Knot Categorification

from Mirror Symmetry

Part II

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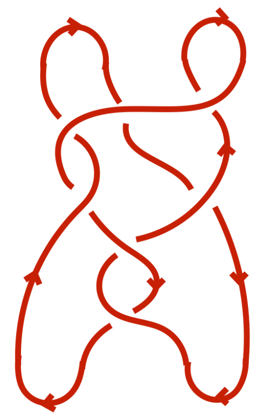
Recall that, underlying

$$U_{\mathfrak{q}}(^L\mathfrak{g})$$

quantum group invariants of a link



is conformal field theory with

$$\widehat{^L\mathfrak{g}}_\kappa$$

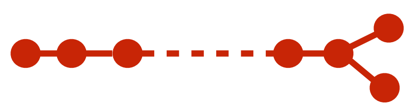
Lie algebra symmetry.

We learned that for

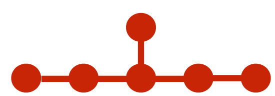
$$^{L}\mathfrak{g}$$

which is simply laced so that $^{L}\mathfrak{g} = \mathfrak{g}$

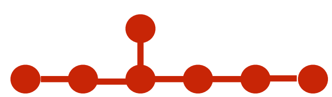
$\mathfrak{g} = A_n$



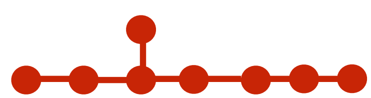
$\mathfrak{g} = D_n$



$\mathfrak{g} = E_6$

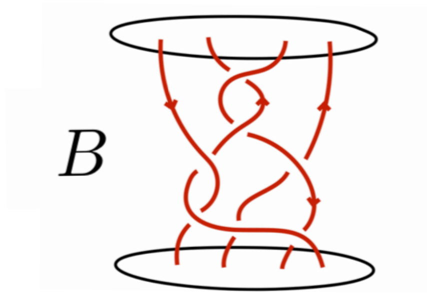


$\mathfrak{g} = E_7$



$\mathfrak{g} = E_8$

the $U_{\mathfrak{q}}({}^L\mathfrak{g})$ braiding matrices



are categorified by

the derived category of T -equivariant coherent sheaves

$$\mathscr{D}_{\mathcal{X}} = D^b Coh_{\mathrm{T}}(\mathcal{X})$$

of a certain very special hyper-Kahler manifold

$$\mathcal{X} = \mathrm{Gr}^{\vec{\mu}}{}_{\nu}$$

$$\mathscr{D}_{\mathcal{X}} = D^b Coh_{\mathrm{T}}(\mathcal{X})$$

the derived category of $\mathrm{T}-$ equivariant coherent sheaves

is a category whose objects are B-type branes

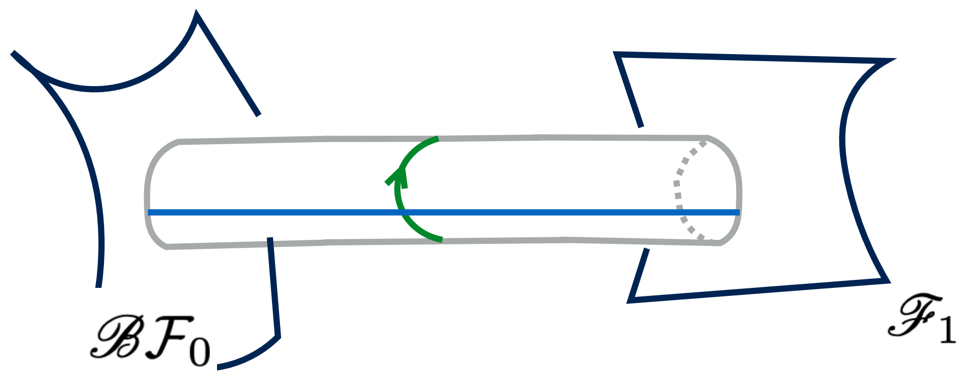
of the sigma model on $\mathcal{X}$,

working equivariantly with respect to

$\mathrm{T}.$

