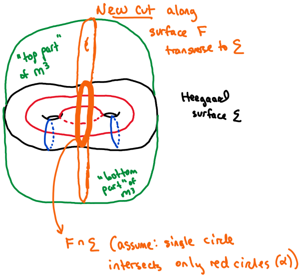

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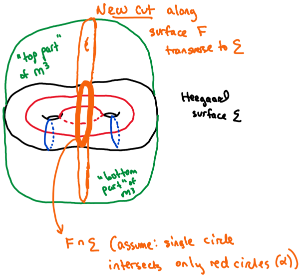

Surface F: represented by certain type of diagram $\mathcal{Z}$ drawn in the circle $F \cap \Sigma$; gets assigned a dg algebra $\mathcal{A}(\mathcal{Z})$ ("bordered strands algebra")
General pieces M_1, M_2: represented by "partial Heegaard diagrams" drawn in $M_1 \cap \Sigma$ and $M_2 \cap \Sigma$; get assigned A_∞ modules over $\mathcal{A}(\mathcal{Z})$

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Here the plot thickens... slight lie to talk about $HF(M^3)$ for closed M^3, since you need to pick a *basepoint* in M^3 (assumed to come from *basepoint in the Heegaard surface* Σ).

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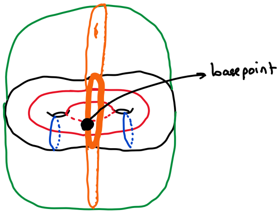
Since there are two cuts, intersecting transversely, tempting to think: bordered HF could be viewed as based on *some ansatz about what HF / SW theory assigns to a circle*.

Here the plot thickens... slight lie to talk about $HF(M^3)$ for closed M^3, since you need to pick a *basepoint* in M^3 (assumed to come from *basepoint in the Heegaard surface* Σ).

Bordered HF: LOT put their basepoint on this circle:



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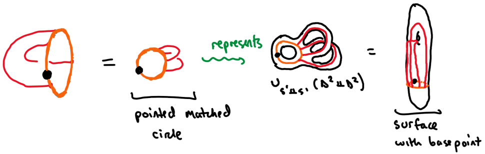
Correspondingly: for LOT, the diagram $\mathcal{Z}$ representing F is a "pointed matched circle:"



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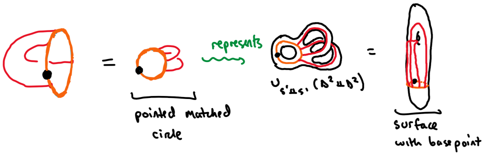

So: is bordered HF really based on an ansatz about what's assigned to a *pointed* circle?

Sutured Floer homology

Higher representations and cornered Heegaard Floer homology

Andrew Manion (joint with Raphaël Rouquier)

Outline

Background: 4d SW theory

Background: Extended TQFT

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Tensor products and cornered HF

From the perspective proposed here: this extra basepoint data needs to get worse before it can get better

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Juhasz '06: generalized $\widehat{HF}(M^3)$ to *sutured* M^3 (compact M^3 with boundary and certain decorations on the boundary; defined by Gabai '83, related to taut foliations)

In fact, Juhasz' sutured Floer homology is a joint generalization of $\widehat{HF}(M^3)$ for closed 3-manifolds and $\widehat{HFK}(K)$ for knots

Sutured Floer homology (continued)

Sutured 3-manifold (for us:) M^3 with its boundary decomposed into two regions R_+ and R_- along a 1-manifold Γ (the "sutures")

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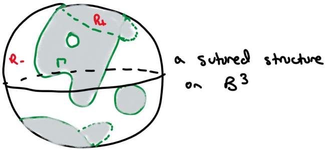

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↳(meridional sutures)

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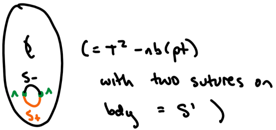

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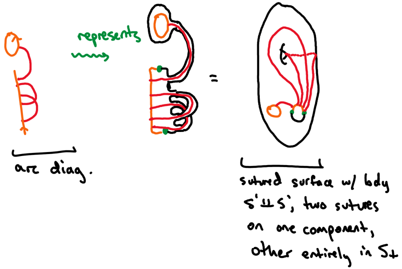

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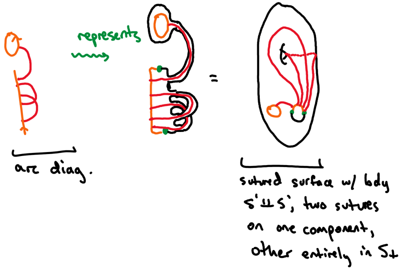

