Knot Categorification
from Mirror Symmetry

Part IV

Mina Aganagic

UC Berkeley

Chapter VI

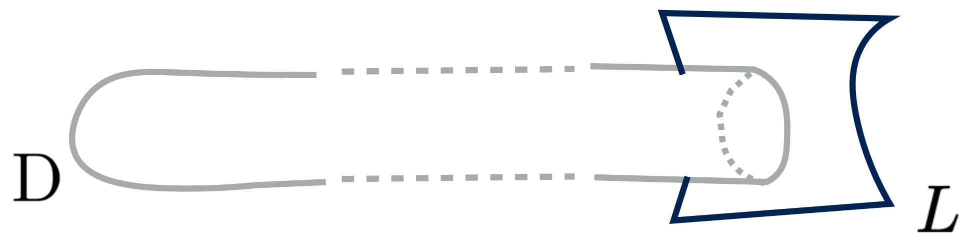
The category of A-branes on

$$(Y, W)$$

Corresponding to a solution of the

Knizhnik-Zamolodchikov equation

is an A-brane at the boundary of D at infinity,



The brane is an object of the category of A-branes

$$\mathscr{D}_Y = D(\mathcal{FS}(Y, W))$$

the derived Fukaya-Seidel category of Y with potential W

The objects of the derived Fukaya-Seidel category

$$\mathscr{D}_Y = D(\mathcal{FS}(Y, W))$$

are simply "graded" Lagrangians on Y

The grading is what is needed to define

cohomological and equivariant degrees of both the objects

and morphisms between them.

The cohomological, or Maslov grading is

the lift of the phase of

$$\Omega^{\otimes 2} = |\Omega^{\otimes 2}| e^{i2\varphi}$$

to a real valued function on the Lagrangian

$$2\varphi : L \to \mathbb{R}.$$

If we denote by $\widetilde{L}$ an A-brane with a specific lift,

we will denote by $\widetilde{L}[d]$ the brane whose lift differs by $2\pi i d$

$$2\varphi|_{\widetilde{L}[d]} = 2\varphi|_{\widetilde{L}} + 2\pi i d$$

In our case,

$$\mathscr{D}_Y = D(\mathcal{FS}(Y, W))$$

should be thought of as a category of

equivariant A-branes,

due to the fact W is multi-valued.

The equivariant gradings on

$$\mathscr{D}_Y = D(\mathcal{FS}(Y, W))$$

come from a collection of one forms on Y

$$c^0 = dW^0/2\pi i, \qquad c^a = dW^a/2\pi i \;\in\; H_1(Y, \mathbb{Z})$$

with integer periods related to the potential by

$$W = \lambda_0 W^0 + \sum_{a=1}^{\mathrm{rk}} \lambda_a W^a$$

where $\mathfrak{q} = e^{2\pi i \lambda_0}$.

The equivariant grading of a Lagrangian is the choice of the lift of phase of

$$e^{-W}$$

where

$$W = \lambda_0 W^0 + \sum_{a=1}^{\mathrm{rk}} \lambda_a W^a$$

to a real valued function on Lagrangians on Y

similarly to what we did for the Maslov gradings.

Changing the lift is the equivariant degree shift operation:

$L \to L\{\vec{n}\}$ with $\vec{n} \in \mathbb{Z}^{\mathrm{rk}+1}$

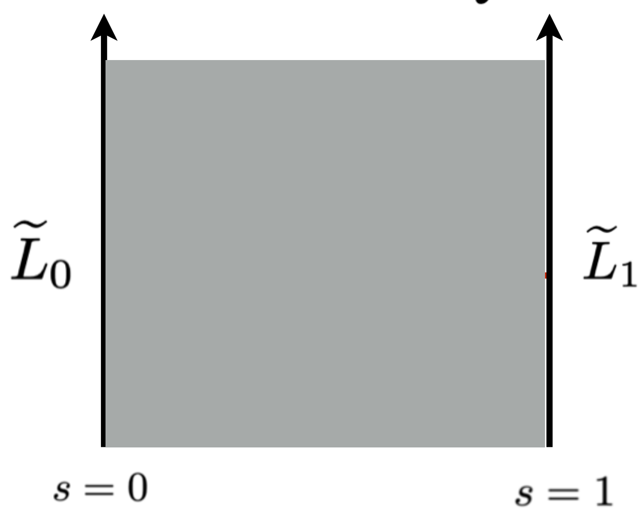
$$W|_{L\{\vec{n}\}} = W|_L + 2\pi i \vec{\lambda} \cdot \vec{n}$$

The space of morphisms between a pair of branes

$$Hom^{*,*}_{\mathscr{D}_Y}(\widetilde{L}_0, \widetilde{L}_1) = \mathrm{Ker}\, Q/\mathrm{Im}\, Q.$$

is the space of supersymmetric ground states,

the cohomology of the supercharge Q of theory on an interval,



with pairs of A-branes as boundary conditions,

in parallel to what we had on the B-side.

The spaces of morphisms, or supersymmetric ground states

$$Hom^{*,*}_{\mathscr{D}_Y}(\widetilde{L}_0, \widetilde{L}_1) = \mathrm{Ker}\, Q/\mathrm{Im}\, Q.$$

are defined by Floer theory,

which is modeled after Morse theory approach to supersymmetric

quantum mechanics.

The starting point is the Floer complex, which is the vector space

$$CF^{*,*}(L_0, L_1) = \bigoplus_{\mathcal{P} \in L_0 \cap L_1} \mathbb{C}\mathcal{P}.$$

the space of perturbative ground states,

spanned by the intersection points of the two Lagrangians,

and graded by cohomological (or Maslov) and the equivariant degrees.

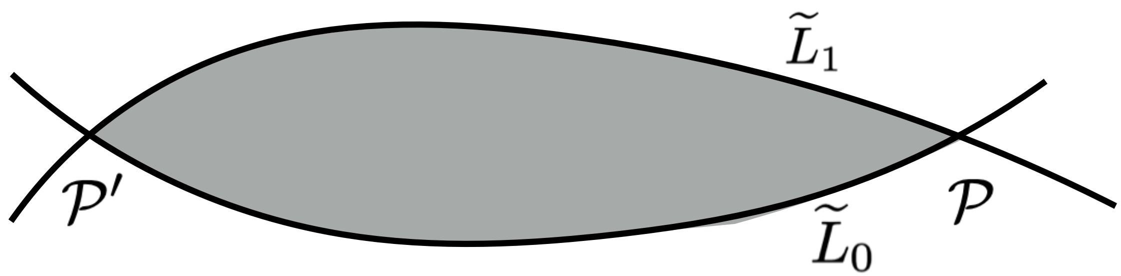
The vector space becomes a complex via the action of the differential

$$Q : CF^{*,*}(L_0, L_1) \to CF^{*+1,*}(L_0, L_1)$$

obtained by counting holomorphic disk instantons

$$y: \ \mathrm{D} \to Y$$

which map the incoming end of the strip to $\mathcal{P}$, the outgoing end to $\mathcal{P}'$



which have cohomological, or Maslov degree one

$$\mathrm{ind}(y) = M(\mathcal{P}) - M(\mathcal{P}') = 1$$

and equivariant degree zero.

The space of morphisms between the branes in $\mathscr{D}_Y$

$$Hom^{*,*}_{\mathscr{D}_Y}(\widetilde{L}_0, \widetilde{L}_1) \;=\; HF^{*,*}(L_0, L_1)$$

is the cohomology of the complex

$$Q : CF^{*,*}(L_0, L_1) \to CF^{*+1,*}(L_0, L_1)$$

known as Floer cohomology,

$$HF^{*,*}(L_0, L_1) = H^*(CF^{*,*}(L_0, L_1), Q)$$

This comes about as follows.

The theory on the strip can be described as a

supersymmetric quantum mechanics with an infinite dimensional

target space which is the space of maps

$$y: \; [0,1] \to Y$$

obeying the boundary conditions,

with a real potential

$$h(y) = \frac{1}{2} \int_{\mathrm{D}} y^* \omega + \int_{[0,1]} \mathrm{Re}\, W \; ds$$

as explained in works of Iqbal, Hori and Vafa and Gaiotto, Moore and Witten.

The space perturbative ground states,

is the space of critical points $0 = \frac{\delta h}{\delta \overline{y}^b}$ of the potential.

The critical point equations are the "soliton" equations:

$$\frac{d}{ds} y = X_W$$

where X_W is the vector field on Y generated by the Hamiltonian

$$H_W = \mathrm{Re}\, W$$

and subject to the boundary conditions.

Solutions to

$$\frac{d}{ds}y = X_W$$

are paths $y: [0,1] \to Y$ which are time one

flows of the Hamiltonian H_W which start somewhere

on L_0 and end somewhere on L_1 .

Finding such paths is the same as finding intersection points

$$\mathcal{P} \in L_{0,H_W} \cap L_1$$

where L_{0,H_W} is obtained by from L_0 by time one flow

of the Hamiltonian $H_W = \mathrm{Re}\, W$

since these are the initial conditions for the flows.

A crucial aspect of the theory is invariance

under deformations of its Lagrangians

by arbitrary Hamiltonian symplectomorphisms.

This let's us undo the deformation from L_0 to L_{0,H_W}

to get the Floer complex

$$CF^{*,*}(L_0, L_1) = \bigoplus_{\mathcal{P}\in L_0\cap L_1} \mathbb{C}\mathcal{P}.$$

generated simply by intersection points of the Lagrangians we started with

$\mathcal{P}\in L_0\cap L_1$

The fact the theory is essentially independent of

$$W$$

reflects the fact that in supersymmetric quantum mechanics

the cohomology of the supercharge Q is independent of the potential.

Changing the potential by Δh acts by conjugation

$$Q \to Q' = e^{\Delta h} Q e^{-\Delta h}$$

so Q and Q' have the same cohomology.

More precisely, the cohomology remains invariant as long as the perturbation does not result in any states coming in or going to infinity.

This is the case if the Lagrangians we consider are compact. Then, the intersection points cannot run off to infinity and the theory would be independent of

$$W$$

were it single valued.

The only effect of a non-single valued potential such as ours is to introduce equivariant gradings in the theory.

An intersection point $P \in L_0 \cap L_1$ at which

$$W(\mathcal{P})|_{L_1\{\vec{n}\}} = W(\mathcal{P})|_{L_0}$$

for some equivariant grading

$$\vec{J} = \vec{n}$$

is defined to be the generator of

$$CF^*(L_0, L_1\{\vec{n}\}) \equiv CF^{*,\vec{n}}(L_0, L_1)$$

where $*$ denotes its Maslov grading.

The graded Euler characteristic of the theory

$$Hom^{*,*}_{\mathscr{D}_Y}(\widetilde{L}_0, \widetilde{L}_1) = \bigoplus_{m\in\mathbb{Z}, \vec{d}\in\mathbb{Z}^{\mathrm{rk}+1}} Hom_{\mathscr{D}_Y}(\widetilde{L}_0, \widetilde{L}_1[m]\{\vec{d}\})$$

defined by

$$\chi(\widetilde{L}_0, \widetilde{L}_1) = \sum_{m,\vec{d}} (-1)^m e^{2\pi i \vec{\lambda}\cdot\vec{d}} \dim Hom_{\mathscr{D}_Y}(\widetilde{L}_0, \widetilde{L}_1[m]\{\vec{d}\})$$

computes the "equivariant" intersection number,

$$\chi(\widetilde{L}_0, \widetilde{L}_1) = \sum_{\mathcal{P}\in L_0\cap L_1} (-1)^{M(\mathcal{P})} e^{2\pi i \vec{\lambda}\cdot\vec{J}(\mathcal{P})}$$

of Lagrangians.

The differential is the first in the sequence of maps

$$\mu^k : CF^{*,*}(\widetilde{L}_{k-1}, \widetilde{L}_k) \otimes \ldots \otimes CF^{*,*}(\widetilde{L}_0, \widetilde{L}_1) \to CF^{*,*}(\widetilde{L}_0, \widetilde{L}_k)$$

which one gets by taking the domain of $y : \mathrm{D} \to Y$ to have

k incoming strips and one outgoing one which map to intersection points

$\mathcal{P}_i \in CF^{*,*}(L_{i-1}, L_i)$ and $\mathcal{P}_{k+1} \in CF^{*,*}(L_0, L_k)$

The maps that contribute to μ^k the product have Maslov (or fermion) number

$$ind(y) = M(\mathcal{P}_{k+1}) - \sum_{i=1}^{k} M(\mathcal{P}_i) = 2 - k$$

and equivariant degree zero:

$$\vec{J}(y) = \vec{J}(\mathcal{P}_{k+1}) - \sum_{i=1}^{k} \vec{J}(\mathcal{P}_i) = 0$$

Of particular importance is the second of these

because it gives rise to a product of morphisms in the derived category

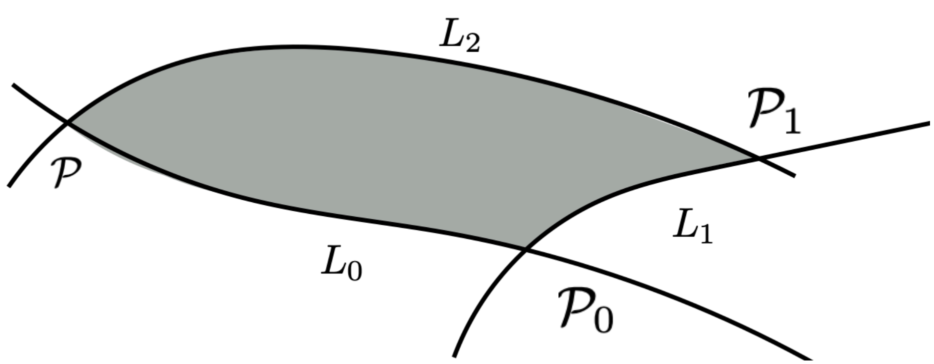
which is the associative product on Floer cohomology groups

$$HF^{*,*}(L_0, L_1) \otimes HF^{*,*}(L_1, L_2) \to HF^{*,*}(L_0, L_2)$$

mapping

$$[\mathcal{P}_0] \cdot [\mathcal{P}_2] = [\mu^2(\mathcal{P}_0, \mathcal{P}_1)] \in HF^{*,*}(L_0, L_2)$$

The product preserves all degrees.