The braiding matrix element between a pair of conformal blocks

$$(\mathfrak{B}\mathcal{V}_0|\mathcal{V}_1)$$



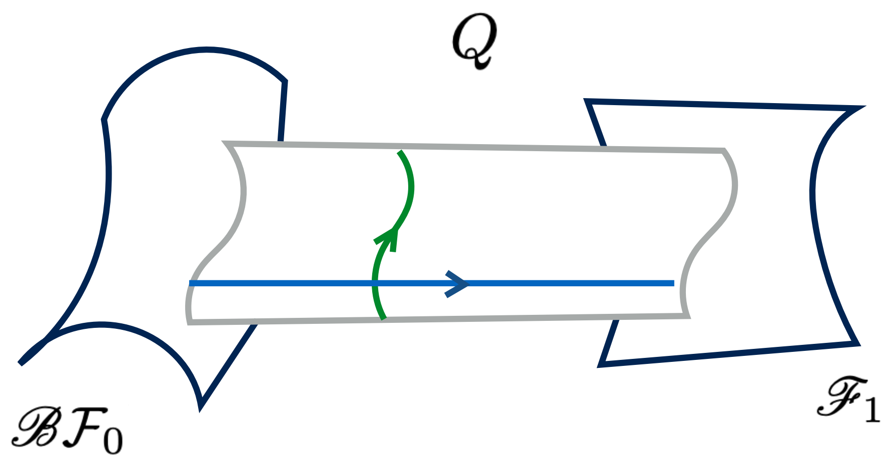
is the index of supercharge Q of the sigma model on $\mathcal{X}$

with a pair of B-type branes

whose vertex functions are the conformal blocks

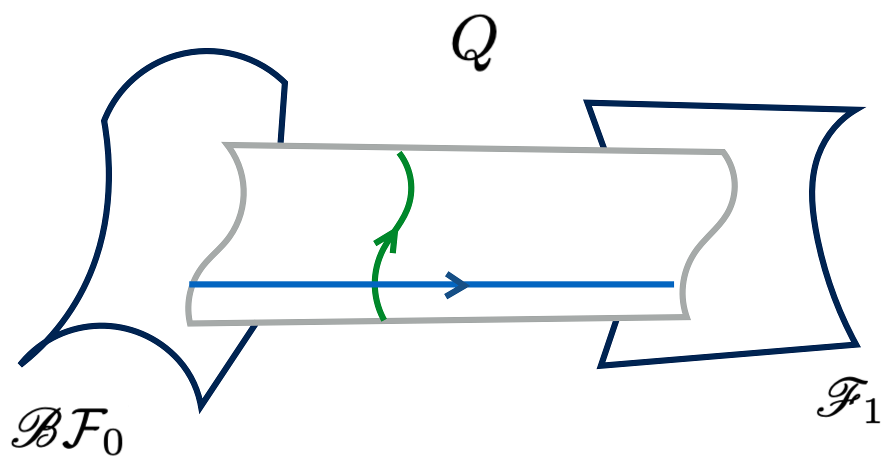
$\mathcal{V}_1 = \mathcal{V}[\mathcal{F}_1]$ and $\mathcal{V}_0 = \mathcal{V}[\mathcal{F}_0]$

The index of the supercharge



is per definition categorified by its cohomology.

The cohomology of the supercharge



is computed by

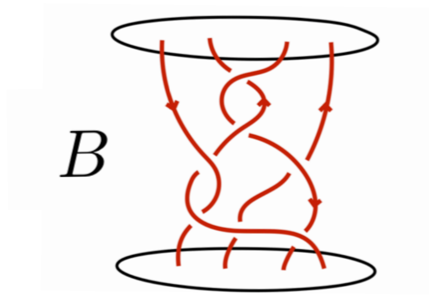
$$\mathscr{D}_{\mathcal{X}} = D^b Coh_{\mathrm{T}}(\mathcal{X})$$

as

$$Hom^{*,*}_{\mathscr{D}_{\mathcal{X}}}(\mathscr{B}\mathcal{F}_0, \mathcal{F}_1)$$

the space of morphisms from $\mathscr{B}\mathcal{F}_0$ to $\mathscr{F}_1$

The braid



is a path in complexified Kahler moduli of

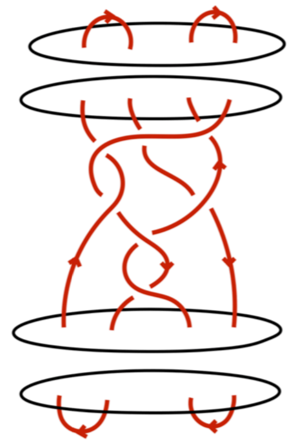
$$\mathcal{X}$$

and determines an auto-equivalences of the category of B-type branes

$$\mathscr{B} : D^b Coh_{\mathrm{T}}(\mathcal{X}) \to D^b Coh_{\mathrm{T}}(\mathcal{X})$$

The quantum invariants of links should be categorified by

$$\mathscr{D}_{\mathcal{X}} = D^b Coh_{\mathrm{T}}(\mathcal{X})$$



since they too can be expressed as matrix elements of the braiding matrix

$$(\mathfrak{U}_1, \mathfrak{B}\, \mathfrak{U}_0)$$

between pairs of conformal blocks.

