What languages do black holes speak?

Gleb Aminov

C. N. Yang Institute for Theoretical Physics
Simons Center for Geometry and Physics

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Plan

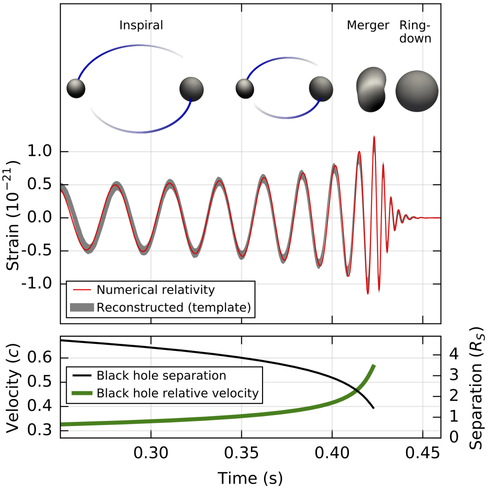
1. Detectors of gravitational waves
2. Ringdown phase and quasinormal modes (QNMs)
3. First analytic approach to QNMs: instanton corrections
4. Second analytic approach to QNMs: a new underlying recursive structure
5. Multiple polylogarithms and more

References

- **G. Aminov, A. Grassi, Y. Hatsuda**. "Black Hole Quasinormal Modes and Seiberg-Witten Theory". Ann. Henri Poincaré 23, 1951–1977 (2022). https://arxiv.org/abs/2006.06111
- **G. Aminov, P. Arnaudo, G. Bonelli, A. Grassi, A. Tanzini**. "Black hole perturbation theory and multiple polylogarithms". https://arxiv.org/abs/2307.10141
- **Gleb Aminov and Paolo Arnaudo**. Work in progress.
- **LIGO Scientific Collaboration and Virgo Collaboration**. "GW150914: First results from the search for binary black hole coalescence with Advanced LIGO"
- **NASA** illustration of LISA, taken from http://lisa.jpl.nasa.gov/gallery/lisa-waves.html
- **Mr. Gantano**, background from https://www.pptgrounds.com/3d/8265-space-galaxy-backgrounds

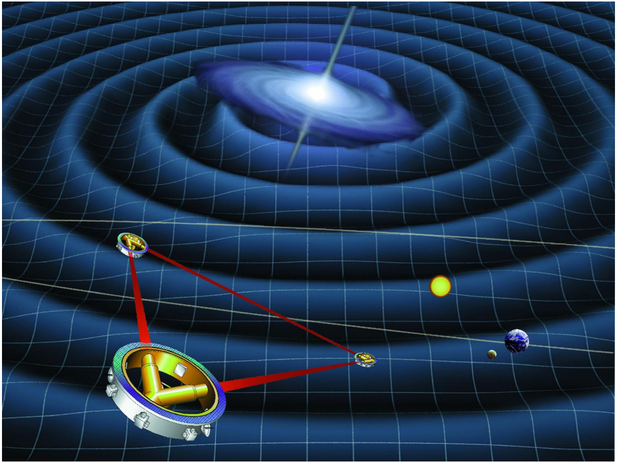
LIGO and Virgo interferometer

- GW150914:



Laser Interferometer Space Antenna (LISA)

- Expected to measure and distinguish between different QNMs



Ringdown phase and quasinormal modes

- QNMs are defined by:

$$\psi''(z) + a_1(z)\,\psi'(z) + a_0(z)\,\psi(z) = 0$$

AND

Boundary Conditions (BCs) at $z = z_1,\; z = z_2$

Different kinds of black holes

- 4d Schwarzschild black holes in three different backgrounds
- For ordinary black holes:

