Knot Categorification from Mirror Symmetry

Part V

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Chapter X

String theory origin

The two dimensional theories we have been

discussing turn out to originate directly from string theory.

In a sense, string theory origins of our construction

were stumbled upon by mathematicians

many years ago,

in a related, but somewhat different context

than what we have been asking so far.

Knizhnik-Zamolodchikov equation

$$\kappa\, a_\ell \frac{\partial}{\partial a_\ell}\,\mathcal{V} = \sum_{j\neq\ell} r_{\ell i}(a_\ell/a_j)\,\mathcal{V}.$$

associated to the affine Lie algebra

$$\widehat{{}^L\mathfrak{g}_\kappa}$$

has a generalization which is even more striking from some perspectives.

The deformation is based on replacing the affine Lie algebra

$$\widehat{{}^L\mathfrak{g}}$$

by the quantum affine algebra,

$$U_\hbar(\widehat{{}^L\mathfrak{g}})$$

which is its $\hbar$-deformation.

The quantum affine algebra

$$U_\hbar(\widehat{{}^L\mathfrak{g}})$$

is related to the affine Lie algebra

$$\widehat{{}^L\mathfrak{g}}$$

by the same construction that relates the quantum group

$$U_\mathfrak{q}({}^L\mathfrak{g})$$

to the finite Lie algebra

$${}^L\mathfrak{g}$$

The quantum Knizhnik-Zamolodchikov equation

is a regular difference equation

$$\mathcal{V}(a_1,\dots pa_\ell,\dots a_n) = \mathcal{R}_{\ell\ell-1}(pa_\ell/a_{\ell-1})\cdots\mathcal{R}_{\ell 1}(pa_\ell/a_{\ell-1})(\hbar^\rho)_\ell$$

$$\times\mathcal{R}_{\ell n}(a_\ell/a_n)\dots\mathcal{R}_{\ell\ell+1}(a_\ell/a_{\ell+1})\mathcal{V}(a_1,\dots a_\ell,\dots a_n)$$

which reduces to the Knizhnik-Zamolodchikov equation

$$\kappa\, a_\ell \frac{\partial}{\partial a_\ell}\mathcal{V} = \sum_{j\neq\ell} r_{\ell i}(a_\ell/a_j)\,\mathcal{V}.$$

in the conformal limit.

While the right hand side of the Knizhnik-Zamolodchikov equation

involves the classical R-matrix

$$r_{ij}(a_i/a_j)$$

the right hand side of the quantum Knizhnik-Zamolodchihov equation

involves the R-matrix

$$\mathcal{R}_{ij}(a_i/a_j)$$

that intertwines two evaluation representations of the quantum-affine algebra.

The step p of the difference equation

$$\mathcal{V}(a_1,\ldots pa_\ell,\ldots a_n) = \mathcal{R}_{\ell\ell-1}(pa_\ell/a_{\ell-1})\cdots\mathcal{R}_{\ell 1}(pa_\ell/a_{\ell-1})(\hbar^\rho)_\ell$$

$$\times\mathcal{R}_{\ell n}(a_\ell/a_n)\ldots\mathcal{R}_{\ell\ell+1}(a_\ell/a_{\ell+1})\mathcal{V}(a_1,\ldots a_\ell,\ldots a_n)$$

is related to κ and $\hbar$ by

$$p = \hbar^{-\kappa}$$

We recover KZ equation from the qKZ equation

in the limit the deformation parameter

$$\hbar$$

goes to one, with the level κ of $\widehat{{}^L\mathfrak{g}}_\kappa$ fixed.

This is a conformal limit in the sense that, while

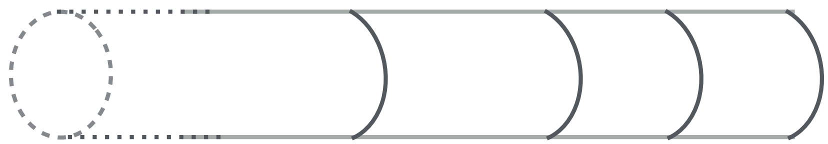
$$\widehat{{}^L\mathfrak{g}}$$

contains the Virasoro algebra as a subalgebra,

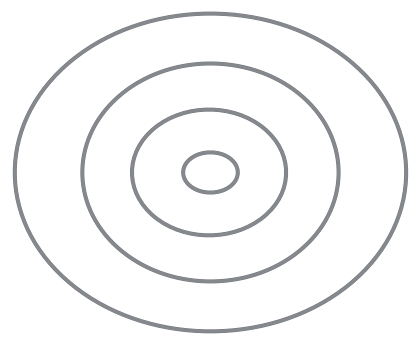
$$U_\hbar(\widehat{{}^L\mathfrak{g}})$$

does not.

The fact that deformation breaks conformal symmetry manifests itself in the fact that working on a cylinder



and the plane



are no longer equivalent from the outset.
The latter leads to the Yangian deformation of $\widehat{L\mathfrak{g}}$ instead of the quantum affine algebra.

The equation was discovered in the 80's

by I. Frenkel and Reshetikhin.

They showed that not only the equation itself,

but much of the rest of the structure of conformal field theory deforms,

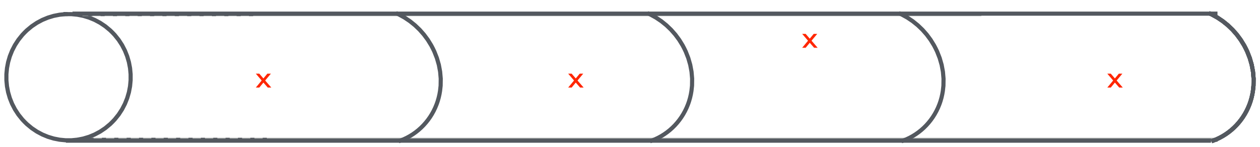
even though conformal invariance is broken.

Like in the conformal case,

the solutions of the qKZ equation may obtained by as correlators of

chiral vertex operators,

$$\mathcal{V}(a_1,\ldots,a_\ell,\ldots,a_n) = \langle\lambda|\ \Phi_{V_1}(a_1)\cdots\Phi_{V_\ell}(a_\ell)\cdots\Phi_{V_n}(a_n)\ |\lambda'\rangle$$



$\mathcal{A}$

except all the operators are q-deformed.

In particular, just like in conformal case,

the chiral vertex operators,

$$\Phi_{V_i}(a_i)$$

act as intertwiners between pairs of Verma module representations.

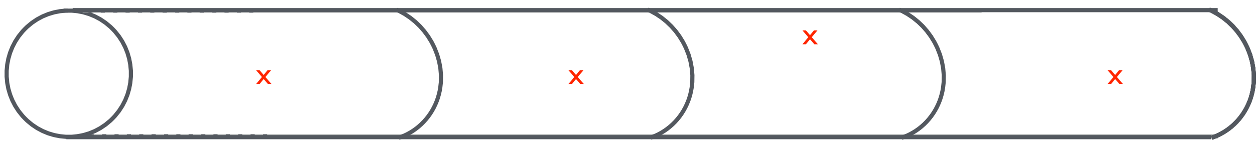
$\langle \lambda_i |$  $| \lambda_{i+1} \rangle$

$$\Phi_{V_i}(a_i) : V_{\lambda_i} \;\rightarrow\; V_i(a_i) \otimes V_{\lambda_{i+1}}$$

We will call the solutions of the qKZ equation

obtained in this way

$$\mathcal{V}(a_1,\ldots,a_\ell,\ldots,a_n) = \langle\lambda|\; \Phi_{V_1}(a_1)\cdots\Phi_{V_\ell}(a_\ell)\cdots\Phi_{V_n}(a_n)\;|\lambda'\rangle$$



$\mathcal{A}$

the q-conformal blocks of

$$U_\hbar(\widehat{{}^L\mathfrak{g}})$$

The reason quantum Knizhnik-Zamolodchikov equation

and its solutions are remarkable

is that they had absolutely no a-priori reason to exist.