Working with the derived Fukaya-Seidel category,

$$\mathscr{D}_Y = D(\mathcal{FS}(Y, W))$$

as opposed to Fukaya-Seidel category $\mathcal{FS}(Y, W)$ itself

is actually simpler.

For one $\mathscr{D}_Y$ has far fewer objects,

as any deformation of branes that does not change the amplitudes is

an equivalence.

By contrast, in $\mathcal{FS}(Y, W)$, one only gets

Hamiltonian isotopies of individual branes.

For example, starting from a pair of branes

$$L_0[1] \oplus L_1$$

intersecting transversely over $\mathcal{P} \in L_0 \cap L_1$ and oriented so that

$$\mathcal{P} \in Hom_{\mathscr{D}_Y}(L_0, L_1)$$

we can get a new brane which is the connected sum

$$L_2 = L_1 \# L_0[1]$$

obtained by gluing together at $\mathcal{P}$.

The differential acting on $CF^{*,*}(N, L_2)$ for any Lagrangian N

is simply the Floer differential.

The brane L_2 is equivalent,

as the object of the derived category to a complex

$$Cone(\mathcal{P}) = L_0 \xrightarrow{\mathcal{P}} L_1$$

which stands for the brane $L_0[1] \oplus L_1$ where we deform the

Floer differential on $CF^{*,*}(N, Cone(\mathcal{P}))$ by $\mu^2(\mathcal{P}, \quad)$

Physically, the two branes both describe to giving an expectation value to an open string tachyon at

$$\mathcal{P} \in L_0 \cap L_1$$

one large, the other small.

Their equivalence means that

$$Hom_{\mathscr{D}_Y}(N, L_2) \quad \text{and} \quad Hom_{\mathscr{D}_Y}(N, Cone(\mathcal{P}))$$

are the same for any N .

One can, in good situations, generate the entire derived category

by a sequence of cone maps, direct sums, and degree shifts

from a finite set of branes.

Our theory admits an additional simplification,

which is not typically present.

Since the target of our theory is

$$Y = \pi^*(Sym^{\vec{d}}(\mathcal{A})\backslash F_0)$$

with

$$Sym^{\vec{d}}\mathcal{A} = \otimes_{a=1}^{\mathrm{rk}} Sym^{d_a}\mathcal{A}$$

the theory is a close cousin of

Heegard-Floer theory

The target of Heegard-Floer theory is simply

$$Y_{HF} = Sym^d(\mathcal{A})$$

It is associated to

$${}^L\mathfrak{g} = \mathfrak{gl}_{1|1}$$

and leads to the Alexander polynomial,

a close relative of the Jones polynomial which we get for

$${}^L\mathfrak{g} = \mathfrak{su}_2$$

The fact that the

${}^L\mathfrak{g} = \mathfrak{gl}_{1|1}$ and the ${}^L\mathfrak{g} = \mathfrak{su}_2$

theories are such close relatives

has a simple string theory explanation

as we will see in the last lecture.

To begin with, recall that a point in

$$Y_{HF} = Sym^d(\mathcal{A})$$

d -touple of unordered points on $\mathcal{A}$

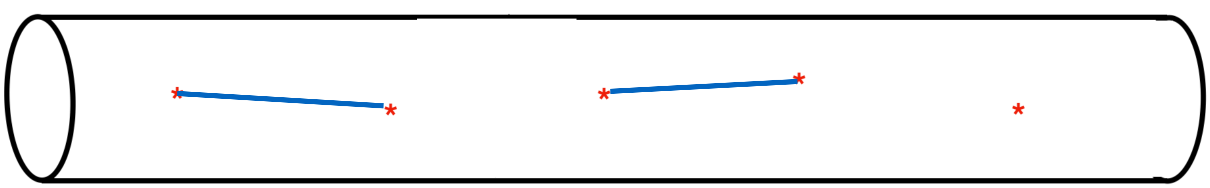
Since a point in

$$Y_{HF} = Sym^d(\mathcal{A})$$

is a d-touple of unordered points on $\mathcal{A}$

so Lagrangians in Y_{HF} are products of d one dimensional Lagrangians on $\mathcal{A}$,

$$L = L_1 \times \ldots \times L_d$$



$d = 2$

Correspondingly, on

$$Y_{HF} = Sym^d(\mathcal{A})$$

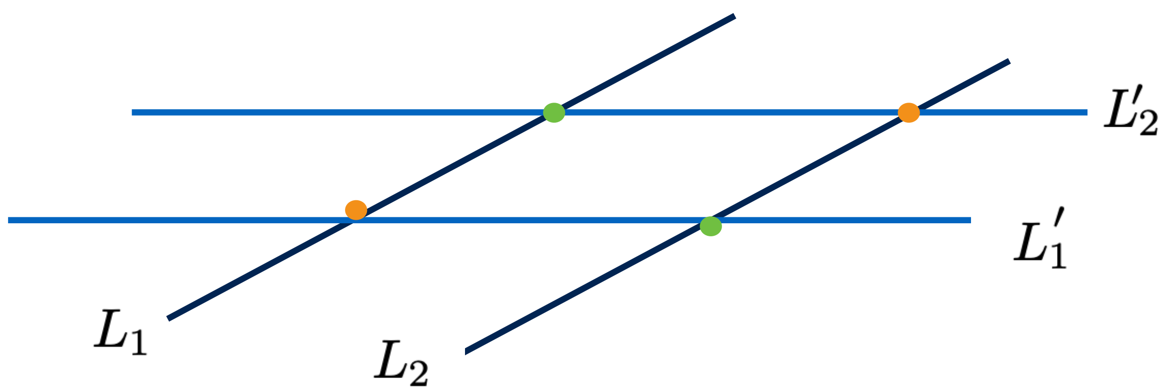
an intersection point of a pair of Lagrangians on

$L = L_1 \times \ldots \times L_d$ and $L' = L'_1 \times \ldots \times L'_d$

is a d -touple of intersection points of d one dimensional Lagrangians on $\mathcal{A}$

taken up to permutation

$$\mathcal{P} = (p_1, \ldots, p_d), \qquad p_\alpha \in L_\alpha \cap L'_{\sigma(\alpha)}$$



In Heegard-Floer theory, one gets to rephrase the A-model

$$y : \mathrm{D} \to Y_{HF}$$

in terms of counting to holomorphic curves

$$S \subset \mathrm{D} \times \mathcal{A}$$

with a pair of projections:

to D as a d -fold cover and to $\mathcal{A}$

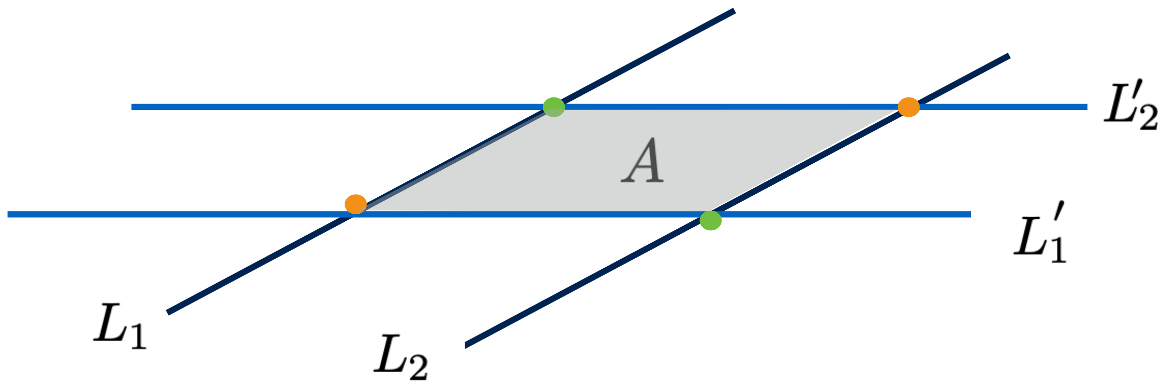
as a domain with boundaries on one dimensional Lagrangians

which is known as the "cylindrical approach" to Floer theory.

A holomorphic map from a disk to Y_{HF}

$$y : \mathrm{D} \rightarrow Y_{HF}$$

projects with non-negative multiplicities, to domains A



on $\mathcal{A}$, with boundaries on the one dimensional Lagrangians

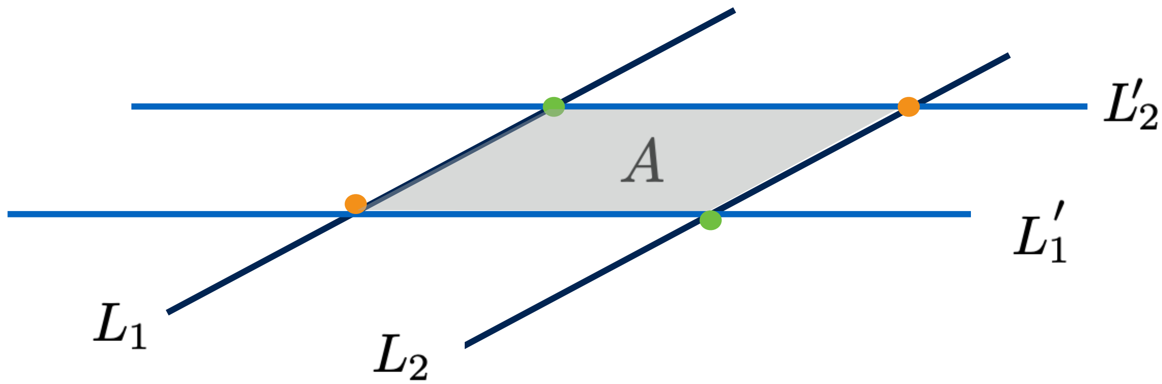
and vertices at their intersection points.

The cylindrical approach to Floer theory reduces the problem of counting holomorphic maps

$$y : \mathrm{D} \to Y_{HF}$$

to a well defined, but hard problem in complex analysis, one for each domain A which amounts to applications of the Riemannian mapping theorem.

As a simple piece of it, one can read off



from the domain A the Maslov index,

and the equivariant degree of the map to

$$y : \mathrm{D} \to Y_{HF}$$

For example Maslov index of a disk is

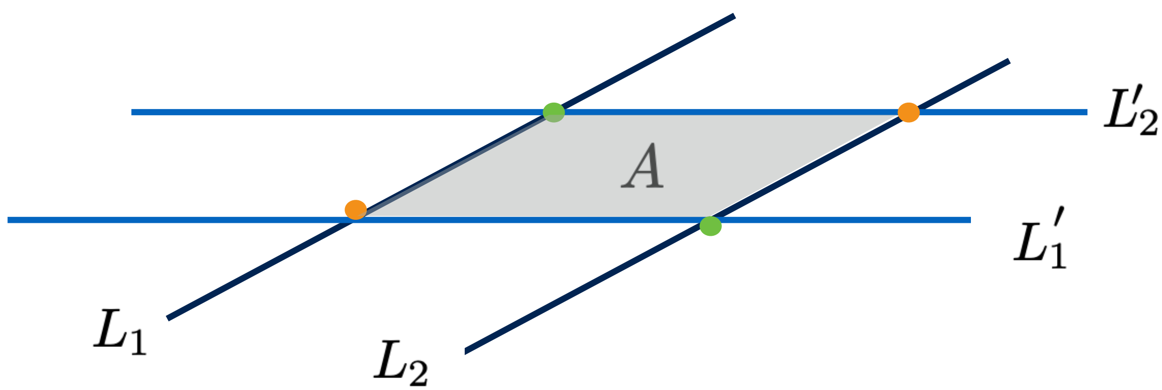
$$ind_{HF}(y) = 2e(A) + i(A)$$

where

$$e(A) = \chi(A) - \#\text{acute}/4 + \#\text{obtuse}/4,$$

and

$$i(A) = n_{\mathcal{P}}(A) + n_{\mathcal{P}'}(A) - e(A)$$



The above disk has Maslov index one.

In the theory based on

$$Y = \pi^*(Sym^{\vec{d}}(\mathcal{A})\backslash F_0)$$

a generic point in on Y is a

$\vec{d} = (d_1, \ldots, d_{\mathrm{rk}})$ -touple

of points on $\mathcal{A}$ which are labeled by $\mathrm{rk} = \mathrm{rank}({}^L\mathfrak{g})$

simple positive roots of ${}^L\mathfrak{g}$

$$Sym^{\vec{d}}\mathcal{A} = \otimes_{a=1}^{\mathrm{rk}} Sym^{d_a}\mathcal{A}$$

and are otherwise are identical.

Correspondingly,

Lagrangians in Y are products of $\vec{d}$ one dimensional Lagrangians on $\mathcal{A}$,

and the intersection points of a pair of Lagrangians

are $\vec{d}$ -uples of intersections of one dimensional Lagrangians

taken up to permutations.

The cylindrical approach to Floer theory extends to Y

because holomorphic maps to Y

$$y : \mathrm{D} \to Y = \pi^*(Y_0 \backslash F^0)$$

correspond to holomorphic maps

$$y^0 = \pi \circ y \quad : \mathrm{D} \to Y^0$$

which pull W^0 back to are regular function on D .

Because of this, the maps that the theory counts are different from Heegard-Floer theory even in the $^L\mathfrak{g} = \mathfrak{su}_2$ case.