The first step is to find objects of

$$\mathscr{D}_{\mathcal{X}} = D^b Coh_{\mathrm{T}}(\mathcal{X})$$

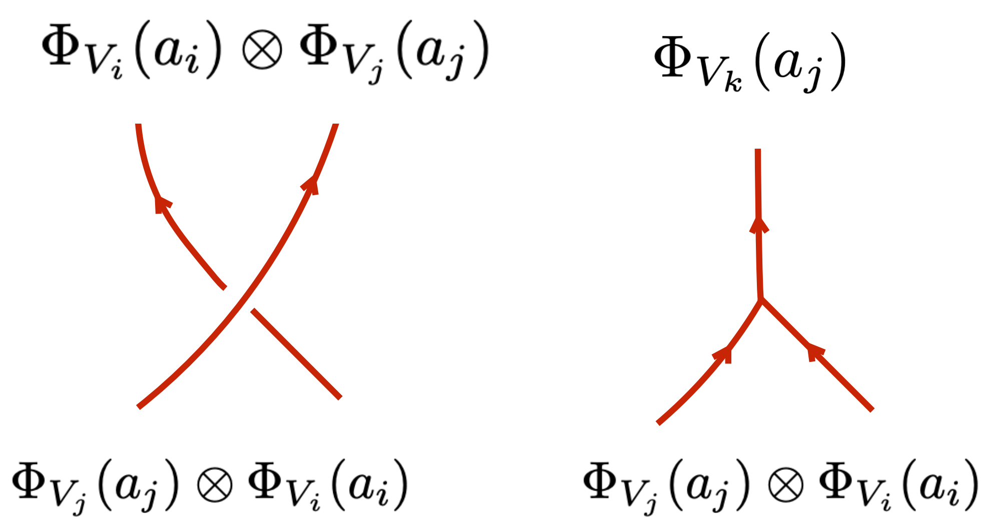
whose vertex functions are conformal blocks

in which pairs of vertex operators fuse to trivial representation.

This makes use of a classic result in conformal field theory,

which is that

fusion diagonalizes braiding.



In looking for objects of

$$\mathscr{D}_{\mathcal{X}} = D^b Coh_{\mathrm{T}}(\mathcal{X})$$

whose vertex functions are conformal blocks corresponding to:

we will discover that not only braiding,

but also fusion has a geometric interpretation in terms of

$$\mathscr{D}_{\mathcal{X}} = D^b Coh_{\mathrm{T}}(\mathcal{X})$$

Chapter III

Fusion from

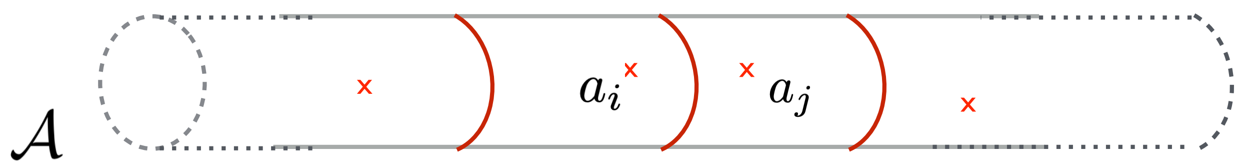
$$\mathscr{D}_{\mathcal{X}} = D^b Coh_{\mathrm{T}}(\mathcal{X})$$

As a pair of vertex operators

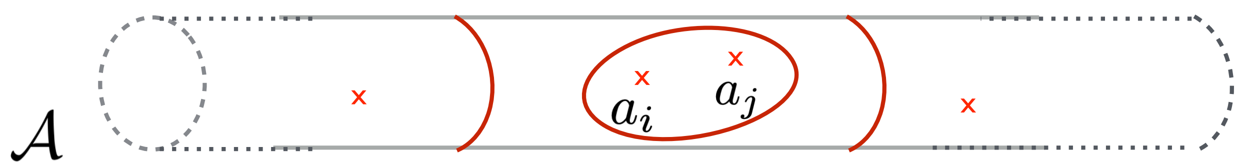
approach, one gets a new natural basis of conformal blocks.

Rather than using conformal blocks as obtained by

sewing chiral vertex operators as follows:



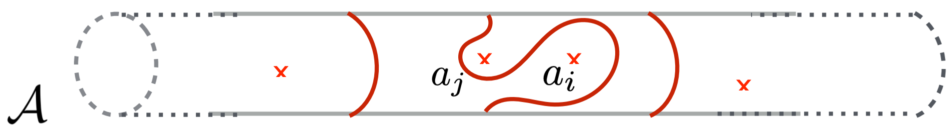
it is more natural to first bring them together, and sew like this instead.