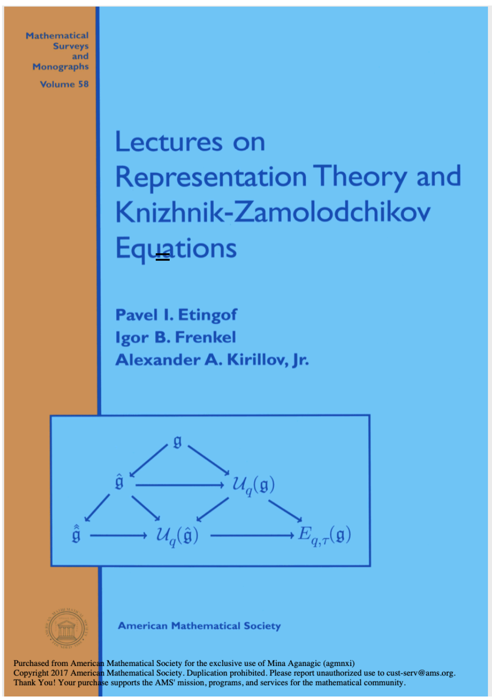
There is no a-priori natural way to a break conformal invariance
and deform the KZ equation to a difference equation.

There are many ways to do it,
and no there is not reason to expect anything good to come out of it.

Why was it possible to break conformal invariance,
yet preserve all of its underlying structure?

Mathematicians of the past have worried about this.

In the preface to their book describing the theory of the Knizhnik-Zamolodchikov equation and its q-deformation its authors, Etingof, Frenkel and Krillov

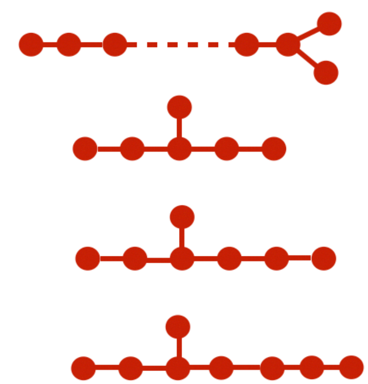


admit to falling prey to the "addictive charm" of the "q-disease" and defend themselves by saying that all eventually realize that the q-case is much more interesting than the original.

The reason this structure exists is that it originates

from a very remarkable string theory in six dimensions

labeled by a choice of a simply laced Lie algebra



$\mathfrak{g} = A_n$

$\mathfrak{g} = D_n$

$\mathfrak{g} = E_6$

$\mathfrak{g} = E_7$

$\mathfrak{g} = E_8$

This six dimensional string theory has supersymmetry of (0,2) type,

and other than having to pick a simply laced Lie algebra,

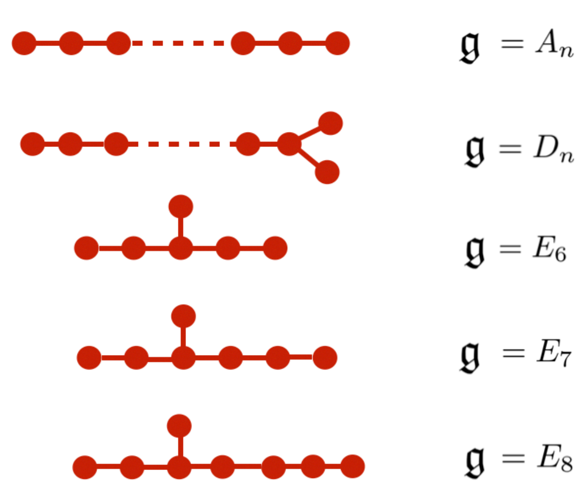
it is completely unique.

Any string theory breaks conformal invariance,

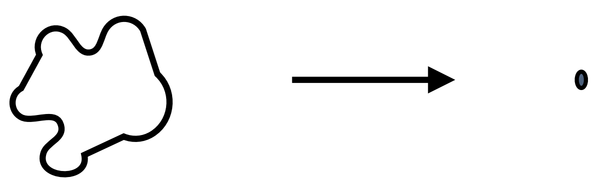
because it has a scale,

which is the characteristic size of the string.

The six dimensional string theory



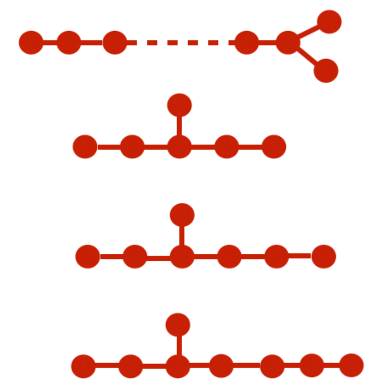
has a point particle limit, in which it becomes a six dimensional



(0,2) theory which turns out to have conformal symmetry.

That this six dimensional (0,2) conformal field theory,

labeled by a choice of a Lie algebra



$\mathfrak{g} = A_n$

$\mathfrak{g} = D_n$

$\mathfrak{g} = E_6$

$\mathfrak{g} = E_7$

$\mathfrak{g} = E_8$

should know everything there is to know about ….

.....the $U_{\mathfrak{q}}({}^L\mathfrak{g})$ quantum link and three manifold invariants

and their categorification

has been anticipated by physicists, since the pioneering works of

Ooguri and Vafa

in 1999, the same year when Khovanov's

paper on categorification of the Jones polynomial appeared,

and of Gukov, Vafa and Schwarz in 2004,

who made the connection between the two works.

The difficulty of getting the physics to bear on the

categorification problem

is that the six dimensional theory is

extremely difficult to understand directly,

even for physicists

incomparably more so than, for example, Yang-Mills theory,

since it does not have a classical limit to use as the starting point.

This makes it hard to get an angle on the problem

that is precise and general enough

to give a unified framework for knot categorification that comes from physics,

as the works of Ooguri and Vafa and Gukov, Schwartz and Vafa

told us to expect.

It turned out that, very surprisingly,

thinking about the six dimensional string theory

rather than the six dimensional conformal field theory

opens up a window into the problem.

This is in part an illustration of the observation of

Etingof, Frenkel and Kirilov,

of the q-case is in someways more interesting yet.

The six dimensional string theory is obtained from IIB string theory on an ADE surface singularity of type

$$\mathfrak{g}$$

by taking a limit which keeps only the degrees of freedom supported at the singularity and decouples the 10d bulk.