In the closest, ${}^L\mathfrak{g} = \mathfrak{su}_2$ case

the branch points of the projection of the holomorphic curves

$$S \subset \mathrm{D} \times \mathcal{A}$$

to D as a d -fold cover

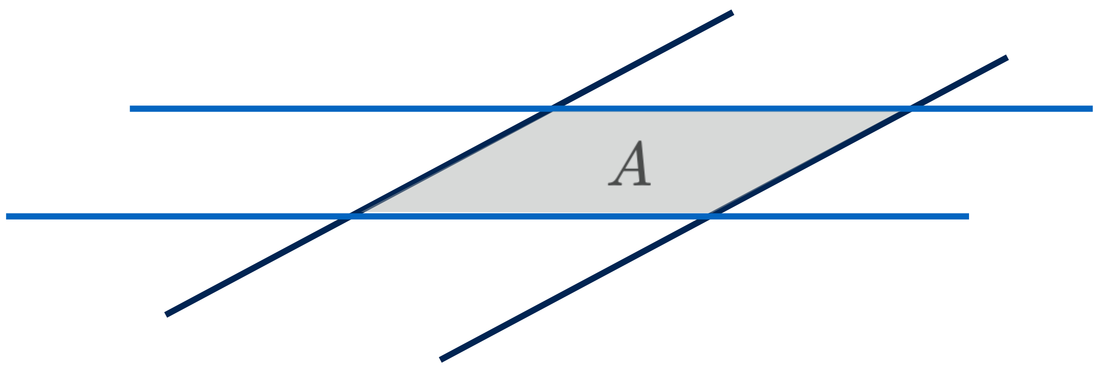
must map to punctures on $\mathcal{A}$

As a result of the restriction, the Maslov index of a disk becomes

$$\operatorname{ind}[y] = 2e(A)$$

where

$$e(A) = \chi(A) - \#\text{acute}/4 + \#\text{obtuse}/4,$$



The above disk has Maslov index two viewed as the index of maps to Y

Chapter VII

Thimble algebras

The theory

$$\mathscr{D}_Y = D(\mathcal{FS}(Y, W))$$

turns out to be solvable exactly.

It is equivalent to the derived category of modules

of an ordinary associative algebra

$$A$$

which one compute.

The potential W has critical points which are isolated and non-degenerate.

They are labeled by the weights

$$\mathcal{C} \in (V_1 \otimes \ldots \otimes V_n)_{\text{weight } \nu}$$

of the ${}^L\mathfrak{g}$ representation the conformal blocks take values in.

This mirrors the fact T -acts on $\mathcal{X}$

with isolated fixed points, which get associated to weights of ${}^L\mathfrak{g}$

via the Geometric Satake correspondence.

$$\text{weight } \nu = \text{highest weight } - \sum_{a=1}^{\text{rk}} d_a \, {}^L e_a$$

The critical point equations

$$\partial_y W(\mathcal{C}) = 0$$

are a variant of famous Gaudin type Bethe ansatz equations

$$\sum_i \frac{\langle \mu_i, {}^L e_a \rangle}{y_{a,\alpha} - a_i}\, a_i - \sum_{(b,\beta)\neq(a,\alpha)} \frac{\langle {}^L e_b, {}^L e_a \rangle}{y_{a,\alpha} - y_{b,\beta}} (y_{a,\alpha} + y_{b,\beta})/2 = \lambda_a/\lambda_0,$$

from works of Feigin, Frenkel and Reshetikhin.

For every a critical point $\mathcal{C} \in Y$ of the potential

$$\partial_y W(\mathcal{C}) = 0$$

we get a pair of "left" and "right" thimbles

$$T_{\mathcal{C}}, \; I_{\mathcal{C}} \; \in \; \mathscr{D}_Y$$

which are, respectively, the set of all initial conditions for upward and downward gradient flows of $\mathrm{Re} W$ on which $\mathrm{Im} W$ is constant.

The set of thimbles depend on the chamber in equivariant parameter space

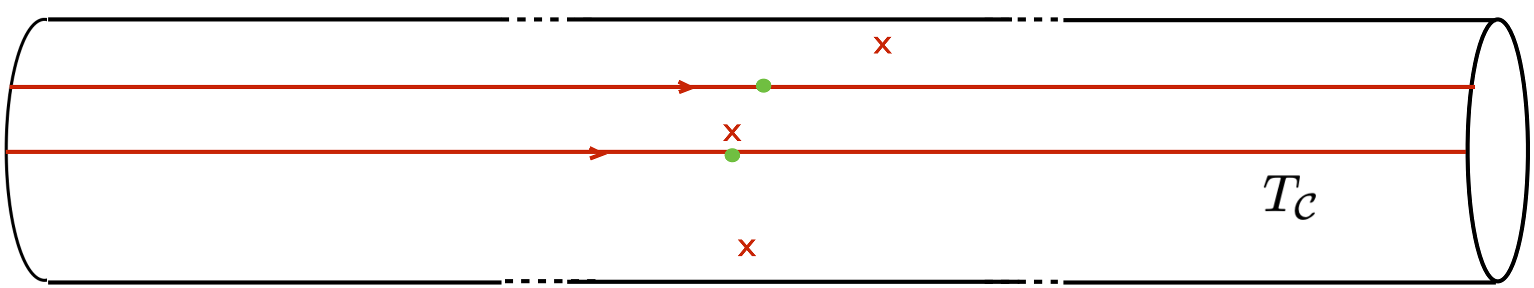
$$(\vec{\lambda}, \kappa)$$

There is a choice of chamber in which the left thimbles

$$T_{\mathcal{C}}$$

are products of real line Lagrangians,

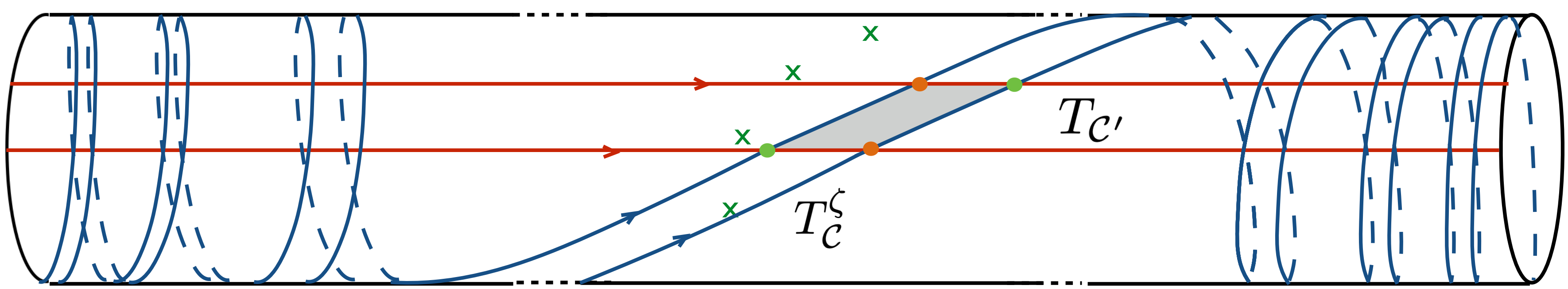
$$T_{\mathcal{C}} = T_{i_1} \times T_{i_2} \times \ldots \times T_{i_D}$$



By results of Seidel, these thimbles generate

the wrapped Fukaya category of

Y with potential W



with Hom's between them defined in the usual way, by

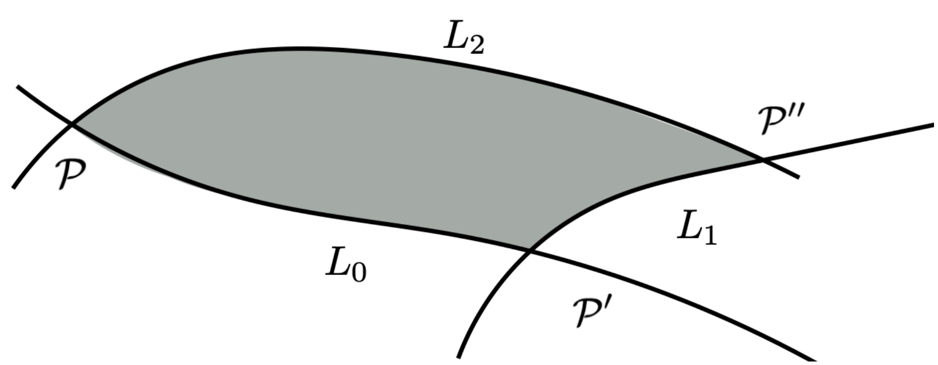
turning on a quadratic Hamiltonian near both infinite ends on $\mathcal{A}$.

The thimbles $T = \oplus_{\mathcal{C}} T_{\mathcal{C}}$ generate an algebra

$$A = Hom^*_{\mathscr{D}_Y}(T,T) = \bigoplus_{\mathcal{C},\mathcal{C}'} \bigoplus_{\vec{J}\in\mathbb{Z}^2} Hom_{\mathscr{D}_Y}(T_{\mathcal{C}}, T_{\mathcal{C}'}\{\vec{J}\})$$

which inherits the product from Floer theory

$$A \times A \;\to\; A$$



The algebra A is an ordinary associative algebra

$$A = \bigoplus_{\vec{n}\in\mathbb{Z}^{\mathrm{rk}+1}} \bigoplus_{\mathcal{C},\mathcal{C}'} Hom_{\mathscr{D}_{\mathrm{Y}}}(T_{\mathcal{C}}, T_{\mathcal{C}'}\{\vec{n}\})$$

graded only by equivariant degrees,

since all the algebra elements have cohomological degree zero

$$Hom_{\mathscr{D}_{\mathrm{Y}}}(T_{\mathcal{C}}, T_{\mathcal{C}'}[k]\{\vec{n}\}) = 0, \text{ for all } k \neq 0, \text{ and all } \vec{n}.$$

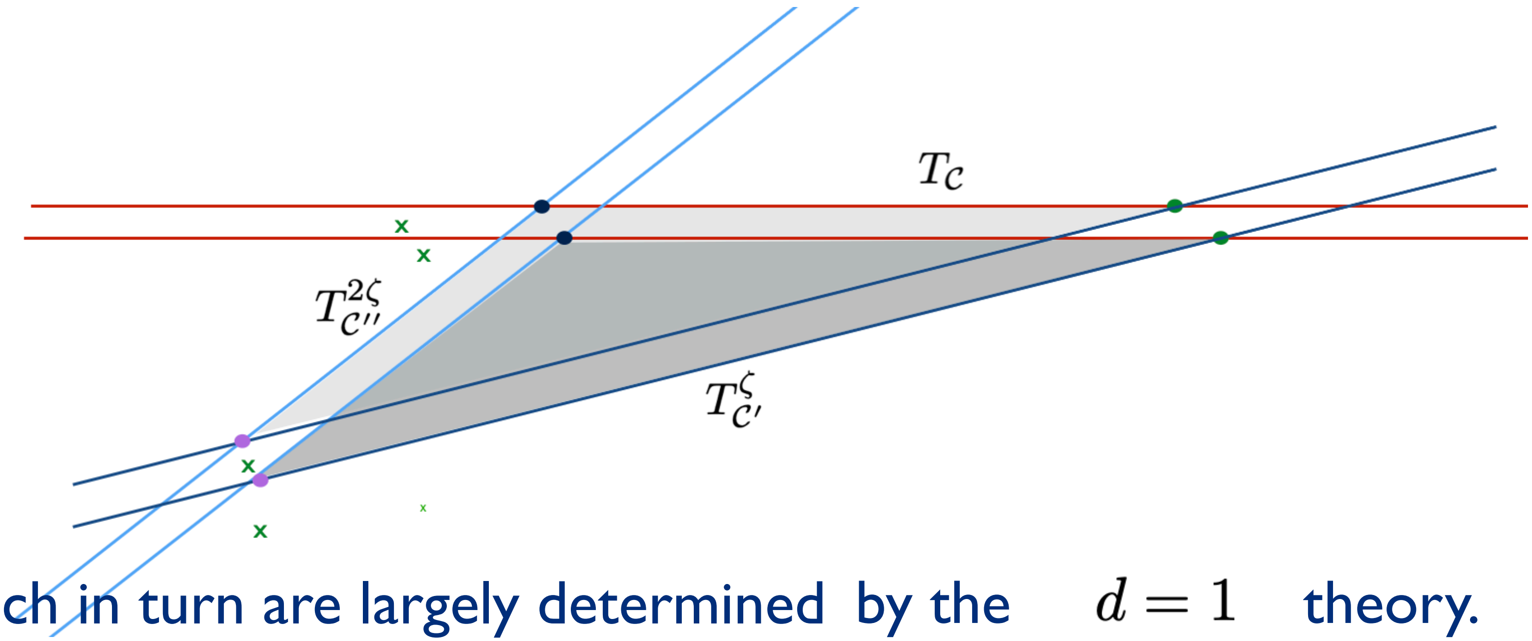
in particular, the action of the differential is trivial.

While there are not many holomorphic disk counts

that one can evaluate explicitly,

all the ones that can contribute to algebra products

are computable, since they come from products of triangles



which in turn are largely determined by the $d = 1$ theory.

The calculation is analogous to that by Auroux, in the context of

Heegard-Floer theory.

The algebra A can always be thought of a path algebra of a quiver

whose nodes correspond to the critical points $\mathcal{C}$

and where paths from node $\mathcal{C}$ to node $\mathcal{C}'$ encode

$$\bigoplus_{\vec{n}\in\mathbb{Z}^{\mathrm{rk}+1}} Hom_{\mathscr{D}_Y}(T_{\mathcal{C}}, T_{\mathcal{C}'}\{\vec{n}\})$$

For us these quivers always have closed loops,

in contrast to simpler theories with coming from

single valued potentials.

This is results in richer representation theory, and richer derived category.