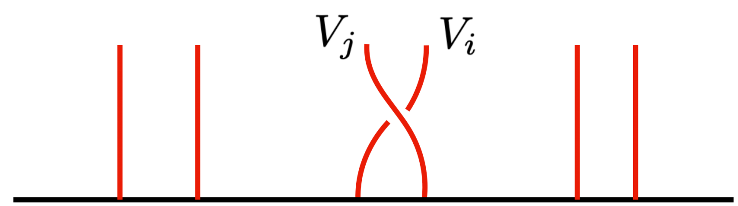


Both choices of basis span the space of solutions

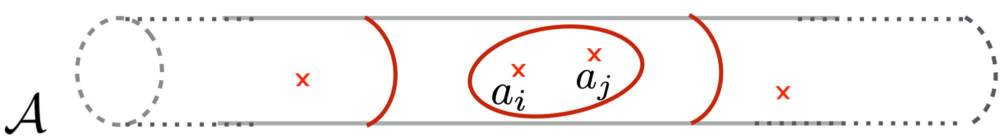
to the Knizhnik-Zamolodchikov equation, but

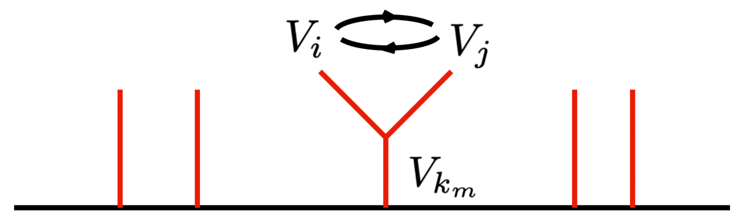
unlike in the basis we started with





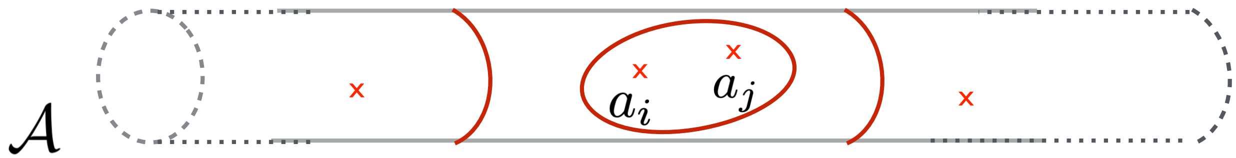
in the fusion basis, braiding acts diagonally,





As we bring the vertex operators together

$$a_i \to a_j$$



and the Riemann surface develops a very long neck,

describes replacing a pair of chiral vertex operators by a single one

$$\Phi_{V_i}(a_i) \otimes \Phi_{V_j}(a_j) \quad \to \quad (a_i - a_j)^{h_k - h_i - h_j} \Phi_{V_k}(a_j)$$

and where $h_i = c_i/\kappa$ are conformal dimensions of vertex operators.

Because V_i and V_j are minuscule,

the possible choices of fusion products

$$\Phi_{V_i}(a_i) \otimes \Phi_{V_j}(a_j) \quad \to \quad (a_i - a_j)^{h_k - h_i - h_j} \Phi_{V_k}(a_j)$$

are labeled by representations

$$V_i \otimes V_j = \bigoplus_{m=0}^{max} V_{k_m}$$

that occur in the tensor product,

with no multiplicities on the right hand side.

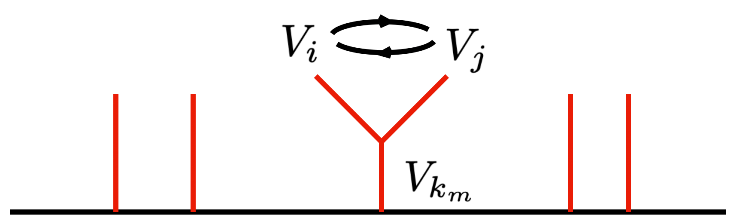
This gives a basis of solutions of the KZ equation

whose behavior as $a_i \to a_j$ is,

$$\mathcal{V}_k = (a_i - a_j)^{h_k - h_i - h_j} \times \text{finite}$$

where "finite" stands for terms $\mathcal{O}(a_i - a_j)$ that are non-vanishing and regular.

They are eigenvectors of braiding



with eigenvalue $e^{-\pi i(h_k - h_i - h_j)} = \mathfrak{q}^{\frac{1}{2}(c_i + c_j - c_k)}$

since $h_i = c_i/\kappa$ and $\mathfrak{q} = e^{2\pi i/\kappa}$

This behavior of conformal blocks has a geometric origin.

Vertex function

$$\mathcal{V}[\mathcal{F}]$$

which gives rise to conformal blocks, is a very close cousin of

"central charge"

$$\mathcal{Z}^0[\mathcal{F}]$$

of the brane, which computes its complexified mass.