One wants to study the six dimensional (2,0) little string theory on

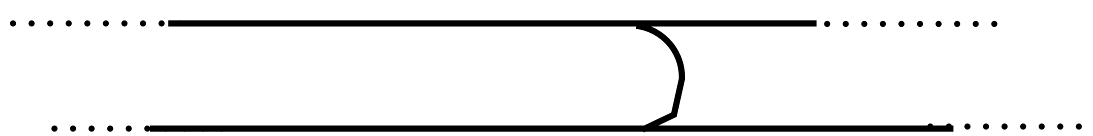
$$M_6 \approx \mathcal{A} \times D \times \mathbb{C}$$

where

$$\mathcal{A} = S^1 \times \mathbb{R}$$

is the Riemann surface where the conformal blocks live,

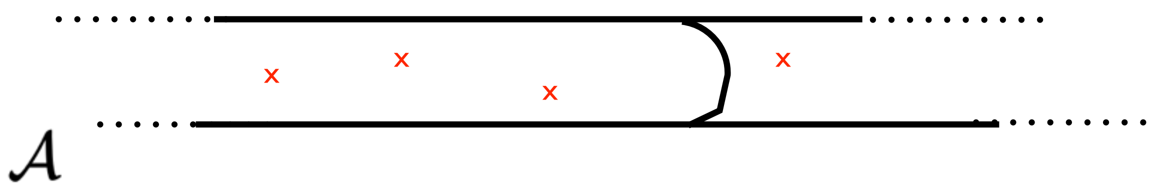
$\mathcal{A}$



$\mathbb{R} \times \mathbb{C}$ is the space where the monopoles live

and D is the domain curve of the 2d theories we had so far.

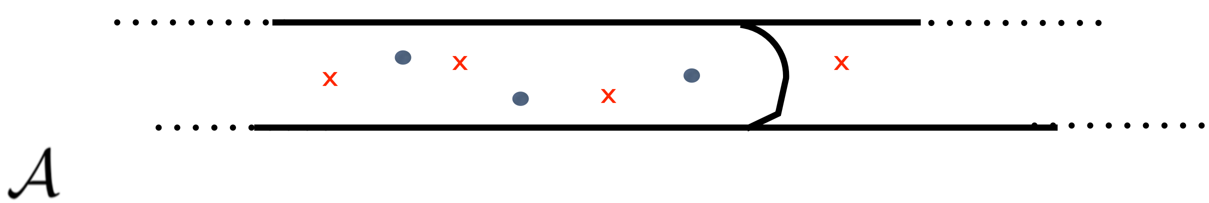
The vertex operators on the Riemann surface



come from a collection of defects in the little string theory,

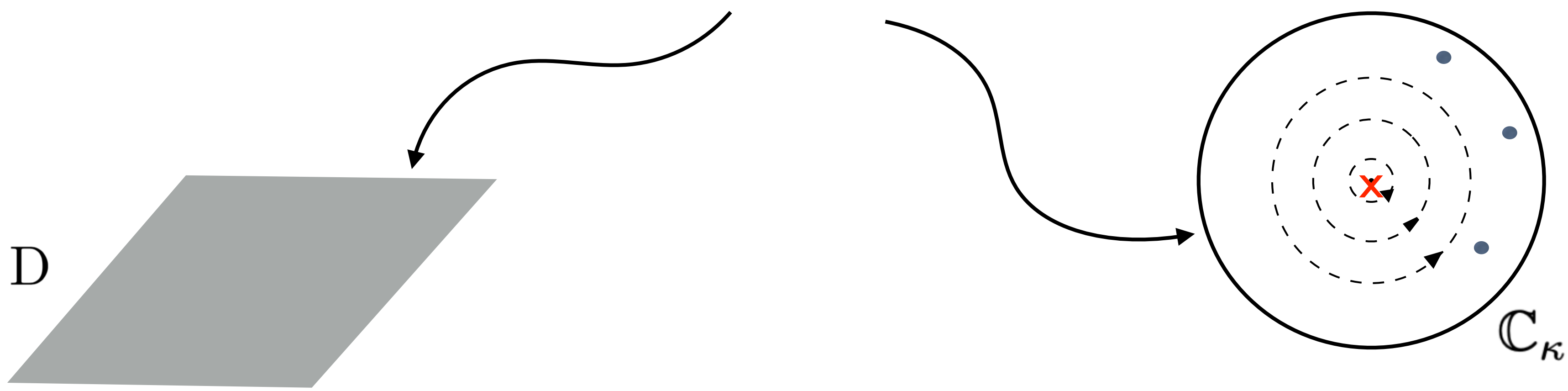
which are inherited from D-branes of the ten dimensional string.

The D-branes needed D3 branes of IIB string which lead to



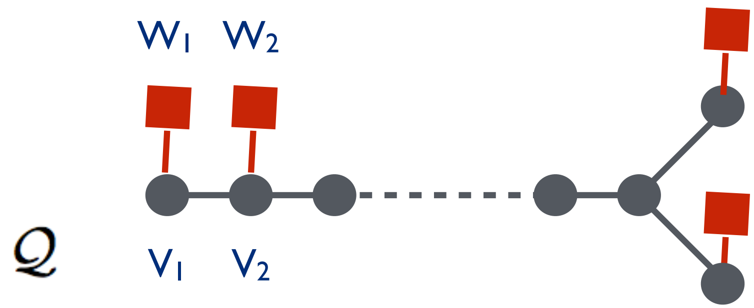
two dimensional defects of the six dimensional theory on

$$\mathcal{A} \times \mathrm{D} \times \mathbb{C}_\kappa$$



supported on D and points on $\mathbb{C}_\kappa$

The theory on the D-branes is the quiver gauge theory

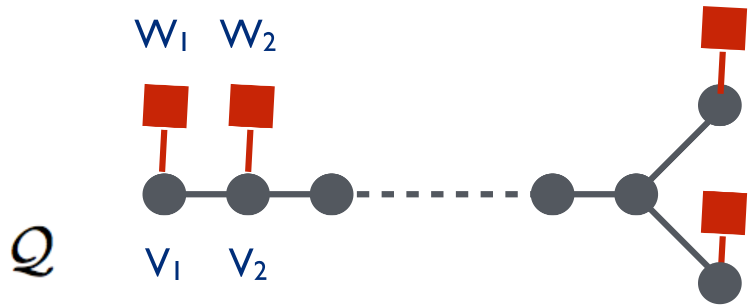


This is a consequence of the familiar description of

D-branes on ADE singularities

due to Douglas and Moore in '96.

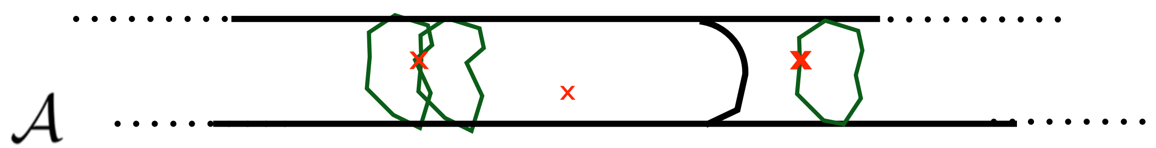
The theory on the D-branes supported on D



is a three dimensional quiver gauge theory on

$$\mathrm{D} \times S^1$$

rather than a two dimensional theory on D



due to string winding modes.

One of the results of my prior work with Andrei Okounkov is that

one can get the fundamental solution of the qKZ equation

which span the space of conformal blocks of the

$$U_\hbar(\widehat{{}^L\mathfrak{g}})$$

from either the

the Coulomb branch $\mathcal{X}$

the Higgs branch $\mathcal{X}^\vee$

of the 3d gauge theory.