For Y which is the equivariant mirror of the A_{m-1} surface

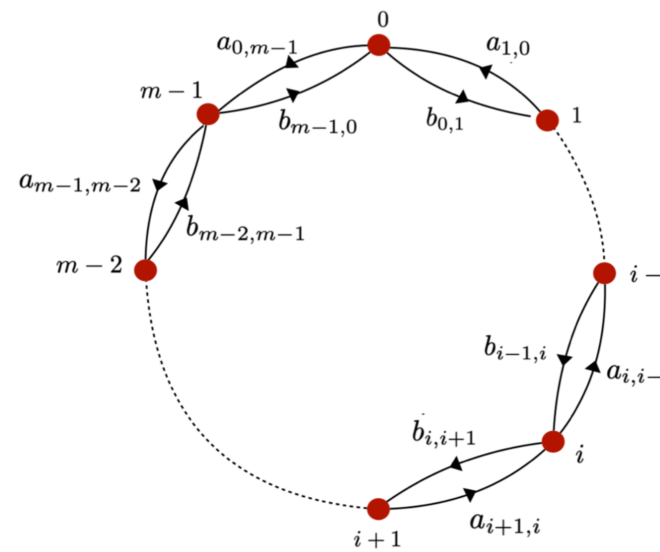
the algebra is the path algebra of the quiver

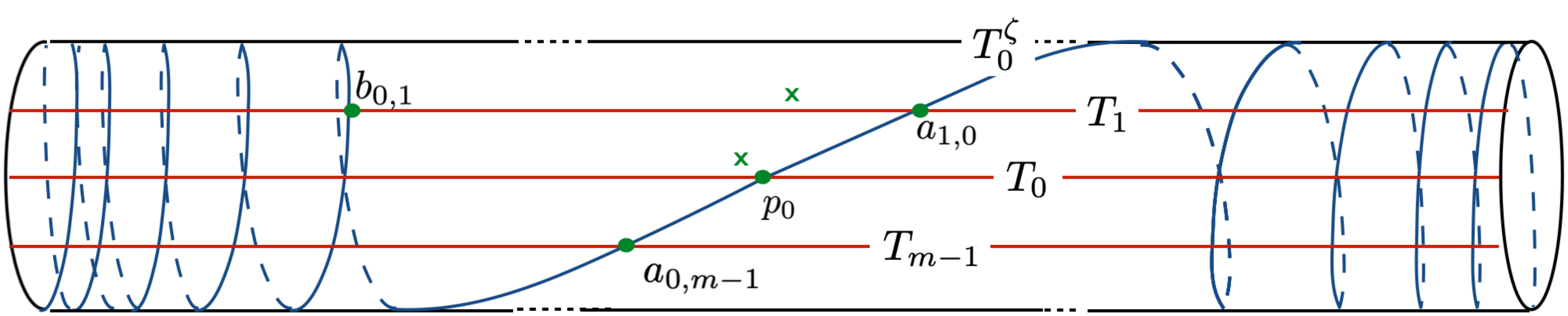
with relations

$$a_{i+1,i}\, b_{i,i+1} = 0, \quad \forall i.$$

which capture thimble intersection points and

relations between them:



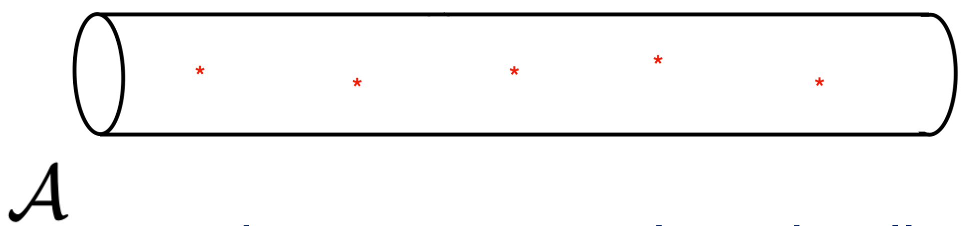


For general

$$Y = \pi^*(Sym^{\vec{d}}(\mathcal{A})\backslash F_0)$$

the algebra

$$A = Hom^*_{\mathscr{D}_Y}(T,T)$$

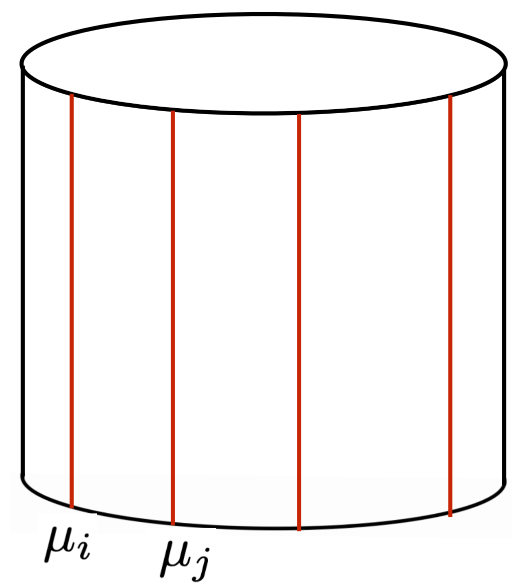


is best represented graphically.

Start with a cylinder

$$\mathrm{C} = S^1 \times [0,1]$$

with m red strands colored by highest weights μ_i of representations V_i

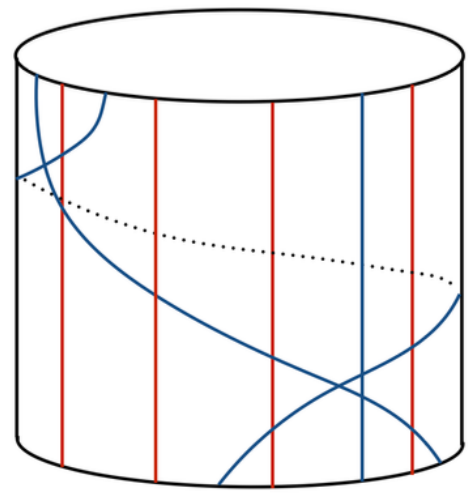


The order of the strands is the order of punctures on the S^1 in $\mathcal{A}$.

The algebra element in

$$Hom_{\mathscr{D}_Y}(T_{\mathcal{C}}, T_{\mathcal{C}'}\{\vec{n}\})$$

is a configuration of $\vec{d} = (d_a)_{a=1}^{\mathrm{rk}}$ blue strings



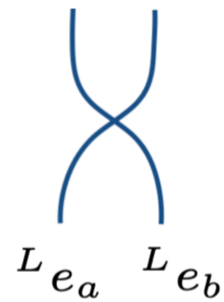
each labeled by simple positive root $^L e_a$ of $^L \mathfrak{g}$.

The configuration at the bottom is determined by $T_{\mathcal{C}}$,

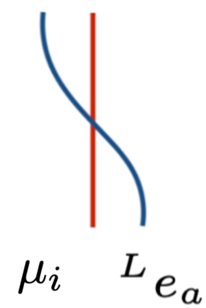
the configuration at the top by $T_{\mathcal{C}'}$.

The c^0-equivariant degrees are obtained by counting intersections:

$$\mathfrak{q}^{-\langle {}^L e_a, {}^L e_b \rangle/2}$$ for

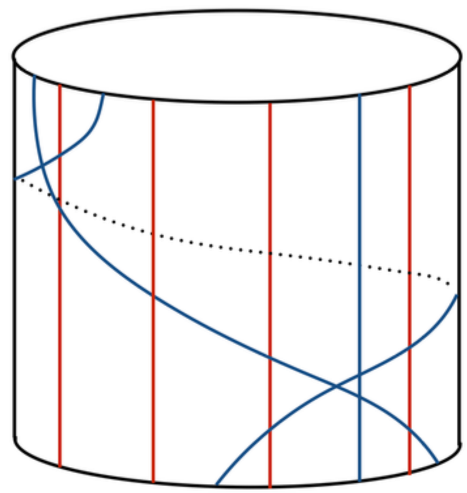


$$\mathfrak{q}^{\langle \mu_i, {}^L e_a \rangle}$$ for



$$\mathfrak{q}^{0}$$ for

The c^a-equivariant degrees capture the winding numbers



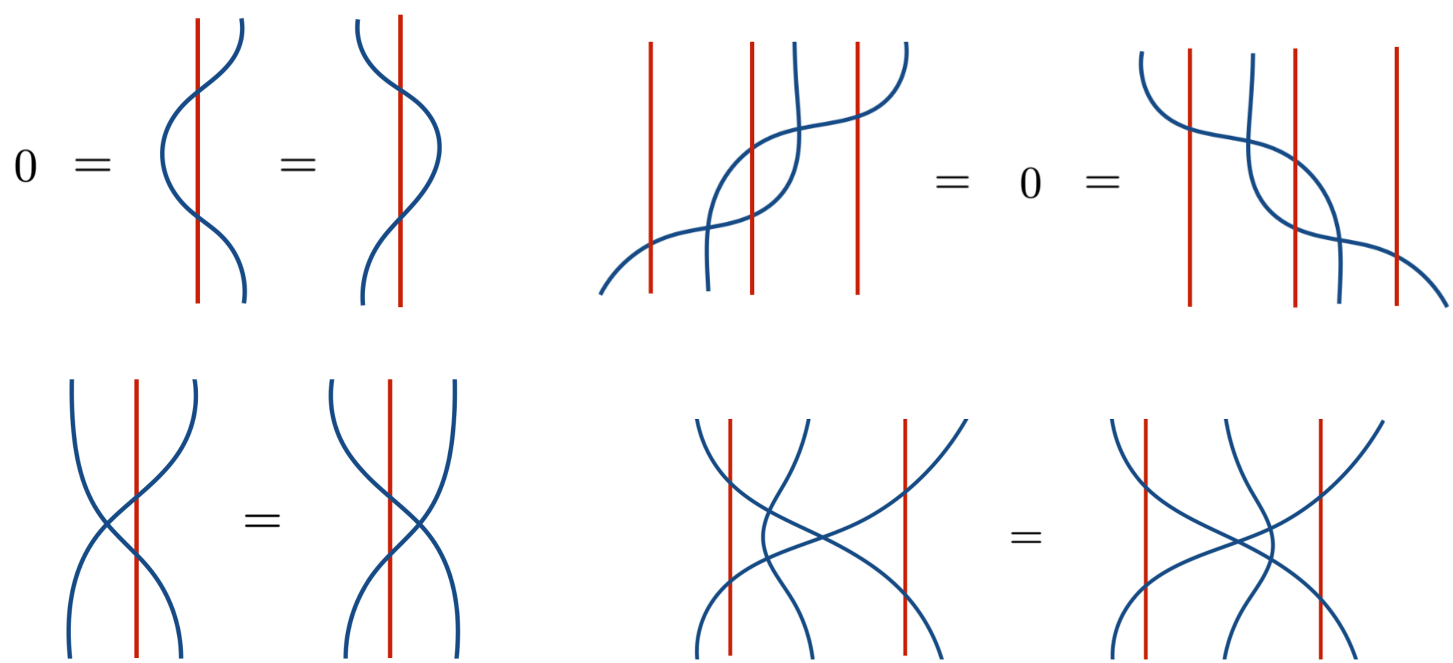
of strands colored by L_{e_a} around the cylinder.

Algebra multiplication

$$A \times A \;\to\; A$$

is stacking cylinders.

The relations say that the blue strings need to be taut



or the algebra element vanishes.

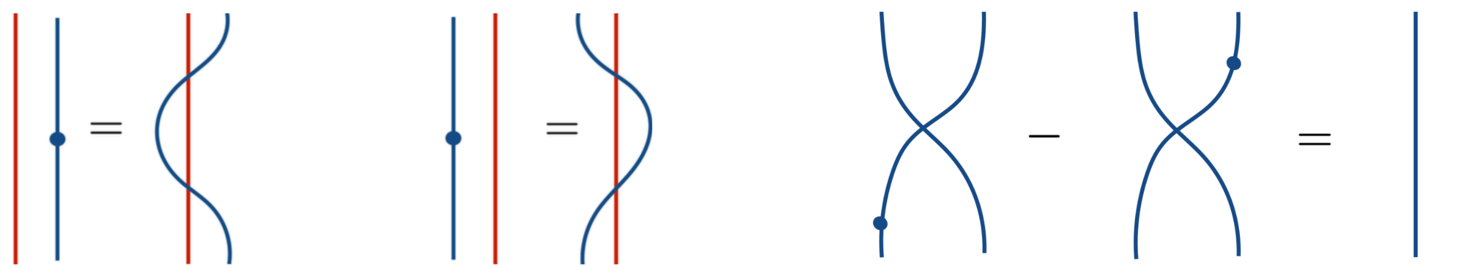
The KLRW algebra

$\mathscr{A}$

of Khovanov, Lauda, Roquier and Webster

are given in similar terms

but with more generators and different relations.



By a result of Webster, it describes the category of branes on $\mathcal{X}$,

so it is naturally more complicated.

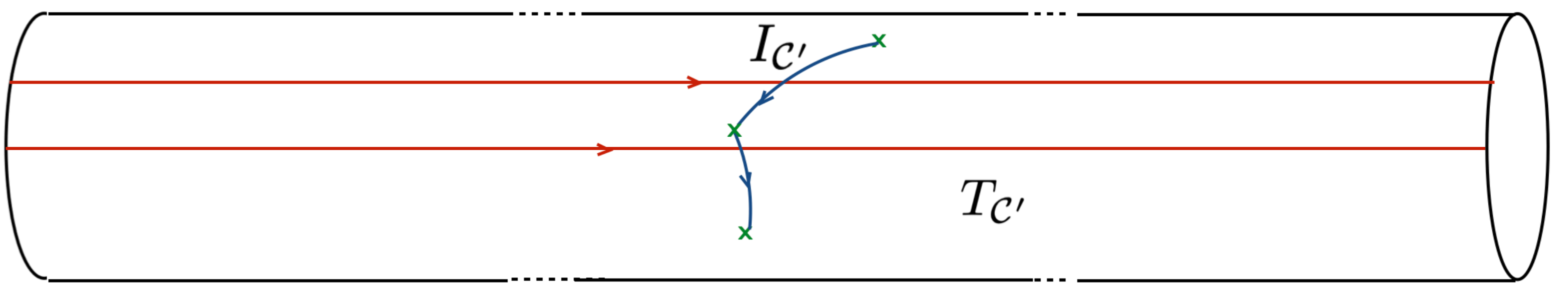
The right thimbles are all compact on $\mathcal{A}$, and dual to the left

$$Hom_{\mathscr{D}_Y}(T_{\mathcal{C}}, I_{\mathcal{C}'}) = \delta_{\mathcal{C},\mathcal{C}'} = Hom_{\mathscr{D}_Y}(I_{\mathcal{C}}, T_{\mathcal{C}'}[d])$$

which implies that

$$I_{\mathcal{C}'}$$

is a simple module of the algebra



the one corresponding to the quiver representation of rank one

for node $\mathcal{C}'$ and zero for all the others.