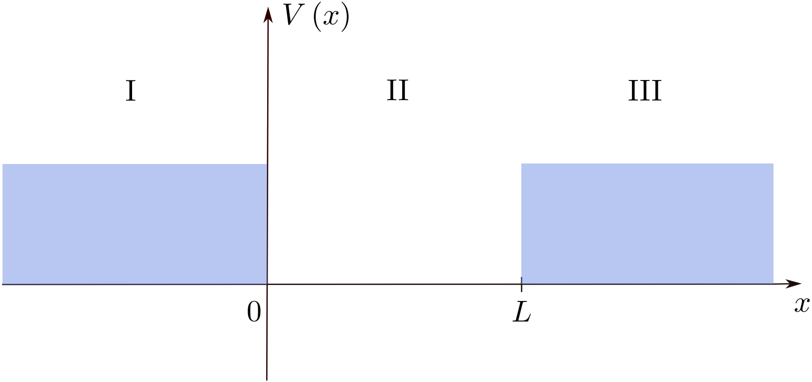
$$\phi''(r) + \frac{f'(r)}{f(r)}\phi'(r) + \frac{\omega^2 - V(r)}{f(r)^2}\phi(r) = 0$$

AND

$$\varphi(r) \sim \begin{cases} \mathrm{e}^{-\imath\omega(r+2M\log(r-2M))} & , \quad r \to 2M \\ \mathrm{e}^{+\imath\omega(r+2M\log(r-2M))} & , \quad r \to \infty \end{cases}$$

Analogy with time-independent Schrödinger equation

- Simple example in QM:



QNMs and Seiberg-Witten theory

- QNM ODEs $\Leftrightarrow$ quantum SW curves:

$$\widehat{H}_{N_f}\psi(z) = E\,\psi(z)\,.$$

- Not all BCs are equivalent!

Schwarzschild	$\Leftrightarrow$	$N_f = 3$, same BCs
SdS_4	$\Leftrightarrow$	$N_f = 4$, same BCs
SAdS_4	$\Leftrightarrow$	$N_f = 4$, different BCs

QNMs and Seiberg-Witten theory

- Quantum periods in SW theory:

$$a = \Pi_A\left(E, m_j, \Lambda\right),$$
$$\frac{\partial}{\partial a}\mathcal{F}^{NS} = \Pi_B\left(E, m_j, \Lambda\right).$$

- Quantization condition

$$\Pi_B = \frac{\partial \mathcal{F}^{NS}}{\partial a} = 2\pi\hbar\left(n+1\right), \quad n = 0, 1, \ldots.$$

- Not enough when BCs do not coincide!

QNMs and Seiberg-Witten theory

- SUSY language needs translation
- Invert the Matone relation

$$E = a^2 - \frac{\Lambda}{2N_c - N_f} \frac{\partial \mathcal{F}^{inst}}{\partial \Lambda}$$

- and use

$$E = -\ell\,(\ell + 1) + 8M^2\omega^2 - \frac{1}{4}.$$

An example

- Compute $a = a\,(M\omega, \ell, s, \Lambda)$, $\Lambda = 2\,i\,\omega$.
- For $\ell = s = 0$ and $M = \frac{1}{2}$:

$$a = \frac{1}{2}\sqrt{8\omega^2 - 1} + \frac{i\,\omega\,\Lambda}{4\sqrt{8\omega^2 - 1}} + O\left(\Lambda^2\right).$$

- Exact expression for Π_b:

$$\Pi_B = \frac{\partial \mathcal{F}^{inst}}{\partial a} - 2\,a\,\log\left[\frac{\Lambda}{\hbar}\right] - 2\,\imath\,\hbar\log\left[\frac{\Gamma\left(1 + \frac{2\imath a}{\hbar}\right)}{\Gamma\left(1 - \frac{2\imath a}{\hbar}\right)}\right]$$
$$- \imath\,\hbar\sum_{j=1}^{3}\log\left[\frac{\Gamma\left(\frac{1}{2} + \frac{m_j - \imath a}{\hbar}\right)}{\Gamma\left(\frac{1}{2} + \frac{m_j + \imath a}{\hbar}\right)}\right].$$

- We solve perturbatively in Λ

$$\Pi_B\left(M\omega, l, s, \Lambda\right) = 2\pi\hbar\left(n+1\right)$$

- and apply the Padé approximant:

Nb	$2M\omega_0(0,0)$
3	0.21453301 – 0.20342058i
8	0.22088781 – 0.20978038i
12	0.22090951 – 0.20979131i
Num	0.22090988 – 0.20979143i