Interval components of $\mathcal{Z}$ $\leftrightarrow$ interval components of S_+; circle components of $\mathcal{Z}$ $\leftrightarrow$ circle components of S_+

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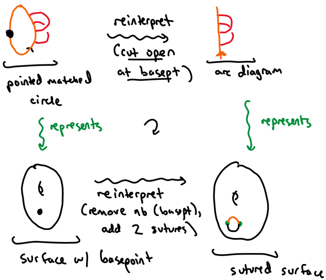

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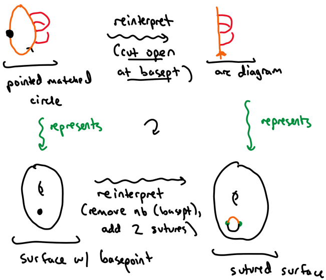

Zarev interprets LOT's invariants of 3-manifolds with boundary as being invariants of certain "bordered sutured 3-manifolds" (look a bit like: 3-manifolds with corners)

Bordered sutured Floer homology (continued)

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Proposed here: another way to view things...

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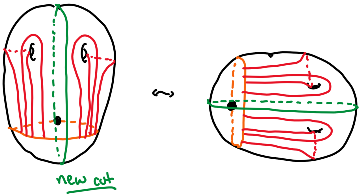

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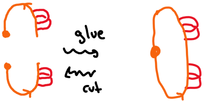

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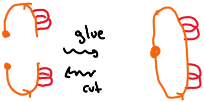

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So: is this related to "what SW / HF assigns to two points, one of which is the basepoint"?

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Strands algebras $\mathcal{A}(\mathcal{Z})$ for various diagrams $\mathcal{Z}$ (representing genus-zero surfaces) have been used to categorify reps of $U_q(\mathfrak{gl}(1|1))$, e.g. $V^{\otimes n}$ where V is the vector / defining representation

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These turn out to be closely related!

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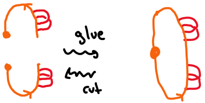
Instead of 

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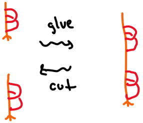

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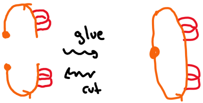
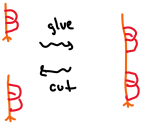
Instead of 

, let's view the gluing operation as 

instead

Have two arc diagrams $\mathcal{Z}_1$ and $\mathcal{Z}_2$, each with a *distinguished interval component*, and we're gluing the distinguished components end-to-end to get a new arc diagram $\mathcal{Z}$ with a distinguished interval component

Cornered Floer from the bordered sutured viewpoint (continued)

Theorem (M.–Rouquier '20)

- *Douglas–Manolescu's sequential algebra-modules can be repackaged to define a 2-action of* $\mathcal{U}$ *on* $\mathcal{A}(\mathcal{Z})$ *for any arc diagram* $\mathcal{Z}$ *with a distinguished interval component, where* $\mathcal{U}$ *is a dg monoidal category introduced by Khovanov '10 (*$\mathcal{U}$ *categorifies positive half of* $U_q(\mathfrak{gl}(1|1))$*)*

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- *There is an operation* $\bigotimes$ *for 2-representations of* $\mathcal{U}$ *such that Douglas–Manolescu's gluing formula becomes* $\mathcal{A}(\mathcal{Z}) \cong \mathcal{A}(\mathcal{Z}_1) \bigotimes \mathcal{A}(\mathcal{Z}_2)$ *as dg algebras*
- *Furthermore, we have* $\mathcal{A}(\mathcal{Z}) \cong \mathcal{A}(\mathcal{Z}_1) \bigotimes \mathcal{A}(\mathcal{Z}_2)$ *as 2-representations of* $\mathcal{U}$ *(allowing iterative use of the gluing formula)*

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This favors the bordered sutured viewpoint on cornered HF

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Cornered Floer from the bordered sutured viewpoint (continued)

This favors the bordered sutured viewpoint on cornered HF

Why? result of the gluing is the *same type of algebraic object* as the two inputs (a 2-representation of $\mathcal{U}$)

This is true when gluing arc diagrams, but not when gluing halves of pointed matched circles

Two types of tensor product

Relatedly: Douglas–Manolescu view their gluing formula as describing the invariant of a closed surface with basepoint, glued from two surfaces with basepoints on their S^1 boundaries along the common S^1