Once more applying the theorem of Seidel,

we get an equivalence of the derived Fukaya-Seidel category

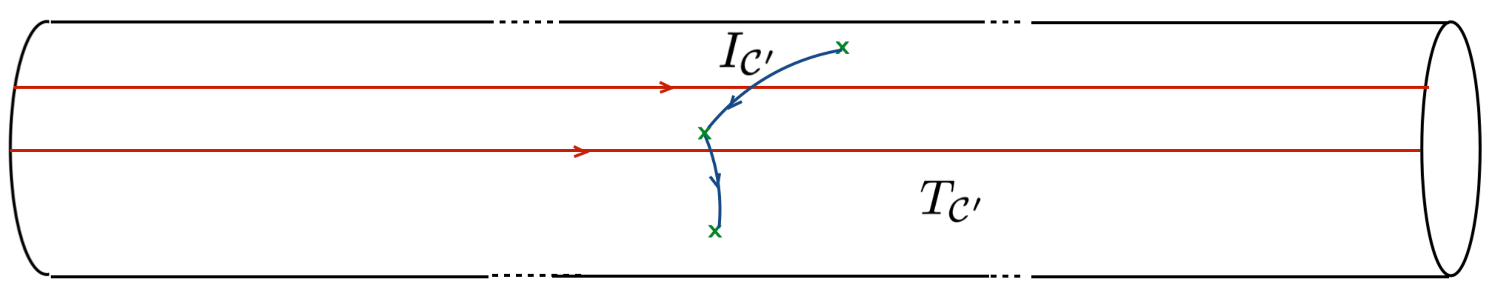
with the derived category of right

$$A^\vee = \; Hom^{*,*}_{\mathscr{D}_Y}(I,I)$$

modules, coming from the functor

$$Hom^{*,*}_{\mathscr{D}_Y}(-,I) : \mathscr{D}_Y \to \mathscr{D}_{A^\vee}$$

where $\displaystyle I = \bigoplus_{\mathcal{C}} I_{\mathcal{C}}$



In our running example, corresponding to

$$Y$$

being the equivariant mirror to

the resolution of an A_{m-1} surface singularity,

the algebra

$$A^{\vee} = Hom^{*,*}_{\mathscr{D}_Y}(I, I)$$

is essentially the algebra studied by Khovanov and Seidel.

The algebraic way to understand the equivalence

$$\mathscr{D}_A \cong \mathscr{D}_Y \cong \mathscr{D}_{A^\vee}$$

should be

Koszul duality

since the equivalences map

$T_{\mathcal{C}}$ -thimbles to projective modules of A

and simple modules of $A^\vee$,

$I_{\mathcal{C}'}$ -thimbles to simple modules of the algebra A

and indecomposable injective modules of $A^\vee$

These algebraic descriptions of

$$\mathscr{D}_Y$$

should make mirror symmetry manifest,

$$X \xleftrightarrow{\text{mirror}} Y$$

in the sense that

$$\mathscr{D}_X \cong \mathscr{D}_A \cong \mathscr{D}_{A^\vee} \cong \mathscr{D}_Y$$

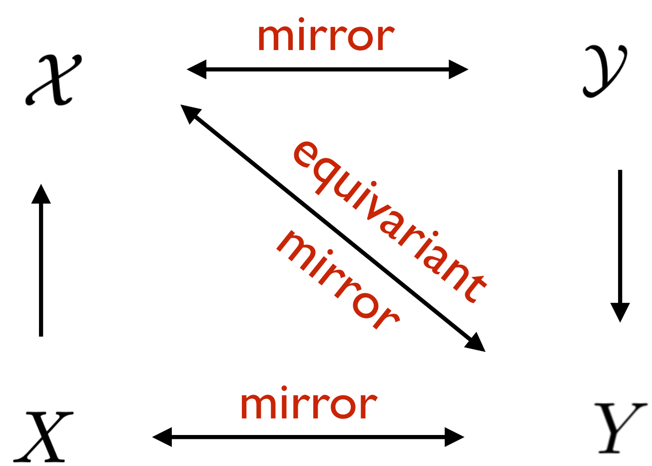
Chapter VIII

Homological link invariants from

$$\mathscr{D}_Y = D(\mathcal{FS}(Y, W))$$

Mirror symmetry

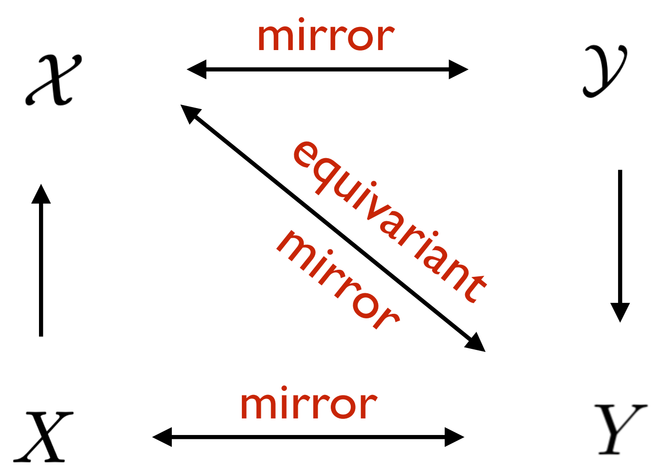
helps us understand exactly which questions we need to ask



to recover homological knot invariants from Y .

Since Y is the ordinary mirror of X ,

we should start by understanding how to recover



homological knot invariants from X , instead of $\mathcal{X}$

Every B-brane on $\mathcal{X}$ which is relevant to us

"comes from"

a B-brane on the core X

via a functor,

$$f_* : \mathscr{D}_X \rightarrow \mathscr{D}_{\mathcal{X}}$$

that interprets a brane F on X, an object of $\mathscr{D}_X$

as a brane $\mathcal{F} = f_* F$ on $\mathcal{X}$, an object of $\mathscr{D}_{\mathcal{X}}$

The functor f_* has an adjoint f^* that goes the other way,

$$\mathscr{D}_{\mathcal{X}} \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathscr{D}_X$$

Adjointness means that given any pair of branes on $\mathcal{X}$ that come from X

the Homs between them, computed upstairs, in $\mathscr{D}_{\mathcal{X}}$

$$\mathcal{F} = f_* F, \quad \mathcal{G} = f_* G$$

agrees with the Hom downstairs, in $\mathscr{D}_X$,

$$Hom^{*,*}_{\mathscr{D}_{\mathcal{X}}}(\mathcal{F}, \mathcal{G}) = Hom^{*,*}_{\mathscr{D}_X}(f^* f_* F, G)$$

after replacing F with $f^* f_* F$

By mirror symmetry, for every pair of B-type branes

$$\mathcal{F} = f_* F, \;\; \mathcal{G} = f_* G$$

on $\mathcal{X}$ which come from X, there is a pair of A-branes

$$k^* k_* L_F \quad , \qquad L_G$$

on Y which are mirror to

$$f^* f_* F \quad , \qquad G$$

such that Hom's on Y agree with those on $\mathcal{X}$.

$$Hom_{\mathscr{D}_{\mathcal{X}}}(\mathcal{F}, \mathcal{G}) = Hom_{\mathscr{D}_{(Y,W)}}(k^* k_* L_F, L_G)$$

The functors k_* and k^* that enter

$$Hom_{\mathscr{D}_{\mathcal{X}}}(\mathcal{F}, \mathcal{G}) = Hom_{\mathscr{D}_{(Y,W)}}(k^* k_* L_F, L_G)$$

relate objects on Y and on $\mathcal{Y}$,

in a way that mirrors f^* and f_* ,

$$\begin{array}{c} \mathscr{D}_{\mathcal{Y}} \\ k^* \big\downarrow \big\uparrow k_* \\ \mathscr{D}_Y \end{array}$$

Their construction, via Lagrangian correspondences,

is joint work with Michael McBreen and Vivek Shende.

One expects that mirror symmetry relating the upstairs theories

$$\mathcal{X} \xleftrightarrow{\text{mirror}} \mathcal{Y}$$

has an algebraic description as well.

$$\mathscr{D}_{\mathcal{X}} \cong \mathscr{D}_{\mathscr{A}} \cong \mathscr{D}_{\mathscr{A}^\vee} \cong \mathscr{D}_{\mathcal{Y}}$$

The algebra

$$\mathscr{A}$$

with property that $\mathscr{D}_{\mathcal{X}} \cong \mathscr{D}_{\mathscr{A}}$ is the endomorphism algebra

$$\mathscr{A} = Hom^*_{\mathscr{D}_{\mathcal{X}}}(\mathcal{P}, \mathcal{P})$$

of the tilting vector bundle on $\mathcal{X}$

$$\mathcal{P} = \bigoplus_{\mathcal{C}} \mathcal{P}_{\mathcal{C}}$$

whose construction is given by works of

Bezrukavnikov and Kaledin using quantization in characteristic p.

By work of Webster, algebra

$$\mathscr{A}$$

should be the cylindrical version of the KRLW algebra

of Khovanov, Lauda, Roquier and Webster

It is given in similar terms as our algebra

$$A$$

but with more generators,

since the blue strands are allowed to carry dots:

which are set to zero in A .

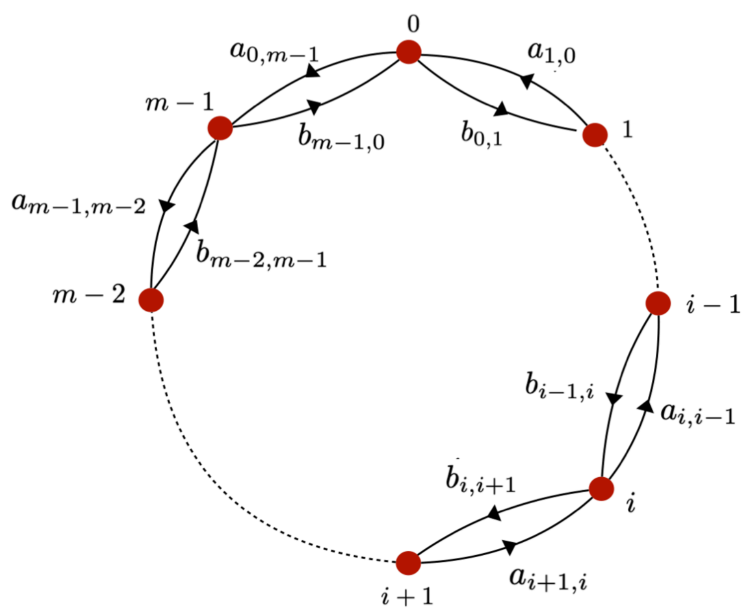
In our running example, one has the equivalence

$$\mathscr{D}_{\mathscr{A}} \cong \mathscr{D}_{\mathcal{X}}$$

with algebra $\mathscr{A}$ which is a path algebra

of the same quiver as for A, with different relations:

$$b_{i,i+1}\, a_{i+1,i} = a_{i,i-1}\, b_{i-1,i},$$



Imposing instead the relation

$$a_{i+1,i}\, b_{i,i+1} = 0, \quad \forall i.$$

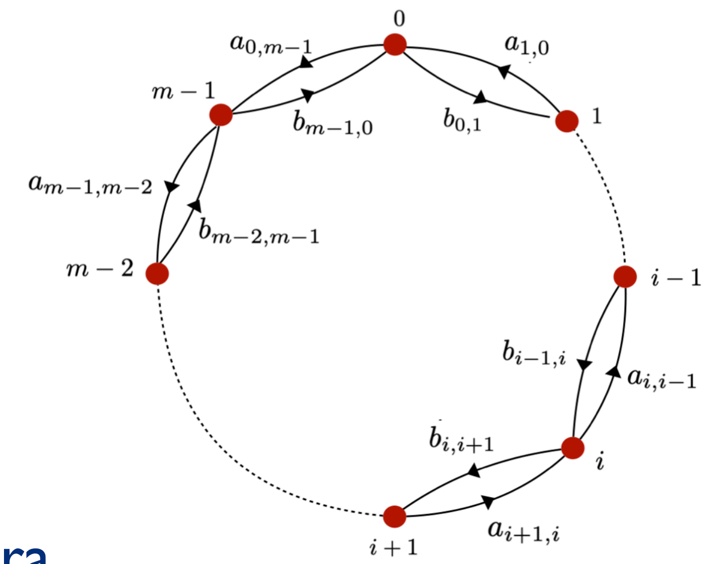
corresponds to restricting $\mathcal{X}$ to its core X

whose category of branes

$$\mathscr{D}_X \cong \mathscr{D}_A$$

is the category of modules of the smaller algebra

$$A = \mathscr{A}/\mathcal{I}$$



In this way, homological mirror symmetry becomes manifest:

$$\mathscr{D}_X \cong \mathscr{D}_A \cong \mathscr{D}_Y$$

This models how mirror symmetry should be understood more generally.