- For Kerr black holes $\rightarrow$ two quantization conditions (for angular and radial parts)

Nb	$M\omega_0$	$M\omega_1$
3	0.1073438 - 0.1016159 i	0.089515 - 0.330273 i
8	0.1105221 - 0.1047959 i	0.086036 - 0.347811 i
11	0.1105330 - 0.1048013i	0.086216 - 0.347686 i
Num	0.1105331 - 0.1048015 i	0.086203 - 0.347664 i

Small ω-expansion

- Are small Λ and small ω expansions the same?

$$\Lambda = 2\, i\, \omega.$$

- Poles in the $\mathcal{F}^{inst}$:

$$\mathcal{F}^{inst} = \frac{2\, i\, \omega^3 \Lambda}{(4\, a^2 + 1)} + O\left(\Lambda^2\right)$$
$$\mathcal{F}_2^{inst} = \frac{\left(14\, \omega^2 + 5\right) \Lambda^2}{128\, (8\, \omega^2 + 3)} \sim \omega^2$$
$$\mathcal{F}_3^{inst} = -\frac{\imath\, \left(2\, \omega^2 + 1\right) \Lambda^3}{512\, \omega\, (8\, \omega^2 + 3)} \sim \omega^2.$$

- Infinitely many instanton corrections need to be resummed!

The case of $SAdS_4$

- Quantum SW periods are not enough!
- Reason: BC is not applied at the singular point
- The singularities are at $z = 0, 1, t, \infty$
- BCs are

$$\psi(z_\infty) = 0, \quad \psi(z) \sim 1 \ \text{ for } z \sim t,$$

$$z_\infty = 1 + t + \sqrt{1 - t + t^2}.$$

- Two local variables:

$$z^L = z, \quad z^R = \frac{t}{z}.$$

Left and Right regions around $z = 1$ and $z = t$

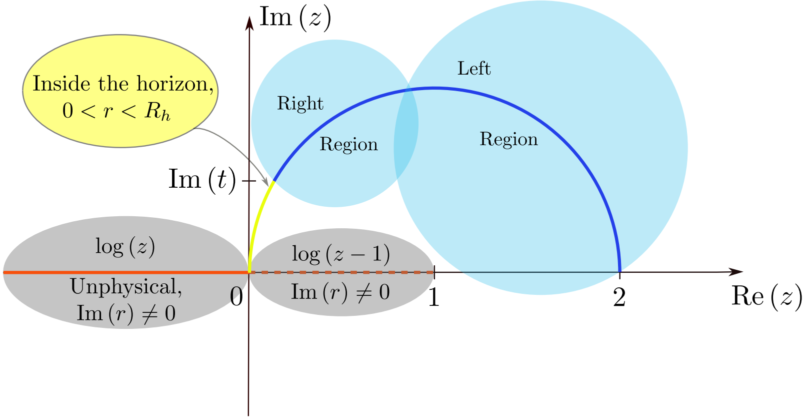


Figure: Branch cuts (red lines) on the complex z plane for anti-de Sitter black holes.

The multi polylog approach

- κ - expansion parameter
- Around z^L, z^R:

$$\psi(z) = f_0(z) + \sum_{K\geq 1} f_K(z)\kappa^K.$$

- $f_0(z)$ and $g_0(z)$ are leading order solutions
- κ is such that f_0, g_0 are elementary functions
- Wronskian - rational function:

$$W_0 \equiv f_0(g_0)' - (f_0)' g_0.$$

The multi polylog approach

- The generic solution:

$$f_K(z) = b_K g_0(z) + c_K f_0(z)$$

$$- g_0(z) \int^z f_0(z') \frac{\eta_K(z')}{W_0(z')} \, \mathrm{d}z'$$

$$+ f_0(z) \int^z g_0(z') \frac{\eta_K(z')}{W_0(z')} \, \mathrm{d}z'.$$

- c_K's can be set to zero!
- b_K's are fixed by BCs and continuity of $\psi(z)$

Multiple polylogarithms for SAdS_4

- Small parameter $t \sim R_h$
- At order $t^K \longrightarrow$ multiple polylogarithms of weight $\leq K$!
- First set of words in the BH dictionary:

$$\mathrm{Li}_{s_1,\ldots,s_n}(z) = \sum_{k_1>k_2>\cdots>k_n\geq 1} \frac{z^{k_1}}{k_1^{s_1}\ldots k_n^{s_n}}.$$