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This type of gluing often corresponds to algebraic tensor products of the form $M \otimes_R N$ where R is a ring (associated to the circle), M is a right R-module, and N is a left R-module (no S^1 boundary on glued surface and no R-action on the tensor product); Douglas–Manolescu phrase their algebraic gluing operation in similar terms

Two types of tensor product (continued)

Another type of tensor product: $M \otimes_k N$ where M and N are left modules over a Hopf k-algebra H; this tensor product carries its own left action of H (defined using coproduct on H)

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The tensor products appearing in expressions like $V^{\otimes n}$ above, and the categorified operation $\bigotimes$, are instances of the second type of tensor product rather than the first (correspondingly, $\mathcal{A}(\mathcal{Z}_1) \bigotimes \mathcal{A}(\mathcal{Z}_2)$ carries a 2-action of $\mathcal{U}$).

Intervals and the open pair of pants

However, this closed pair-of-pants gluing doesn't exactly capture the effect on sutured surfaces of our arc-diagram gluing $\mathcal{Z}_1, \mathcal{Z}_2 \mapsto \mathcal{Z}$

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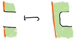
What does? 3 ways to view it:

1. Glue small neighborhood of suture in ∂F_1 to small neighborhood of suture in ∂F_2: 

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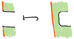
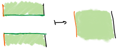
1. Glue small neighborhood of suture in ∂F_1 to small neighborhood of suture in ∂F_2: 
2. View sutured surfaces as cobordisms from S_- to S_+ restricting to id_Λ on the boundary; glue along an interval in id_Λ 

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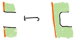
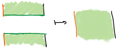
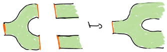
1. Glue small neighborhood of suture in ∂F_1 to small neighborhood of suture in ∂F_2: 
2. View sutured surfaces as cobordisms from S_- to S_+ restricting to id_Λ on the boundary; glue along an interval in id_Λ 
3. - Non-self-gluing case: glue "open pair of pants" to S_+ interval in F_1 and S_+ interval in F_2: 
   - Self-gluing case: glue  to S_+ interval in F

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The open pair-of-pants cobordism makes sense in the open-closed cobordism category but not the 012 (fully extended) cobordism 2-category...

Intervals and the open pair of pants (continued)

Looks like $2\,\mathrm{Rep}(\mathcal{U})$ (2-category) is being assigned to an interval and tensor product comes from open pair of pants

Modulo details, this is reasonable: invariant of circle is "center" / "double" of invariant of interval, quantum group is "close to" double of its positive half

The open pair-of-pants cobordism makes sense in the open-closed cobordism category but not the 012 (fully extended) cobordism 2-category...

however, if one started with a 012 TQFT and used it to define an open-closed TQFT, the open pair of pants gluing (item 3 above) would come from the type of gluing shown in item 2 above

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The circle would be assigned something new, not yet appearing in Heegaard Floer homology as far as I know (call it $2\,\mathrm{Rep}(D(\mathcal{U}))$ for now...)

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...maybe given by tensor product over $\mathcal{U}$ with a $\mathcal{U}$-bimodule category, i.e. a category with two commuting 2-actions of $\mathcal{U}$?

Sutured surfaces (continued)

To a general sutured surface, reasonable guess is to assign a category with commuting 2-actions of $\mathcal{U}$ (for interval components of S_-, S_+) and $D(\mathcal{U})$ (for circle components of S_-, S_+)...

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Call this hypothetical algebra $\mathcal{A}^?(\mathcal{Z})$, given an arc diagram $\mathcal{Z}$; heuristic idempotent count shows $\mathcal{A}(\mathcal{Z})$ is too small to be $\mathcal{A}^?(\mathcal{Z})$

Sutured surfaces (continued)

However: if we suppose that for *all intervals and circles in* S_- and *all circles in* S_+, the hypothetical algebra $\mathcal{A}^?(\mathcal{Z})$ has been "tensored over $\mathcal{U}$ or $D(\mathcal{U})$ with the trivial 2-representation," then heuristic count gives the right number of idempotents for $\mathcal{A}(\mathcal{Z})$

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"Item 2" gluing of sutured surfaces should then correspond to taking tensor products $\mathcal{A}(\mathcal{Z}_1) \bigotimes \mathcal{A}(\mathcal{Z}_2)$ as desired

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Twice-decategorified level: works very nicely and there's a fully extended 2d TQFT recovering the right idempotent counts (work in preparation)

Thanks

Thanks for your time!