One of the lessons from the very early days of mirror symmetry

is that the geometry of $\mathcal{X}$ near a point in its moduli space

where it develops a singularity

is reflected in the behavior of its central charge

$$\mathcal{Z}^0 : K(\mathcal{X}) \to \mathbb{C}$$

The vertex function

$$\mathcal{V}[\mathcal{F}]$$

generalizes the central charge function

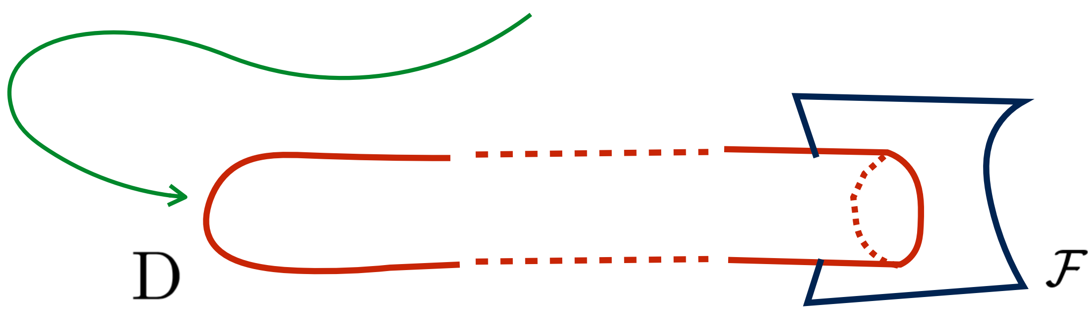
$$\mathcal{Z}^0[\mathcal{F}]$$

in two different ways.

Firstly, the vertex function is a vector

$$\mathcal{V}_\alpha[\mathcal{F}]$$

due to insertions of $\alpha \in \mathrm{H}^*_{\mathrm{T}}(\mathcal{X})$ classes at the origin of D.



and secondly it depends on equivariant parameters

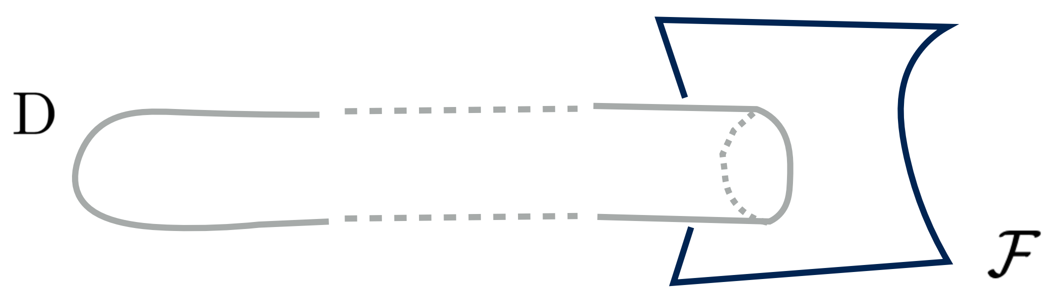
of the T-action on $\mathcal{X}$.

Undoing the first generalization, by placing no insertion at the origin

we get a scalar analog of the vertex function

which is the "equivariant central charge function"

$$\mathcal{Z}[\mathcal{F}] : \quad K_{\mathrm{T}}(\mathcal{X}) \quad \to \quad \mathbb{C}$$

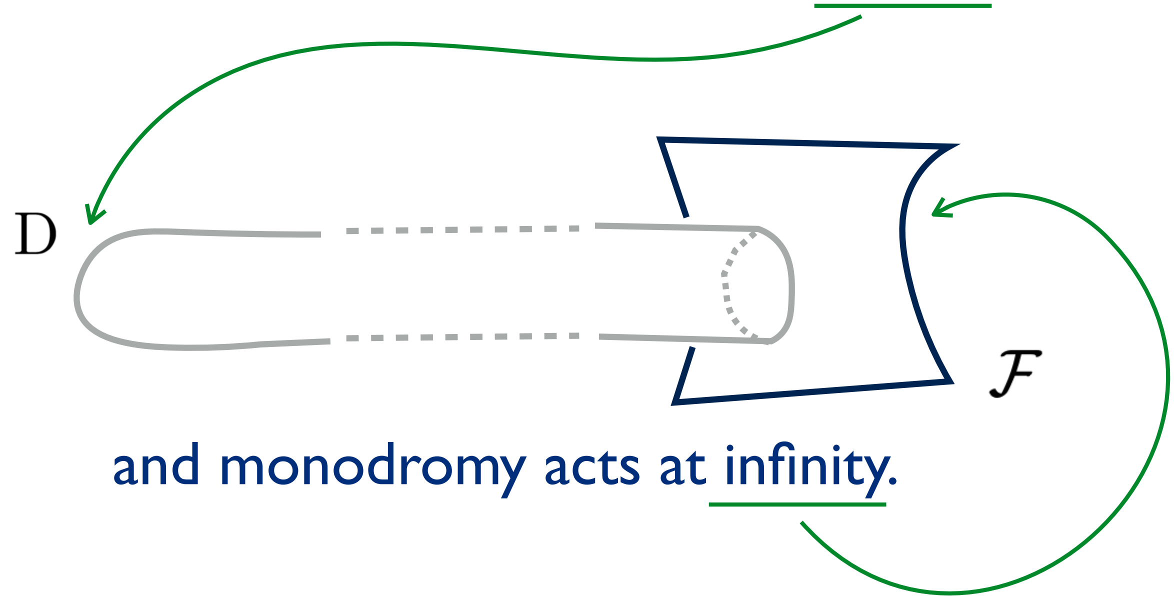


The scalar functions and vector vertex functions

$$\mathcal{Z}[\mathcal{F}] \qquad \text{and} \qquad \mathcal{V}_\alpha[\mathcal{F}]$$

have exactly the same monodromy

since they differ by the data at the origin,



and monodromy acts at infinity.