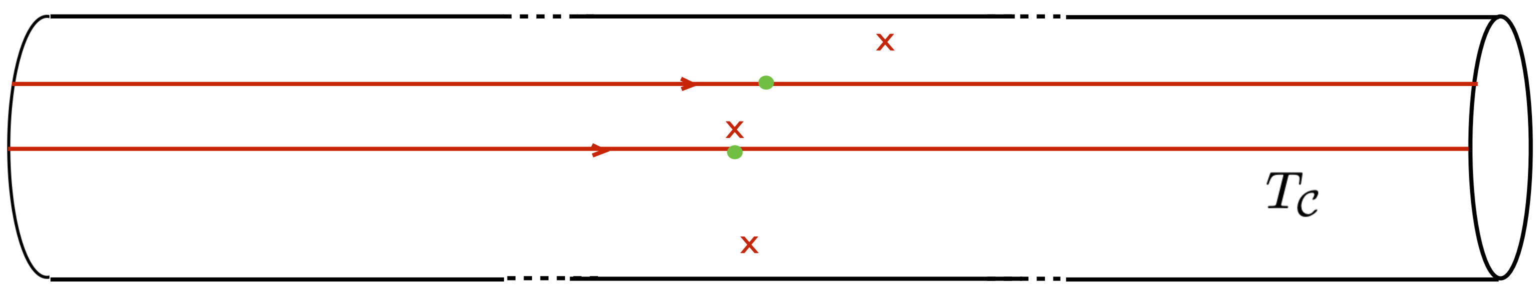
Composing the functor f^* with mirror symmetry we get

$$\mathscr{D}_{\mathcal{X}} \xrightarrow{h^*} \mathscr{D}_Y$$

an exact functor that sends

components of the tilting vector bundles to left thimbles

$$h^* \mathcal{P}_{\mathcal{C}} = T_{\mathcal{C}}$$



The functor, per exactness,

$$\mathscr{D}_{\mathcal{X}} \xrightarrow{\;h^*\;} \mathscr{D}_{Y}$$

takes the complex giving a projective resolution of object in $\mathscr{D}_{\mathcal{X}}$

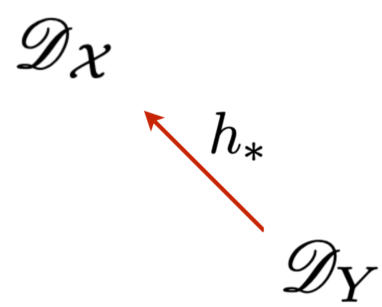
$$\mathcal{F} \cong \ldots \xrightarrow{t_2} \mathcal{F}_2(\mathcal{P}) \xrightarrow{t_1} \mathcal{F}_1(\mathcal{P}) \xrightarrow{t_0} \mathcal{F}_0(\mathcal{P})$$

in terms of the tilting generator to

to the complex of left thimbles resolving its image

$$h^*\mathcal{F} \cong \ldots \xrightarrow{h^* t_2} \mathcal{F}_2(T) \xrightarrow{h^* t_1} \mathcal{F}_1(T) \xrightarrow{h^* t_0} \mathcal{F}_0(T)$$

Similarly, composing the functor f_* with mirror symmetry we get

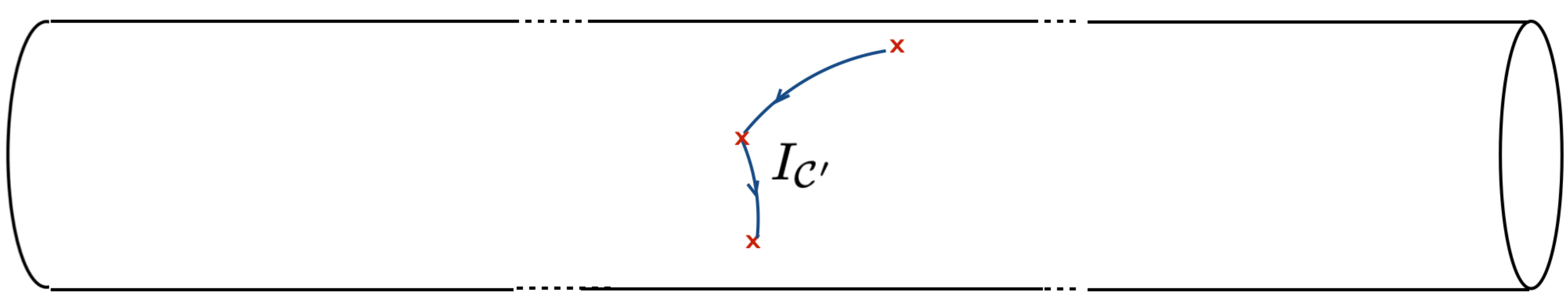


an exact functor h_* that sends

the right thimbles $I_{\mathcal{C}'}$ which are the simple modules of the algebra A

$$h_* I_{\mathcal{C}'} = \mathcal{S}_{\mathcal{C}'}$$

to simple modules $\mathcal{S}_{\mathcal{C}'}$ of the algebra $\mathscr{A}$ upstairs.



These are branes with $Hom_{\mathscr{D}_{\mathcal{X}}}(\mathcal{P}_{\mathcal{C}}, \mathcal{S}_{\mathcal{C}'}) = \delta_{\mathcal{C},\mathcal{C}'}$

The key to applying this to link invariants is that the branes

$$\mathcal{U} \in \mathscr{D}_{\mathcal{X}}$$

that serve as caps and cups in construction of link invariants from

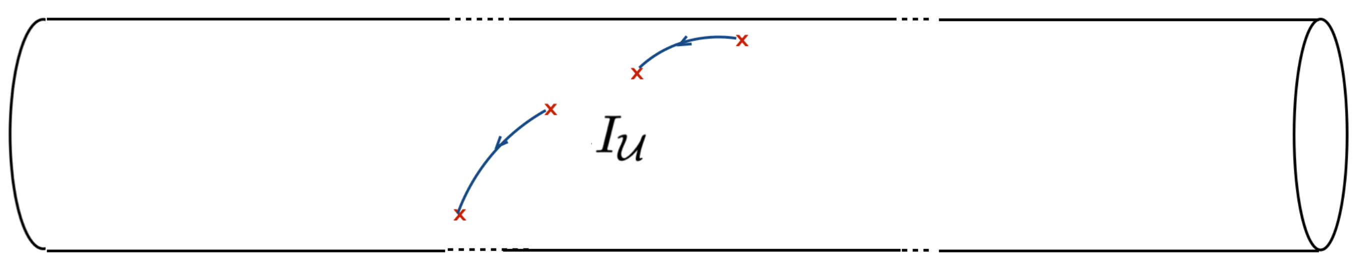
$$\mathscr{D}_{\mathcal{X}}$$

are the branes which are the simples of the algebra $\mathscr{A}$

$$\mathcal{U} = \mathcal{S}_{\mathcal{U}} = h_* I_{\mathcal{U}}$$

which come from right thimbles in $\mathscr{D}_Y$,

the simples of algebra A



This has a simple, but striking consequence,

giving us a new perspective on link homologies.

Equivariant mirror symmetry relates

$$Hom^{*,*}_{\mathscr{D}_{\mathcal{X}}}(\mathscr{B}\mathcal{U},\mathcal{U}) = Hom^{*,*}_{\mathscr{D}_{Y}}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}})$$

where the brane

$$E_{\mathcal{U}} \equiv h^*\mathcal{U} = h^*h_*I_{\mathcal{U}}$$

serves as the downstairs cap (distinct from the downstairs cap $I_{\mathcal{U}}$)

and where

$$\mathscr{B}E_{\mathcal{U}} = h^*\mathscr{B}\mathcal{U} = h^*h_*\mathscr{B}I_{\mathcal{U}}$$

is its image under the braiding functor, which commutes with everything.

By the virtue of the equivalence

$$\mathscr{D}_A \cong \mathscr{D}_Y$$

as any brane in $\mathscr{D}_Y$, the braided cap branes

$$\mathscr{B}E_{\mathcal{U}} \in \mathscr{D}_Y$$

have a projective resolution as a complex ,

$$\mathscr{B}E_{\mathcal{U}} \cong \ldots \xrightarrow{e_1} \mathscr{B}E_1(T) \xrightarrow{e_0} \mathscr{B}E_0(T)$$

every term of which is a direct sum of thimble $T_{\mathcal{C}}$ branes.

The complex

$$\mathscr{B}E_{\mathcal{U}} \cong \ldots \xrightarrow{e_1} \mathscr{B}E_1(T) \xrightarrow{e_0} \mathscr{B}E_0(T)$$

describes how to get the brane starting with the direct sum of thimbles

$$\bigoplus_k \mathscr{B}E_k(T)[k]$$

and taking connected sums which, deforms the differential

from trivial to

$$Q = \sum_k e_k \in A$$

From the complex describing the brane

$$\mathscr{B}E_{\mathcal{U}} \cong \ldots \xrightarrow{e_1} \mathscr{B}E_1(T) \xrightarrow{e_0} \mathscr{B}E_0(T)$$

we get for free a complex of vector spaces

$$0 \to hom_A(\mathscr{B}E_0, I_{\mathcal{U}}\{J\}) \xrightarrow{e_0} hom_A(\mathscr{B}E_1, I_{\mathcal{U}}\{J\}) \xrightarrow{e_1} \ldots .$$

with the action of the differential

$$Q = \sum_k e_k$$

that squares to zero.

The link homology,

$$Hom^*_{\mathscr{D}_Y}(E_{\mathcal{U}}, \mathscr{B}I_{\mathcal{U}}\{\vec{J}\}) = H^*\big(hom_A(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}}\{J\})\big)$$

is the cohomology of this complex.

It means that we get a second, purely classical, description

of knot homology groups

$$Hom^{*,*}_{\mathscr{D}_Y}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}}) = \bigoplus_{J\in\mathbb{Z}} Hom^{*}_{\mathscr{D}_Y}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}}\{J\})$$

which we can read off from the description of the brane

$$\mathscr{B}E_{\mathcal{U}} \cong \ldots \xrightarrow{e_1} \mathscr{B}E_1(T) \xrightarrow{e_0} \mathscr{B}E_0(T)$$

as a bound state of thimbles,

without any further work.

Per construction, the vector space one gets

$$0 \to hom_A(\mathscr{B}E_0, I_{\mathcal{U}}\{J\}) \xrightarrow{e_0} hom_A(\mathscr{B}E_1, I_{\mathcal{U}}\{J\}) \xrightarrow{e_1} \ldots .$$

as the k-term in the complex

is spanned by the intersection points

$$\mathcal{P} \in \mathscr{B}E_{\mathcal{U}} \cap I_{\mathcal{U}}$$

of equivariant degree J and fermion number $M = k$

so it is isomorphic to the Floer complex

$$CF^{*,*}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}}) = \bigoplus_{\mathcal{P} \in \mathscr{B}E_{\mathcal{U}} \cap I_{\mathcal{U}}} \mathbb{C}\mathcal{P}$$

The differential $Q = \sum_k e_k$ constructed classically,

$$\mathscr{B}E_{\mathcal{U}} \cong \ldots \xrightarrow{e_1} \mathscr{B}E_1(T) \xrightarrow{e_0} \mathscr{B}E_0(T)$$

from the geometry of the brane

sums up the action of instantons on it.

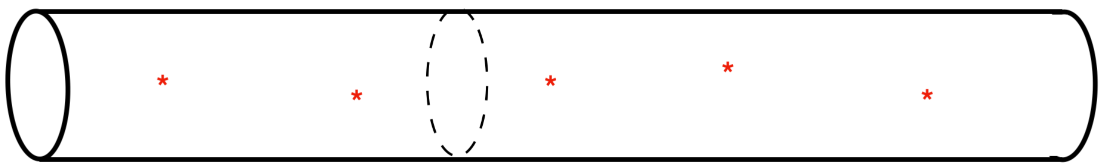
This is how the equivalence

$$\mathscr{D}_A \cong \mathscr{D}_Y$$

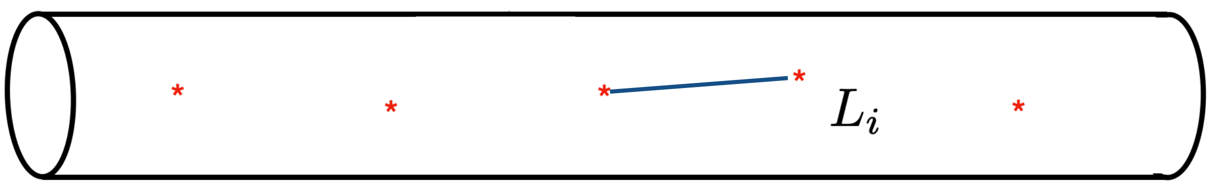
solves the knot categorification problem.

Recall our example, Y the equivariant mirror to

$\mathcal{X}$ which is the A_n surface.