- R_h expansion of the frequencies:

$$\omega = \sum_{k\geq 0} \omega_k R_h^k, \quad \omega_0 = \pm(2\,n+\ell+2)\,.$$

Results for SAdS$_4$ QNMs

- $\omega_k \longrightarrow$ Euler sums and MZVs:

$$\zeta\left(s_1, \ldots, s_n; \epsilon\right) = \sum_{k_1 > k_2 > \cdots > k_n \geq 1} \frac{\epsilon^{k_1}}{k_1^{s_1} \ldots k_n^{s_n}}.$$

- QNMs with $n = 0$ and $\ell \geq 1$:

$$\omega\left(0, \ell, s\right) = \ell + 2 - \frac{2^{2\ell+2}}{\pi} \frac{2\,\ell + s^2}{\ell\left(\ell+1\right)} \frac{\left(\left(\ell+1\right)!\right)^2}{\left(2\,\ell+2\right)!} R_h + \mathcal{O}\left(R_h^2\right).$$

- Software and full results are available at

  https://github.com/GlebAminov/BH_PolyLog.

Scalar sector of gravitational perturbations in SAdS_4

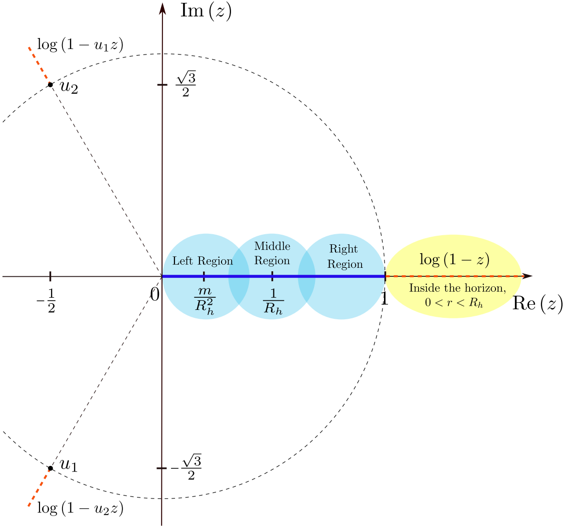
- Small parameter $\alpha = 1/R_h$
- Regular singularities at $z \sim 0, 1, u_1, u_2, \infty$
- Robin BC at spatial infinity:

$$\psi(z) \sim 1 \quad \text{for } z \sim 1,$$
$$\left\{ \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\psi(z)}{z}\right) + \left[\frac{3(1+R_h^2)}{m} + \frac{i\omega}{R_h}\right]\frac{\psi(z)}{z}\right\}\Bigg|_{z=0} = 0.$$

- Apply the multi polylog method in 3 regions:

$$x = \frac{R_h^3}{mr} + \frac{1}{3}, \quad y = \frac{R_h^2}{r}, \quad z = \frac{R_h}{r}.$$

Left, Middle, and Right regions for scalar sector



Low-lying quasinormal frequencies

- AdS/CFT $\longrightarrow$ hydrodynamic modes of 3d CFT
- Frequency expansion:

$$\omega = \sum_{k\geq 0} \omega_k \alpha^k .$$

- Second set of words in the BH dictionary:

$$\mathrm{Li}_{s_1,\ldots,s_k}(z_1,\ldots,z_k) = \sum_{n_1>n_2>\cdots>n_k\geq 1}^{\infty} \frac{z_1^{n_1}\ldots z_k^{n_k}}{n_1^{s_1}\ldots n_k^{s_k}},$$

- where $s_1 = s_2 = \cdots = s_k = 1$.