The q-conformal blocks of

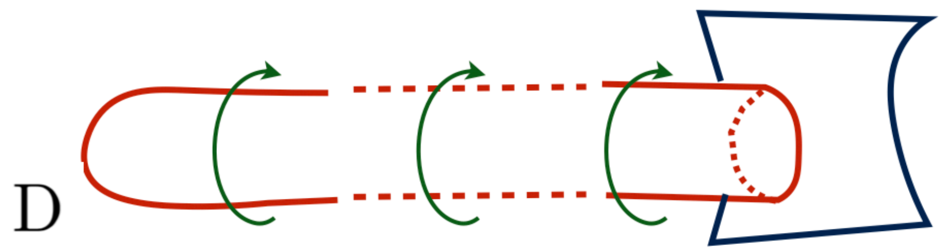
$$U_\hbar(\widehat{{}^L\mathfrak{g}})$$

are the supersymmetric partition functions

$$\mathrm{Tr}(-1)^F p^{S-S_H} \hbar^{S_H-S_V}$$

the three dimensional N=4 supersymmetric gauge theory on

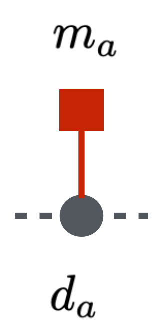
$$\mathrm{D} \times S^1$$



which include holonomies for the R-symmetries

$$U(1)_H \times U(1)_V \in SU(2)_H \times SU(2)_V$$

The ranks of the vector spaces



determine the representation,

$$V_1 \otimes \ldots \otimes V_\ell \otimes \ldots \otimes V_n$$

and a weight ν in that representation which qconformal blocks

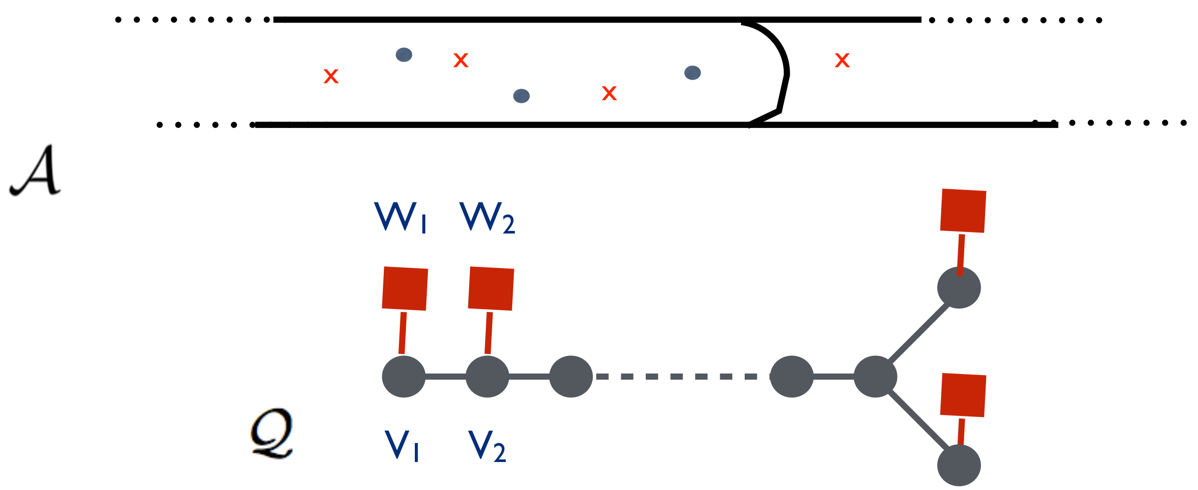
$$\langle\lambda|\ \Phi_{V_1}(a_1)\cdots\Phi_{V_\ell}(a_\ell)\cdots\Phi_{V_n}(a_n)\ |\lambda'\rangle$$

transform in.

Positions of vertex operators

$$\mathcal{V}(a_1, \ldots, a_\ell, \ldots, a_n) = \langle\lambda|\ \Phi_{V_1}(a_1)\cdots\Phi_{V_\ell}(a_\ell)\cdots\Phi_{V_n}(a_n)\ |\lambda'\rangle$$

are the positions of heavy, flavor D3 branes on the Riemann surface.



The highest weight vector of Verma module $\langle\lambda|$

$$\mathcal{V}(a_1,\ldots,a_\ell,\ldots,a_n) = \langle\lambda|\ \Phi_{V_1}(a_1)\cdots\Phi_{V_\ell}(a_\ell)\cdots\Phi_{V_n}(a_n)\ |\lambda'\rangle$$

is the Fayet-Iliopoulos parameter of the 3d theory. It comes from a

complex scalar field in the (2,0) theory which abelianizes it.

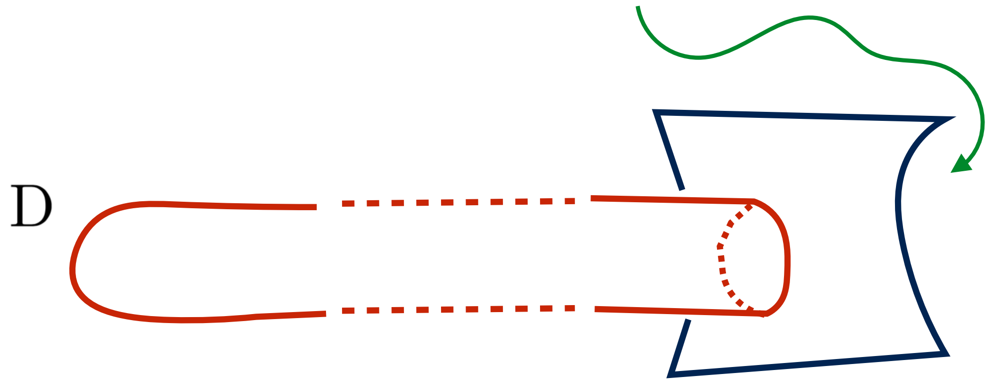
The fact that the theory lives in the broken phase of the (2,0) theory

is important, because it implies the complicated bulk dynamics is not important.

Which solution of the qKZ equation

the partition function computes depends on the choice of boundary condition

at infinity of



The fact that one can obtain the partition function from

either

$\mathcal{X}$ $\qquad\qquad$ $\mathcal{X}^\vee$

is a reflection of

three dimensional mirror symmetry,

which implies

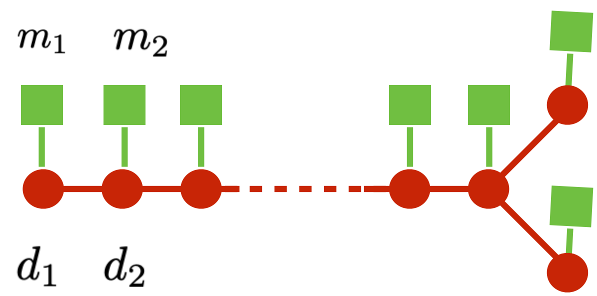
with suitable identification of parameters and boundary conditions,

they are interchangeable

for the kind of questions we are asking.