Aside: Braiding and variations of stability

By it origin in the sigma model to $\mathcal{X}$, the functor

$$\mathscr{B} : D^b Coh_{\mathrm{T}}(\mathcal{X}) \to D^b Coh_{\mathrm{T}}(\mathcal{X})$$

comes from variation of stability condition on

$$D^b Coh_{\mathrm{T}}(\mathcal{X})$$

defined with respect to the central charge function

$$\mathcal{Z}^0(\mathcal{F}) : K(\mathcal{X}) \to \mathbb{C}$$

The stability condition defined with respect to

$$\mathcal{Z}^0(\mathcal{F}) : K(\mathcal{X}) \to \mathbb{C}$$

is known as the Pi stability condition,

discovered by Douglas.

Since $\mathcal{X}$ is also hyper-Kahler,

$$\mathcal{Z}^0(\mathcal{F}) : K(\mathcal{X}) \to \mathbb{C}$$

is extremely simple.

It should lead to a model example of

a Bridgeland stability condition,

generalizing works of Bridgeland and Thomas

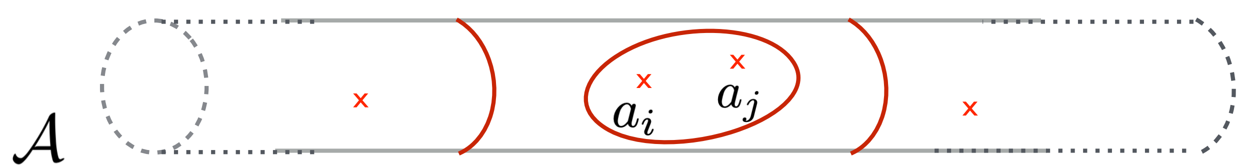
for an ADE surface.

One can explicitly work out the behavior of

central charges,

as we bring the vertex operators together

$$a_i \to a_j$$



Corresponding to a conformal block that vanishes as

$$\mathcal{V}_k \quad = \quad (a_i - a_j)^{h_k - h_i - h_j} \times \text{finite}$$

where $h_i = c_i/\kappa$

is equivariant central charge that turns out to vanish as

$$\mathcal{Z}_k = (a_i - a_j)^{\Delta_k - \Delta_i - \Delta_j} \times \text{finite}$$

where $\Delta_i = c_i/\kappa - d_i$

in terms of

conformal dimensions $c_i \equiv \frac{1}{2}\langle \mu_i, \mu_i + 2\,{}^L\rho \rangle$ and

$d_i \equiv \langle \mu_i, \rho \rangle$

We get from the equivariant central charge

$$\mathcal{Z} : K_{\mathrm{T}}(\mathcal{X}) \to \mathbb{C}$$

the ordinary central charge

$$\mathcal{Z}^0 : K(\mathcal{X}) \to \mathbb{C}$$

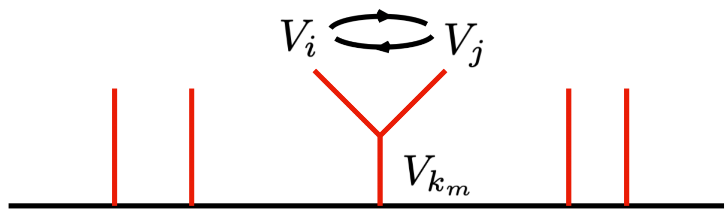
by turning off the equivariant parameters,

which includes sending to infinity the level κ of the affine Lie algebra

$$\widehat{{}^L\mathfrak{g}}_{\kappa}$$

It follows that corresponding to a conformal block which is

is the eigenvector of braiding associated to



is the equivariant central charge that vanishes as

$$\mathcal{Z}_{k_m} = (a_i - a_j)^{C_m/\kappa + D_m} \times \text{finite}$$

and ordinary central charge that vanishes as

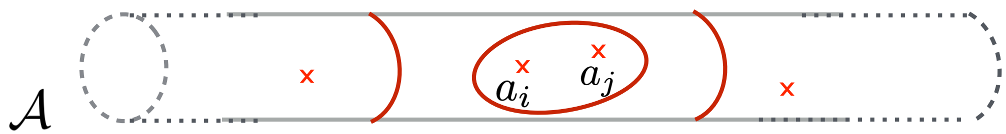
$$\mathcal{Z}^0_{k_m} = (a_i - a_j)^{D_m} \times \text{finite}$$

where

$C_m = c_{k_m} - c_i - c_j$ and $D_m = d_i + d_j - d_{k_m}$

and where D_m is a positive integer.