Hydrodynamic limit

- QNM frequencies of the M2-brane:

$$\mathfrak{w} = \frac{2\,\omega}{3\,R_h}, \quad R_h \to \infty, \quad \ell \to \infty, \quad \frac{2\,\ell}{3\,R_h} \to \mathfrak{q}.$$

- Upon taking the limit:

$$\mathfrak{w} = \sum_{k \geq 1} \mathfrak{w}_k\, \mathfrak{q}^k.$$

- $\mathfrak{w}_{1,2,3}$ agree with the known results
- Expressions for $\mathfrak{w}_4 - \mathfrak{w}_7$ are new!

Results for $\mathfrak{w}_k$

- Hydrodynamic frequencies $\longrightarrow$ colored multiple zeta values
- For $\mathfrak{w}_4$:

$$\mathfrak{w}_4 = -\frac{\sqrt{3}}{16}\left[\mathrm{Li}_{1,1}(u_1, u_1) + u_1 \mathrm{Li}_{1,1}(u_2, u_1) - u_2 \mathrm{Li}_{1,1}(u_1, u_2)\right] + \frac{72\, i\sqrt{3} + 24\, i\,\pi + \pi^2}{384\sqrt{3}} - \frac{12\, i\sqrt{3} + i\,\pi}{64\sqrt{3}} \log(3) + \frac{\sqrt{3}}{128}(u_2 - 3\,u_1)\log(3)^2 .$$

- Full results are available at

  https://github.com/GlebAminov/BH_PolyLog.

Advantages compared to numerical methods

- Speed and Accuracy
- Coefficients can be computed with the desired precision fast

$$
\begin{aligned}
\mathfrak{w}_1 &= \frac{1}{\sqrt{2}}, \\
\mathfrak{w}_2 &= -\frac{i}{4}\ , \\
\mathfrak{w}_3 &= 0.155473446153645..., \\
\mathfrak{w}_4 &= 0.067690388847266... \ \cdot i, \\
\mathfrak{w}_5 &= -0.010733416957692..., \\
\mathfrak{w}_6 &= 0.013959543659902... \ \cdot i, \\
\mathfrak{w}_7 &= -0.016615814626711... .
\end{aligned}
$$

Ordinary Schwarzschild black holes

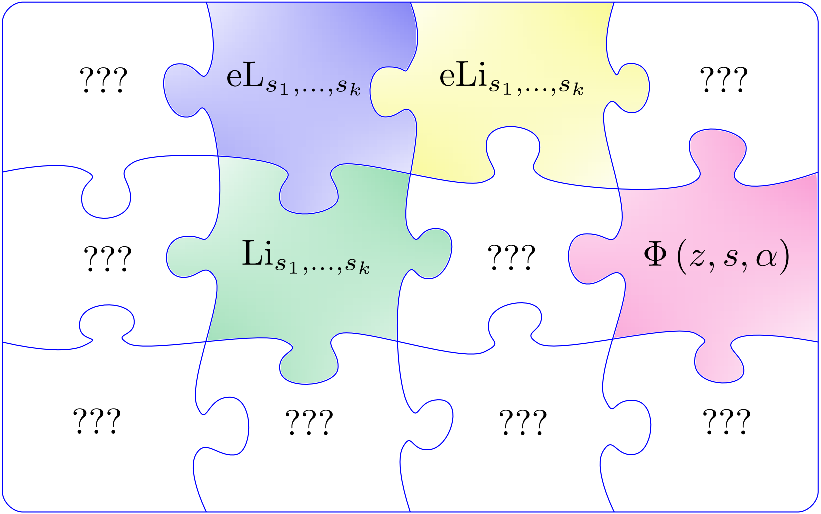
- Small parameter ω
- Regular singularities at $z \sim 0, 1$
- Irregular singularity at $z \sim \infty$
- Near-horizon region:

$$\mathrm{Li}_{s_1,\ldots,s_n}(z)$$

- Near-spatial infinity region:

$$\mathrm{eL}_{s_1,\ldots,s_n}(z) = \sum_{k_1>k_2>\cdots>k_n\geq 1} \frac{1}{k_1^{s_1}\ldots k_n^{s_n}} \frac{z^{k_1}}{k_1!}.$$

There is more to explore!



Thank you!