For the theory defined by the quiver $\mathcal{Q}$,

$\mathfrak{g} = D_n$



the Coulomb branch is our

$$\mathcal{X} = \mathrm{Gr}^{\vec{\mu}}{}_{\nu}$$

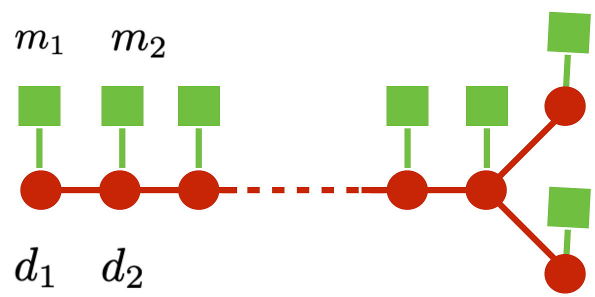
thus giving $\mathcal{X}$ a third interpretation, in addition to monopole moduli space,

and the resolution of transversal slice in affine Grassmannian.

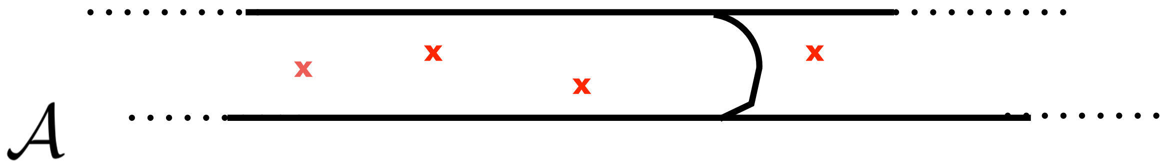
The Higgs branch is the Nakajima quiver variety

$$\mathcal{X}^{\vee} = T^{*}\mathrm{Rep}\,\mathcal{Q} /\!/\!/\!/ G_{\mathcal{Q}}$$

$\mathcal{Q}$



The positions of vertex operators on



are equivariant parameters of

$$\mathcal{X}^\vee = T^*\mathrm{Rep}\,\mathcal{Q}////G_{\mathcal{Q}}$$

and the Kahler parameters of

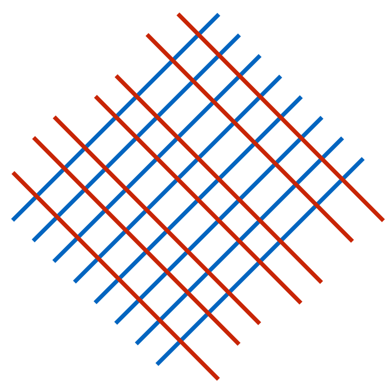
$$\mathcal{X} = \mathrm{Gr}^{\vec{\mu}}{}_{\nu}$$

Pursuing this story further,

rather than discovering knot invariants

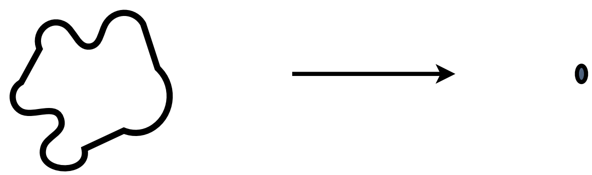
we would discover integrable lattice models,

those of a very general kind.



This story is developed in the work with Andrei.

The conformal, point particle limit of the string theory



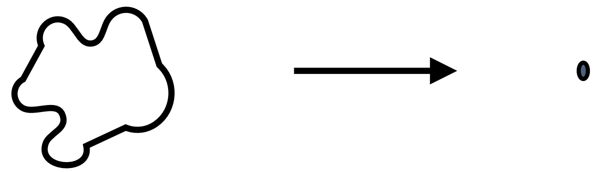
in which it becomes the six dimensional conformal field theory

of type $\mathfrak{g}$

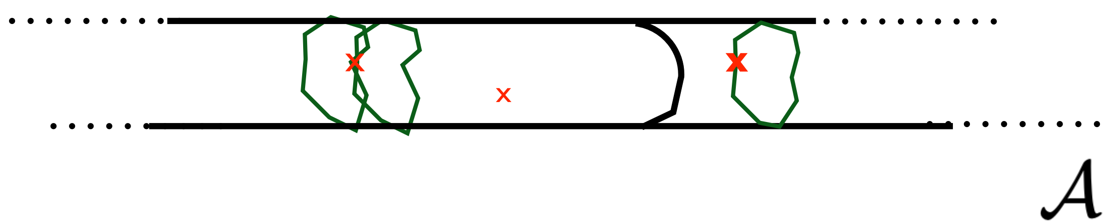
coincides with the conformal limit of the quantum affine algebra

$$U_\hbar(\widehat{{}^L\mathfrak{g}}) \longrightarrow \widehat{{}^L\mathfrak{g}}$$

In the point particle limit,



the winding modes that made the theory
on the defects three dimensional, instead of two,
become infinitely heavy.



As a result, in the conformal limit, the theory on the defects
becomes a two dimensional theory on D

It is surprising, but well understood

that there are different two dimensional limits

a three dimensional gauge theory can have.

The point particle limit of little string theory

specifies which two dimensional limit

of the three dimensional gauge theory on a circle we need to take.

The limit we need is not the one that takes the 3d gauge

to a two dimensional gauge theory with the same Lagrangian

rather, it is the limit that would do this for the 3d mirror theory.

The conformal limit takes

$$U_\hbar(\widehat{{}^L\mathfrak{g}}) \longrightarrow \widehat{{}^L\mathfrak{g}}$$

and the qKZ equation to the corresponding KZ equation.

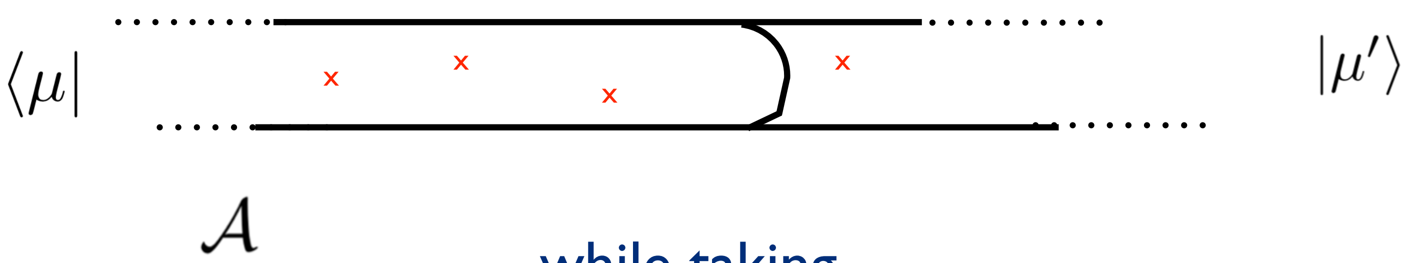
It is amounts to

$$\begin{aligned} \hbar &\to 1 \\ p = \hbar^{-\kappa} &\to 1 \qquad\qquad \kappa, a, \ \mu \text{ fixed} \\ z = \hbar^{\mu} &\to 1 \end{aligned}$$

The limit undoes the $\hbar$ deformation

$$U_\hbar(\widehat{^L\mathfrak{g}}) \quad \rightarrow \quad \widehat{^L\mathfrak{g}}$$

keeps positions of vertex operators fixed



while taking

$$z = \hbar^\mu \rightarrow 1$$

This corresponds to keeping the data of the conformal block fixed.