Mirror to i-th vanishing $\mathbb{P}^1$ in X is the Lagrangian



The functor going up

$$\mathscr{D}_{\mathcal{Y}}$$

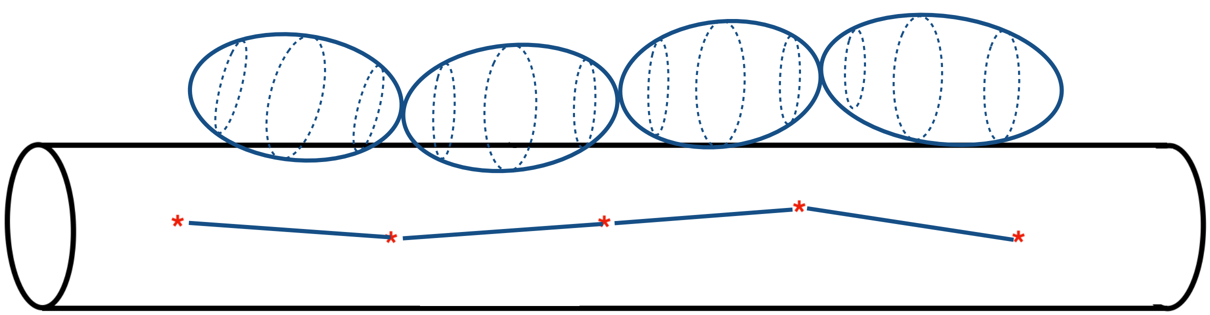
$$\uparrow k_*$$

$$\mathscr{D}_Y$$

amounts to pairing the brane downstairs with the

$$(S^1)^D \subset (\mathbb{C}^\times)^D$$

fiber over it, which is how one gets this picture:



The functor going the other way

$$k^* : \mathscr{D}_{\mathcal{Y}} \to \mathscr{D}_Y$$

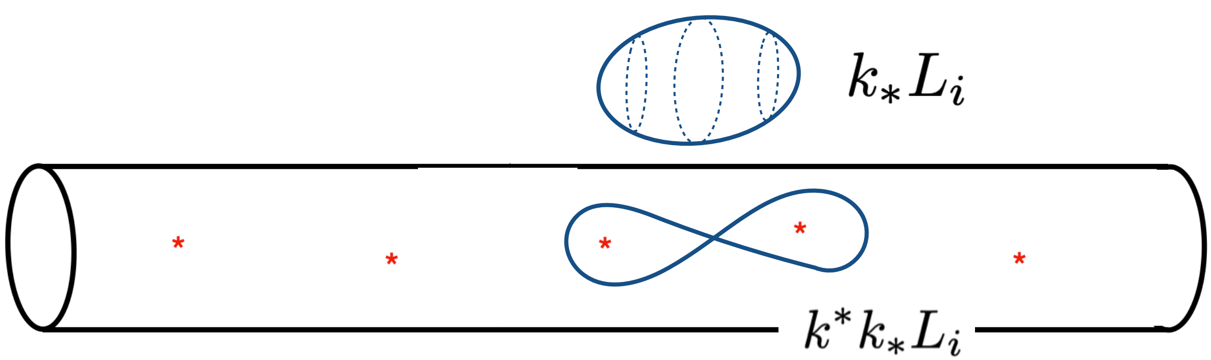
does not send the vanishing sphere $k_* L_i$ back to L_i :

$$k^* k_* L_i \neq L_i$$

Instead, either computing it either from mirror symmetry,

or its via its definition (coming from a Lagrangian correspondence),

one finds a figure eight Lagrangian

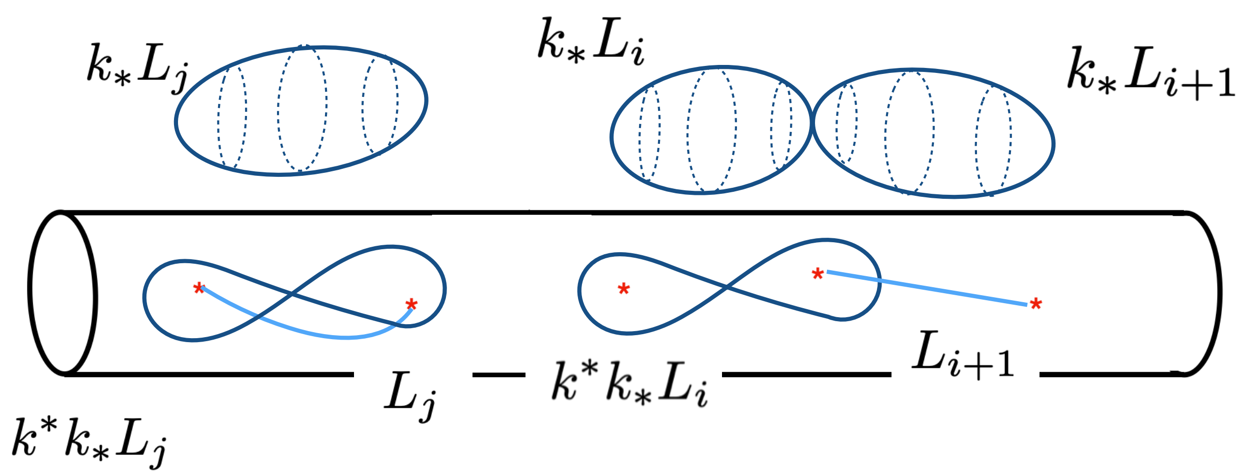


The basic virtue of the pair of adjoint functors,

that one ends up preserving Hom's,

$$Hom_{\mathscr{D}_{\mathcal{Y}}}(k_* L_i, k_* L_j) = Hom_{\mathscr{D}_Y}(k^* k_* L_i, L_j)$$

is manifest for example:



The example we just gave corresponds to

$$^{L}\mathfrak{g} = \mathfrak{su}_2$$

$\mathcal{X}$ is the moduli space of a single smooth $G = SU(2)/\mathbb{Z}_2$ monopole,

in presence of m singular ones.

Taking instead

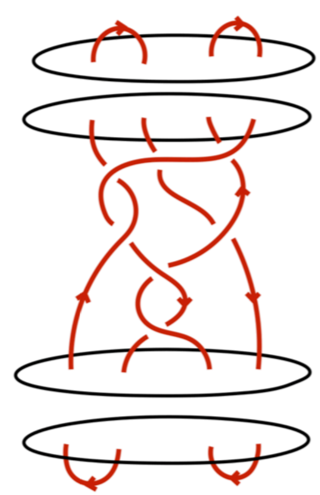
$$\mathcal{X}$$

to be the moduli space of d smooth $G = SU(2)/\mathbb{Z}_2$ monopoles,

in presence of m singular ones.

should give Khovanov homology

of any link obtained by closing of a braid with $m = 2d$ strands.

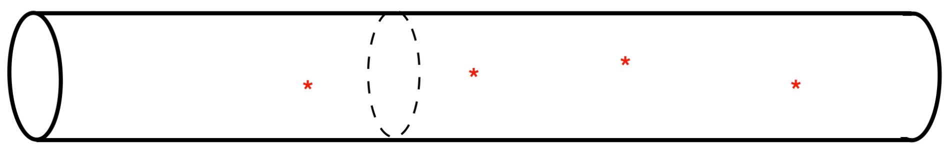


$d = 2$

Its equivariant mirror is

$$Y = \pi^*(Sym^d(\mathcal{A})\backslash F_0)$$

obtained starting with configuration space of d unordered points on



$\mathcal{A}$

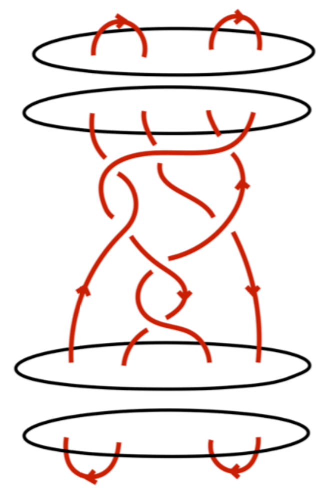
and deleting the locus where any pair of points collide either with themselves,

or with the punctures, with potential

$$W = \lambda_0 W^0 + \lambda_1 W^1$$

In $\mathcal{X}$ the cup and the cap branes are structure sheaves

$$\mathcal{U} = \mathcal{O}_U \in \mathscr{D}_{\mathcal{X}}$$



of a vanishing cycle $U \subset \mathcal{X}$

which is a product of d non-intersecting $\mathbb{P}^1$'s,

$$U = \mathbb{P}^1 \times \ldots \times \mathbb{P}^1$$

This cup/cap brane on $\mathcal{X}$ comes, via the functor

$$\mathscr{D}_{\mathcal{X}} \xleftarrow{h_*} \mathscr{D}_Y \qquad \mathcal{U} = h_* I_{\mathcal{U}}$$

from the cup brane on Y

$I_{\mathcal{U}}$:

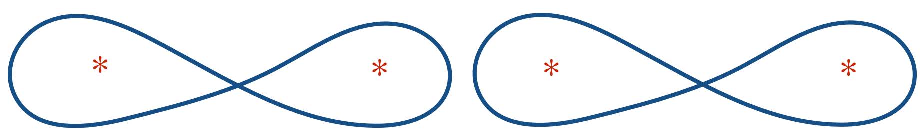
which is a product of d non-intersecting intervals.

This cap brane on Y is the image of $\mathcal{U} = h_* I_{\mathcal{U}}$ under

$$\mathscr{D}_{\mathcal{X}} \xrightarrow{h^*} \mathscr{D}_Y$$

which gives a product of d non-intersecting figure eights:

$E_{\mathcal{U}}$:



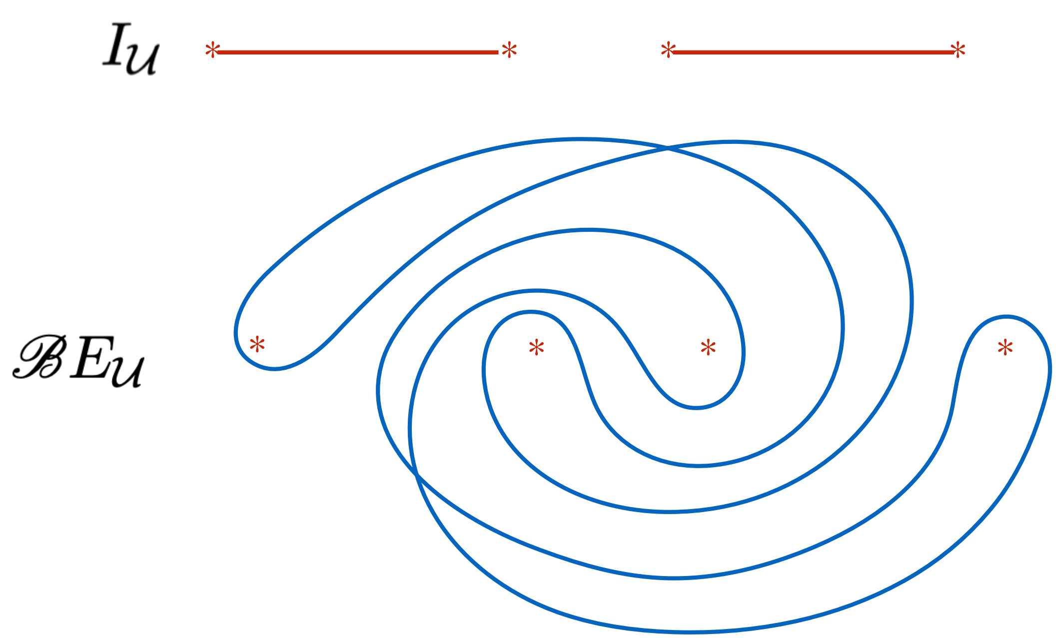
$$E_{\mathcal{U}} \equiv h^* \mathcal{U} = h^* h_* I_{\mathcal{U}}$$

The homological link invariant is

the space of morphisms

$$Hom^{*,*}_{\mathscr{D}_Y}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}}) = \bigoplus_{M\in\mathbb{Z},\vec{J}\in\mathbb{Z}^2} Hom_{\mathscr{D}_Y}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}}[M]\{\vec{J}\})$$

between the pair of branes.

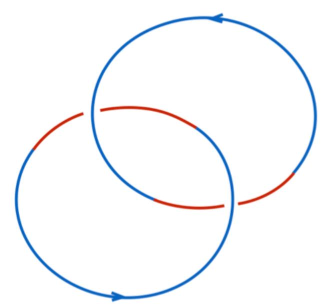


In the Landau-Ginsburg description,

both the Lagrangians and the action of braiding on them are geometric.

so we can start with a projection of a link to a the surface $\mathcal{A}$.

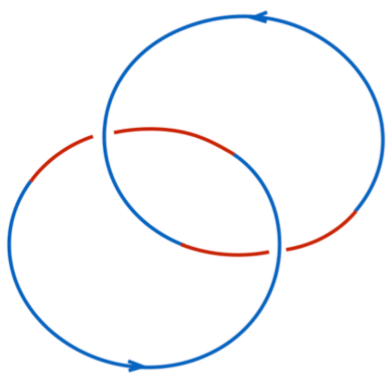
To translate it to a pair of Lagrangians, choose a bicoloring,

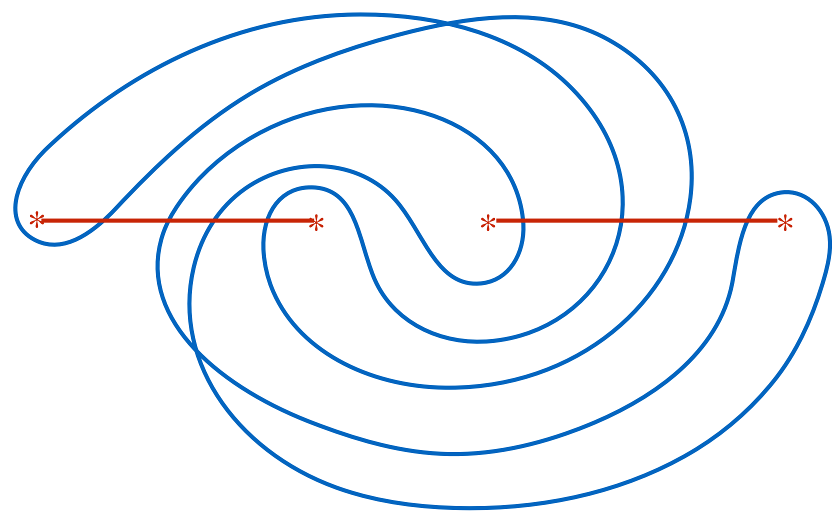


by equal number d of segments of each color,

such that red always underpasses the blue.

The mirror Lagrangians $I_{\mathcal{U}}$ and $\mathscr{B}E_{\mathcal{U}}$ are obtained by replacing all the red segments by simple intervals, and the blue segments by figure eight branes:





The spaces of morphisms

$$Hom^{*,*}_{\mathscr{D}_Y}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}}) = \bigoplus_{M\in\mathbb{Z},\vec{J}\in\mathbb{Z}^2} Hom_{\mathscr{D}_Y}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}}[M]\{\vec{J}\})$$

is apriori defined as the cohomology of the Floer complex,

$$CF^{*,*}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}}) = \bigoplus_{\mathcal{P}\in\mathscr{B}E_{\mathcal{U}}\cap I_{\mathcal{U}}} \mathbb{C}\,\mathcal{P}$$

graded by cohomological (or Maslov) and the equivariant degrees.