We expect thus that, as a pair of vertex operators approach each other



$\mathcal{X}$ should develop a singularity with a collection of vanishing cycles

$$F_{k_m}$$

labeled by representations in the tensor product

$$V_i \otimes V_j = \bigoplus_{m=0}^{max} V_{k_m}$$

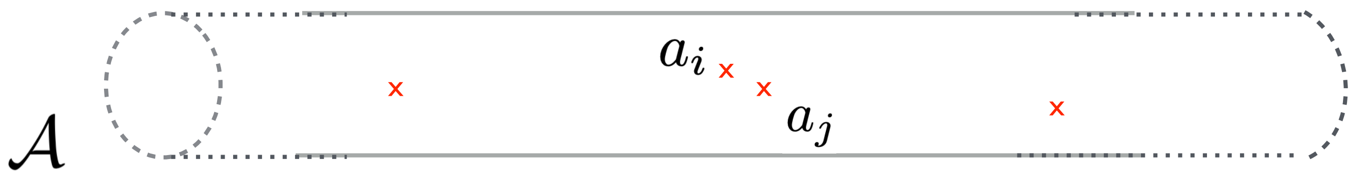
whose dimension is $\dim_{\mathbb{C}} F_{k_m} = d_i + d_j - d_{k_m} \equiv D_m$ which lead

to vanishing:

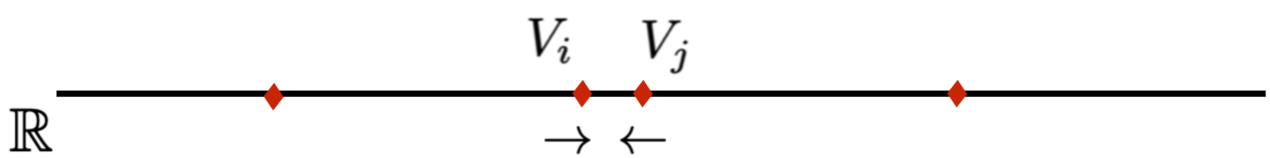
From perspective of

$\mathcal{X}$

bringing a pair of vertex operators together



corresponds to a wall in Kahler moduli

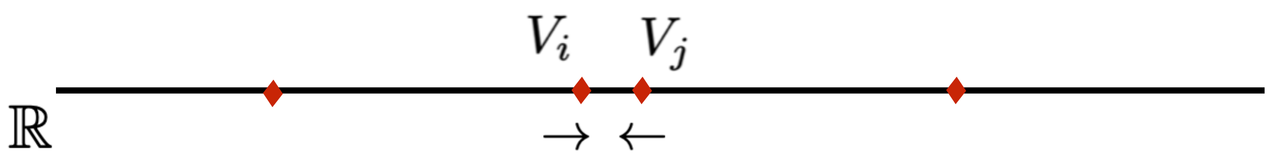


where a pair of singular monopoles come together.

As the monopoles collide,

$$\mathcal{X} = \mathrm{Gr}^{\vec{\mu}}{}_{\nu}$$

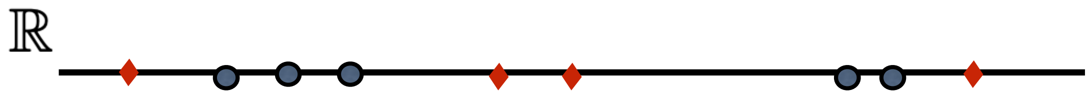
becomes the singular,



with singularities which are due to monopole bubbling phenomena.

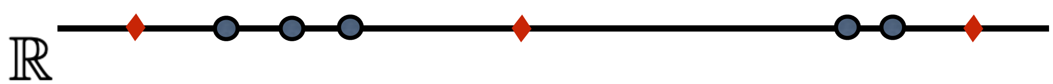
Naively, bringing together singular monopoles of charge

μ_i and μ_j



we get a single singular monopole of charge

$$\mu_{ij} = \mu_i + \mu_j$$



in their place, and so the manifold we started out with

$\mathcal{X} = \mathrm{Gr}^{\vec{\mu}}{}_{\nu}$ should get replaced with $\mathrm{Gr}^{\vec{\mu}_{ij}}{}_{\nu}$

where $\vec{\mu}_{ij}$ is obtained from $\vec{\mu} = (\ldots, \mu_i, \mu_j, \ldots)$

by replacing (μ_i, μ_j) with μ_{ij}

The resulting manifold

$$\mathrm{Gr}^{\vec{\mu}_{ij}}{}_{\nu}$$

would have the same dimension as $\mathcal{X} = \mathrm{Gr}^{\vec{\mu}}{}_{\nu}$ but,

it has additional non-compact directions,

because of the possibility for the monopole bubbling to occur.