The conformal limit treats

$$\mathcal{X} = \mathrm{Gr}^{\vec{\mu}}{}_{\nu} \quad \text{and} \quad \mathcal{X}^{\vee} = T^{*}\mathrm{Rep}\,\mathcal{Q}/\!/\!/\!/G_{\mathcal{Q}}$$

very differently,

since it treats the

z- and the a-variables,

differently:

$$z \to 1, \qquad a \text{ fixed}$$

Kahler for

$$\mathcal{X}^{\vee} = T^{*}\mathrm{Rep}\,\mathcal{Q}/\!/\!/\!/G_{\mathcal{Q}}$$

Kahler for

$$\mathcal{X} = \mathrm{Gr}^{\vec{\mu}}{}_{\nu}$$

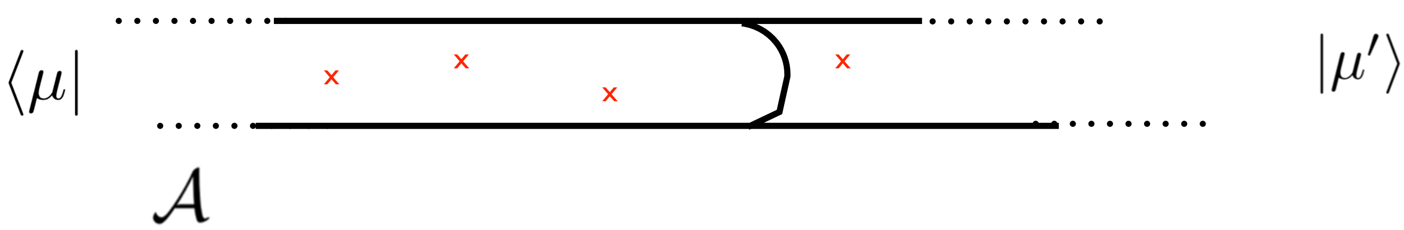
From perspective of the Coulomb branch,

$$\mathcal{X} = \mathrm{Gr}^{\vec{\mu}}{}_{\nu}$$

the limit is perfectly geometric.

Its Kahler variables are the a -variables,

the positions of vertex operators,



which are kept fixed.

The conformal limit,

is not a geometric limit from perspective of the Higgs branch

$$\mathcal{X}^{\vee} = T^{*}\mathrm{Rep}\,\mathcal{Q}/\!/\!/\!/G_{\mathcal{Q}}$$

The limit results in a badly singular space,

since

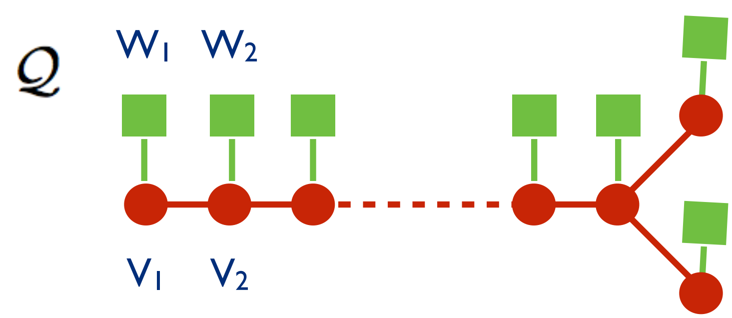
$$z \rightarrow 1$$

is a limit in its Kahler variables.

What one gets instead is the Landau-Ginsburg model

with target

Y and potential W

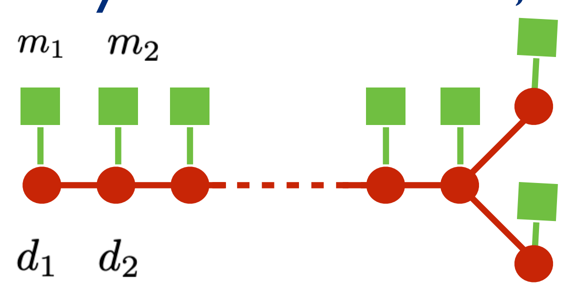


The potential

$$W = \lambda_0 W^0 + \sum_{a=1}^{\mathrm{rk}} \lambda_a \, W^a$$

is a limit of the three dimensional effective

superpotential, which arizes by a semi-classical, i.e. one-loop computation,



and for this reason can be read off from the quiver, as a sum of contributions

associated to its nodes and its arrows.

Chapter XI

This has an extension to non-simply laced Lie algebras, for which

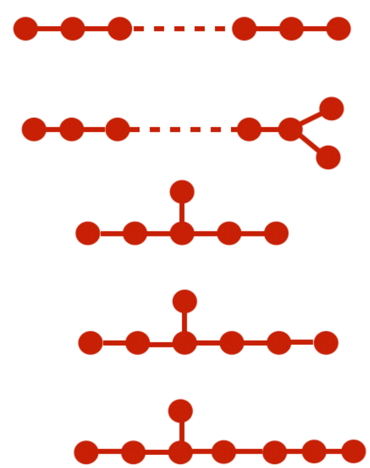
$$ {}^L\mathfrak{g} \neq \mathfrak{g} $$

which is dictated by string theory.

The (2,0) theory in six dimensions

is always associated with

simply laced Lie algebras, of ADE types.