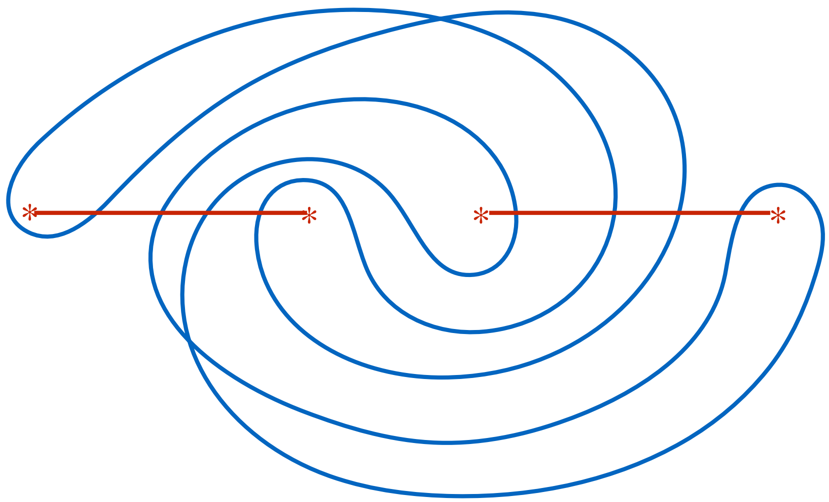
but defining the differential requires counting instantons.

To evaluate the Euler characteristic

$$\chi(E, \mathscr{B}I) = \sum_{M,J\in\mathbb{Z}} (-1)^M \mathfrak{q}^J Hom_{\mathscr{D}_Y}(E, \mathscr{B}I[M]\{J\})$$

one simply counts the intersection points of Lagrangians, keeping track of grading

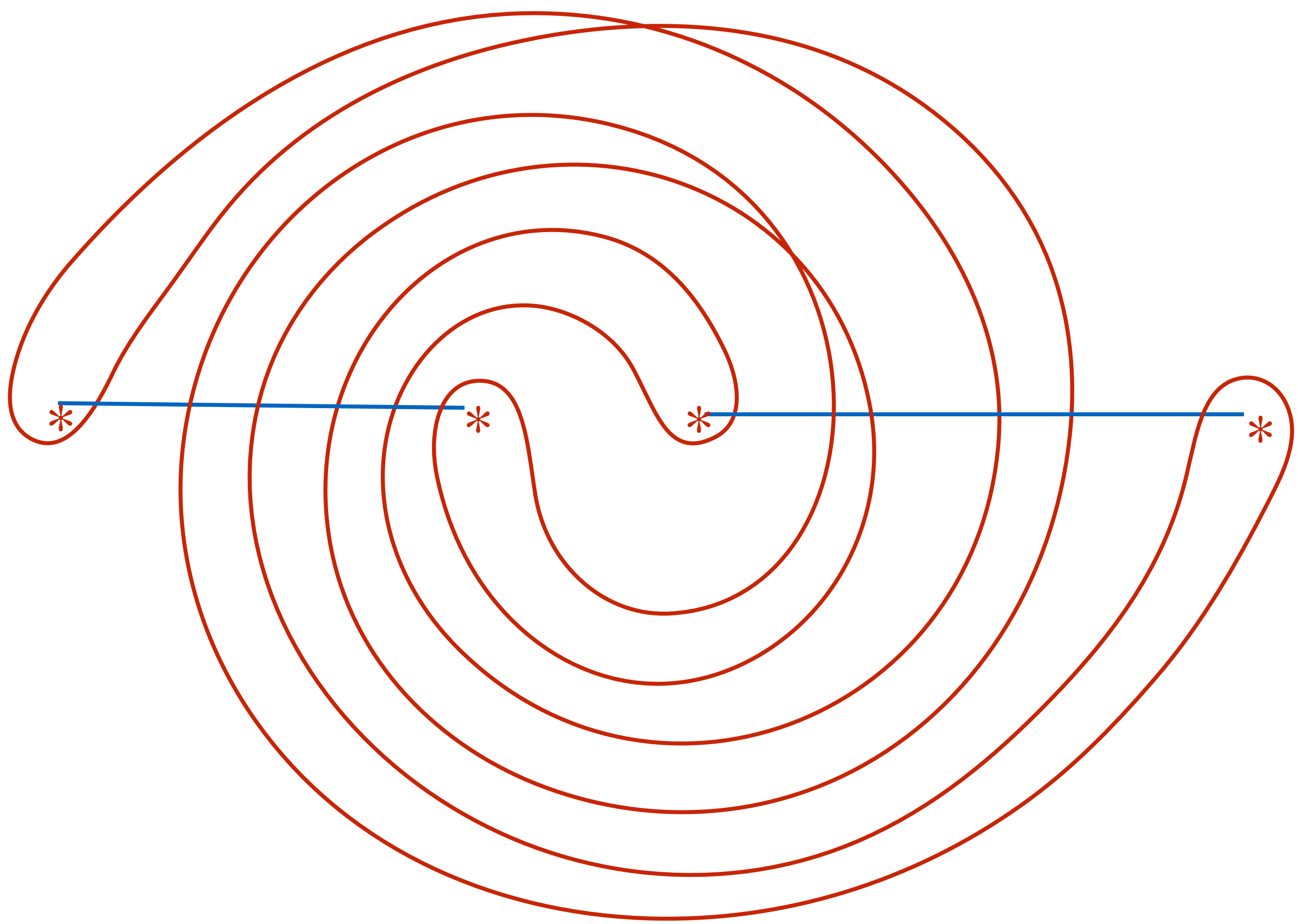
$$\chi(E, \mathscr{B}I) = \sum_{\mathcal{P}\in E\cap\mathscr{B}I} (-1)^{M(\mathcal{P})} \mathfrak{q}^{J(\mathcal{P})}$$



The fact this computes the Jones polynomial is a theorem of Bigelow from the '90s.

As an example,

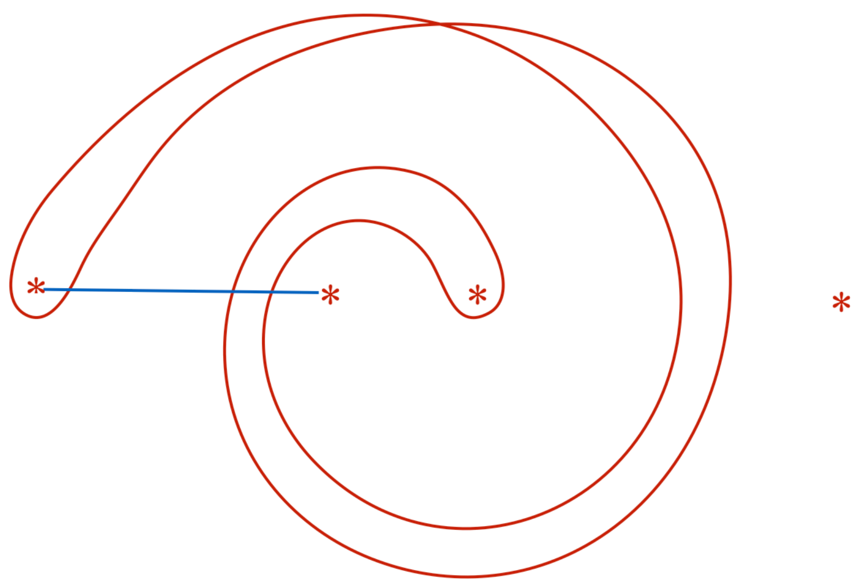
for the trefoil, we would get the following brane configuration:



For simplicity, we can ask about the reduced Khovanov homology

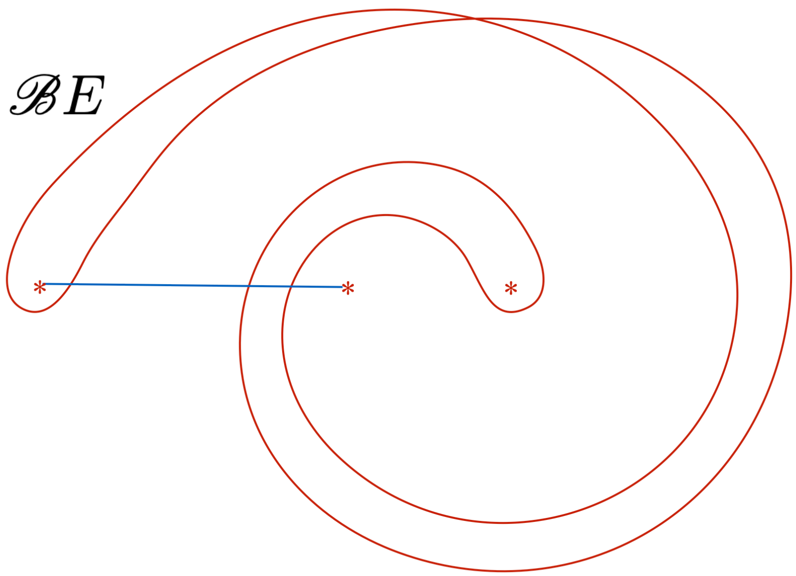
where the unknot homology is set to be trivial.

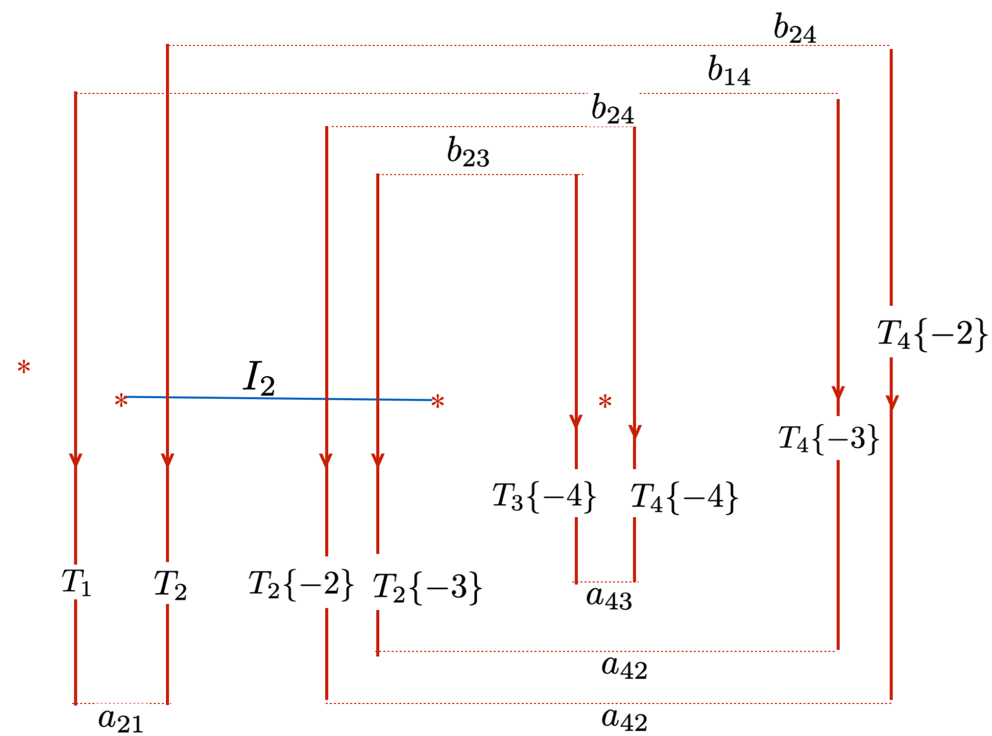
This lets us erase a pair of Lagrangians and simplify the problem to $d = 1$



which puts us in the setting of our running example.

The $\mathscr{B}E$ brane has the following resolution





$$\mathscr{B}E \cong \quad T_3\{-4\} \xrightarrow{\begin{pmatrix} a_{43} \\ b_{23} \end{pmatrix}} \begin{matrix} T_4\{-4\} \\ \oplus \\ T_2\{-3\} \end{matrix} \xrightarrow{\begin{pmatrix} b_{24} & 0 \\ 0 & a_{42} \end{pmatrix}} \begin{matrix} T_2\{-2\} \\ \oplus \\ T_4\{-3\} \end{matrix} \xrightarrow{\begin{pmatrix} a_{42} & 0 \\ 0 & b_{14} \end{pmatrix}} \begin{matrix} T_4\{-2\} \\ \oplus \\ T_1 \end{matrix} \xrightarrow{\begin{pmatrix} b_{24} \\ a_{21} \end{pmatrix}} T_2$$

Out of the complex describing the brane

a small part contributes to the homology

$$\mathscr{B}E \cong \; T_3\{-4\} \xrightarrow{\begin{pmatrix} a_{43} \\ b_{23} \end{pmatrix}} \begin{matrix} T_4\{-4\} \\ \bigoplus \\ T_2\{-3\} \end{matrix} \xrightarrow{\begin{pmatrix} b_{24} & 0 \\ 0 & a_{42} \end{pmatrix}} \begin{matrix} T_2\{-2\} \\ \bigoplus \\ T_4\{-3\} \end{matrix} \xrightarrow{\begin{pmatrix} a_{42} & 0 \\ 0 & b_{14} \end{pmatrix}} \begin{matrix} T_4\{-2\} \\ \bigoplus \\ T_1 \end{matrix} \xrightarrow{\begin{pmatrix} b_{24} \\ a_{21} \end{pmatrix}} T_2$$

and reproduces the reduced Khovanov homology of the trefoil.

$$Hom(\mathscr{B}E, I[k]\{n\}) = H^k(Hom_A(\mathscr{B}E^{\bullet}, I_2\{n\}))$$