$\mathfrak{g} = A_n$

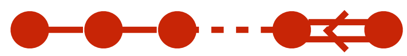
$\mathfrak{g} = D_n$

$\mathfrak{g} = E_6$

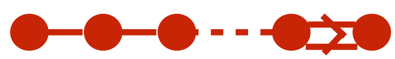
$\mathfrak{g} = E_7$

$\mathfrak{g} = E_8$

The non-simply laced Lie algebras

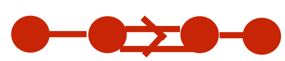


$$\mathfrak{g} = C_n$$



$$\mathfrak{g} = B_n$$

$$\mathfrak{g} = G_2$$



$$\mathfrak{g} = F_4$$

arize only upon compactification

One uses the fact that the non-simply laced Lie algebra

$$\mathfrak{g}$$

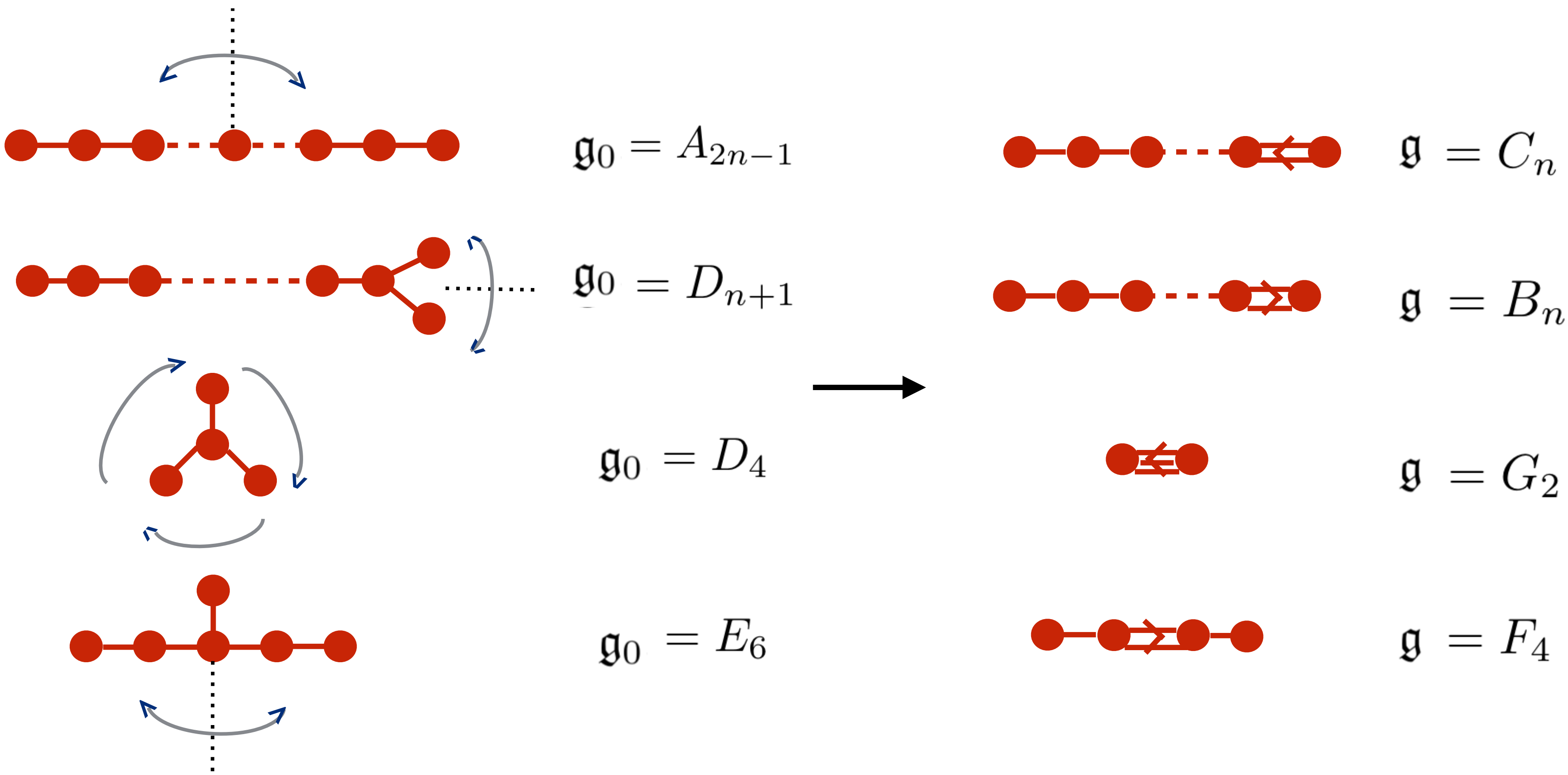
can be obtained from a simply laced Lie algebra

$$\mathfrak{g}_0$$

using an outer automorphism H of its Dynkin diagram

$$(\mathfrak{g}_0, H) \to \mathfrak{g}$$

H acts as an involution of the Dynkin diagram of $\mathfrak{g}_0$



From this perspective,

elements of the root lattice of

$\mathfrak{g}$

are elements of the root lattice of

$\mathfrak{g}_0$

in a single orbit of H.

A root of $\mathfrak{g}$ is long or short, depending on the length of the H-orbit.

Langlands duality exchanges the root lattice of

$$\mathfrak{g}$$

with the co-weight lattice of

$$^{L}\mathfrak{g}$$

Mapping ,

$\mathfrak{g} = C_n$ to $^{L}\mathfrak{g} = B_n$

$\mathfrak{g} = B_n$ to $^{L}\mathfrak{g} = C_n$

and preserving others.

The co-weight lattice of

$$\mathfrak{g}$$

where the singular monopole charges live

maps to the weight lattice of

$$^{L}\mathfrak{g}$$

where charges of electrically charged particles live.

To get link invariants based on the Lie algebra

$$^{L}\mathfrak{g}$$

one studies little string theory of type

$$\mathfrak{g}_0$$

on

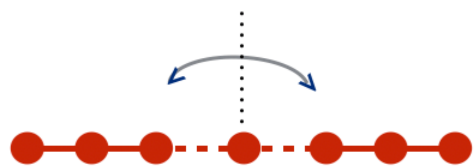
$$M_6 = \mathcal{A} \times \mathrm{D} \times \mathbb{C}_\hbar$$

with an H-twist

The twist

permutes the nodes of the Dynkin diagram of $\mathfrak{g}_0$

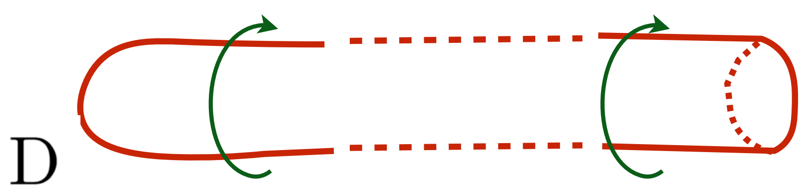
by a generator of H,



$\mathfrak{g}_0 = A_{2n-1}$

as we go once around the origin of the complex D -plane

which supports the defects.



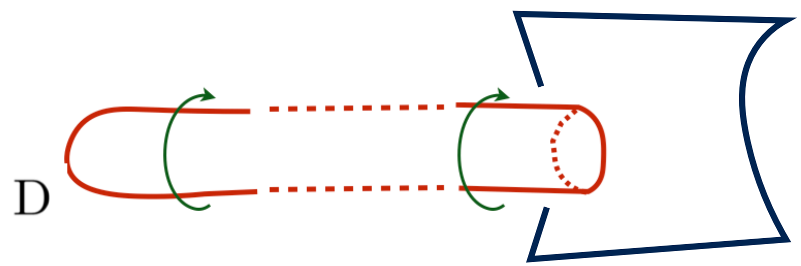
The theory on D-branes is a three dimensional

$\mathfrak{g}_0$ -type quiver gauge theory

on

$\mathrm{D} \times S^1$

with an H- twist around D



When we impose the boundary conditions,

they too need to be compatible with the twist.

In the conformal limit,

the theory

becomes two dimensional.

It has two different descriptions

both of which originate from the

$\mathfrak{g}_0$

theory and involve the H-twist.

The resulting partition functions

should compute conformal blocks of

$$\widehat{{}^L\mathfrak{g}}_\kappa$$

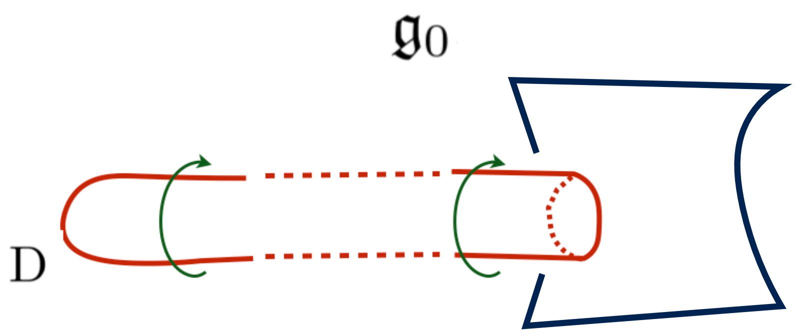
the affine Lie algebra

based on the Langlands dual group.

This is easiest to show directly in the Landau-Ginsburg model

with target Y_0 and potential W_0

both based on



In the B-model on D, only the zero modes contribute,

and since H acts by permutation, one simply identifies the fields

in a single orbit of the H-action.

Restricting to H-invariant sector

gives an effective theory with target

$$Y = (Y_0)^h, \; W = (W_0)^h.$$

The corresponding partition function, of the form

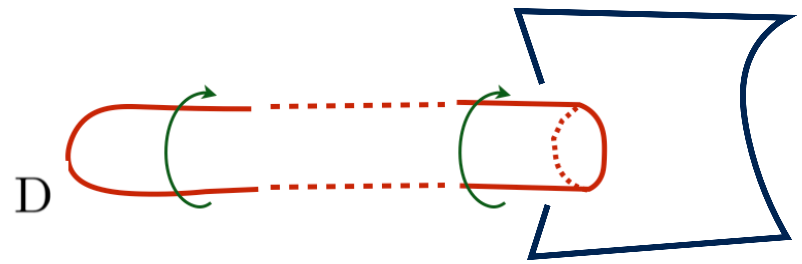
$$\mathcal{V}_a[L] = \int_L \Phi_a \, \Omega \, e^{W_{LG}}.$$

is the integral formula of Feigin and Frenkel for $\widehat{^L\mathfrak{g}}_\kappa$

conformal blocks.

Since the partition function of the (Y_0, W_0) theory on D

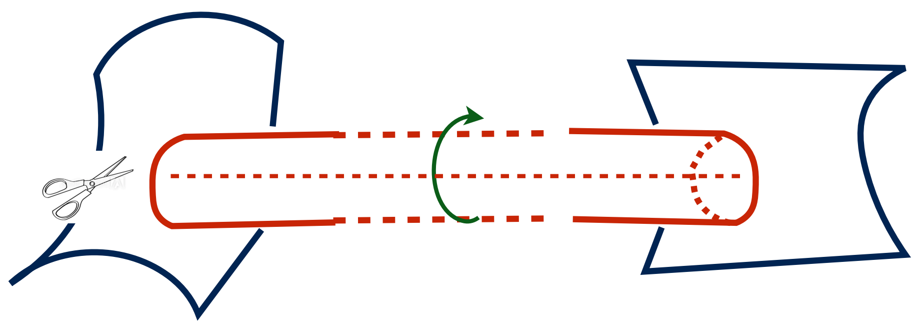
with H-twist



computes the conformal blocks of

$$\widehat{{}^L\mathfrak{g}}_\kappa$$

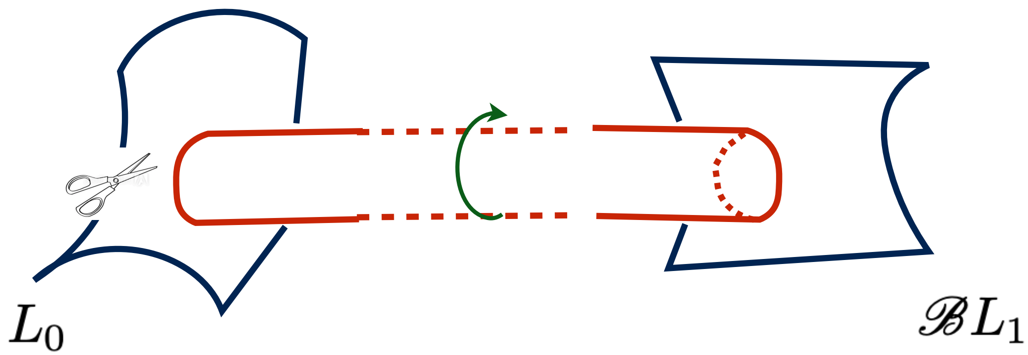
the corresponding annulus partition function



will compute the braiding matrices of

$$U_\mathfrak{q}({}^L\mathfrak{g})$$

The annulus partition function computes intersections



of a pair of Lagrangians

$$\sum_{\mathcal{P}\in L_0\cap \mathscr{B}L_1} (-1)^{F_0(\mathcal{P})}\ \mathfrak{q}^{J_0(\mathcal{P})}\gamma(h)$$

where we now include the twist corresponding to action of H by permutations.

Since H is a symmetry of the Landau-Ginsburg model

it should commute with the action of the differential Q.

It follows that Q should preserve the

subset of fixed points which are H-invariant,

allowing one to define a reduced complex to which only these states contribute.

$$CF^*(L,L')^{H} = \bigoplus_{\substack{\mathcal{P}\in L\cap L', \\ H-\text{invariant}}} \mathbb{C}\,[\mathcal{P}]$$

whose cohomology should give the $U_{\mathfrak{q}}({}^L\mathfrak{g})$ link homology.

Chapter XII

Another approach

There is a third description,
due to Witten.

It describes the same physics,
just from the bulk perspective.

Compactified on a very small circle,

the six dimensional $\mathfrak{g}$ -type (2,0) conformal theory

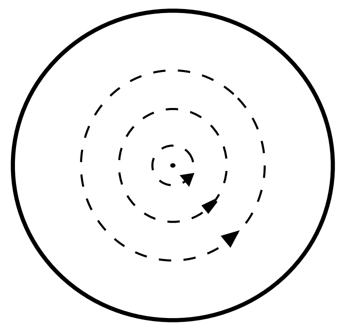
with no classical description,

becomes a $\mathfrak{g}$ -type gauge theory

in one dimension less.

To get a good 5d gauge theory description of the problem,

the circle one shrinks corresponds to S^1 in



so from a six dimensional theory on

$$M_6 \approx \mathcal{A} \times D \times \mathbb{C}$$

one gets a five-dimensional gauge theory on a manifold with a boundary

The five dimensional gauge theory is supported on

$$\widetilde{M}_5 = \widetilde{M}_3 \times \mathrm{D} \quad \text{where} \quad \widetilde{M}_3 = \mathscr{A} \times \mathbb{R}_{\geq 0}$$

It has gauge group

$$G$$

which is the adjoint form of a Lie group with lie algebra $\mathfrak{g}$.

Our two dimensional defects become monopoles

of the 5d gauge theory on

$$\widetilde{M}_5 = \widetilde{M}_3 \times \mathrm{D}$$

supported on D and at points on,

$$\widetilde{M}_3 = \mathcal{A} \times \mathbb{R}_{\geq 0} \ ,$$

along its boundary.

Witten shows that the five dimensional theory on

$$\widetilde{M}_5 = \widetilde{M}_3 \times \mathrm{D}$$

can be viewed as a gauged

Landau-Ginzburg model on D with potential

$$\mathcal{W}_{\mathrm{CS}} = \int_{\widetilde{M}_3} \mathrm{Tr}(A \wedge dA + A \wedge A \wedge A)$$

on an infinite dimensional target space $\mathcal{Y}_{\mathrm{CS}}$

corresponding to $\mathfrak{g}_{\mathbb{C}}$ connections on $\widetilde{M}_3 = \mathcal{A} \times \mathbb{R}_{\geq 0}$

with suitable boundary conditions (depending on the knots).

To obtain knot homology groups in this approach,
one ends up counting solutions to
certain five dimensional equations.

The equations arise in
constructing the Floer cohomology groups
of the five dimensional Landau-Ginzburg theory.

Thus, we end up with three different approaches
to the knot categorification problem,
all of which have the same
six dimensional origin.

They all describe the same physics
starting in six dimensions.

The two geometric approaches,
describe the physics from perspective of the defects that introduce knots
in the theory.

The approach based on the 5d gauge theory,
describes it from perspective of the bulk.

In general,
theories on defects
capture only the local physics of the defect.

In this case,
they capture all of the relevant physics,
due to a version of supersymmetric localization:
in the absence of defects,
the bulk theory is trivial.