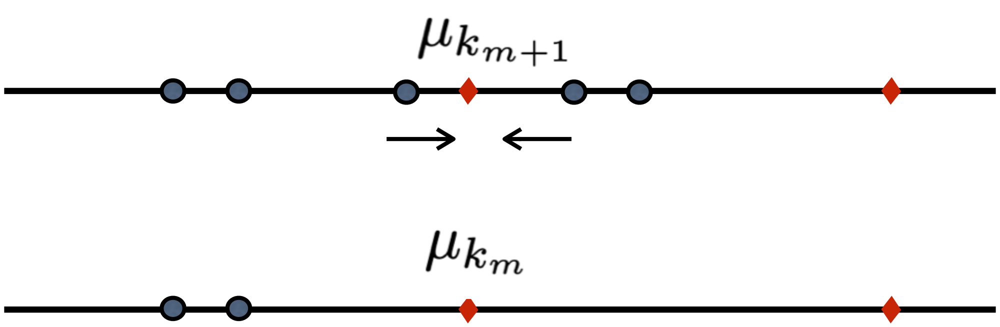
Monopole bubbling occurs

when smooth monopoles concentrate at a location of a singular monopole

whose charge is not minuscule,

and disappear,

leaving behind a singular monopole of lower charge.



To compactify this, one has to add lower dimensional strata,

$$\mathcal{X}^{\times} = \overline{\mathrm{Gr}^{\vec{\mu}}}^{\times}_{ij\,\nu} = \cup_{\mu_k \leq \mu_{ij}} \mathrm{Gr}^{\vec{\mu}_k}{}_{\nu}$$

the endpoints of all possible monopole bubblings:

in $\mathrm{Gr}^{\vec{\mu}_{k_m}}{}_{\nu}$ exactly $\mu_{ij} - \mu_{k_m}$ monopoles have bubbled off.

Kapustin and Witten explained that

the types of monopole bubbling that can occur

$$\mathcal{X}^{\times} = \mathrm{Gr}^{\vec{\mu}_{ij}^{\times}}{}_{\nu} = \cup_{\mu_k \leq \mu_{ij}} \mathrm{Gr}^{\vec{\mu}_k}{}_{\nu}$$

are labeled by representations V_k in the tensor product

$$V_i \otimes V_j = \bigoplus_{m=0}^{max} V_{k_m}$$

which we will order so that

$$\mu_{k_m} \leq \mu_{k_{m+1}}$$

corresponding to more bubbling, the smaller m gets.

This structure turns out to lead to the vanishing cycles

$$F_{k_m} \quad \leftrightarrow \quad V_{k_m}$$

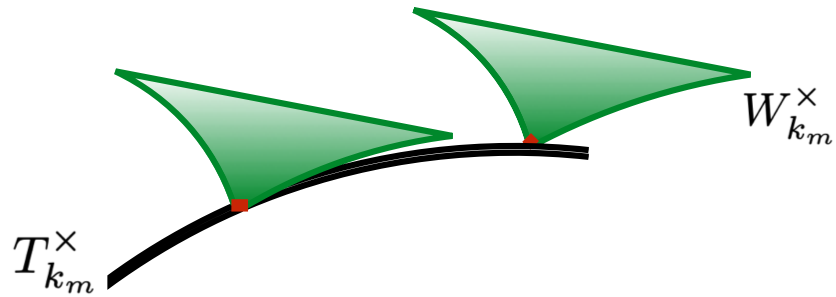
we are after.

Transverse space to the stratum,

$$T^{\times}_{k_m} = \mathrm{Gr}^{\vec{\mu}^{\times}_{k_m}}{}_{\nu} = \cup_{\mu_k \leq \mu_{k_m}} \mathrm{Gr}^{\vec{\mu}_k}{}_{\nu}$$

in the moduli space of monopoles where the singular monopole

of charge μ_{ij} replaced by a singular monopole of charge μ_{k_m}



is itself a moduli space of monopoles:

$$W^{\times}_{k_m} = \mathrm{Gr}^{\mu^{\times}_{ij}}{}_{\mu_{k_m}}$$

it is the moduli space of monopoles whose positions we needed to tune to

so that bubbling of type μ_{k_m} can occur.

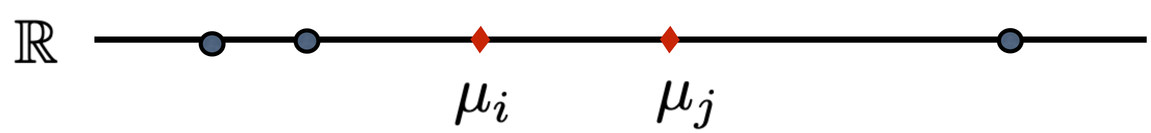
As we resolve the singularities to replace

$$\mathcal{X}^\times = \mathrm{Gr}^{\vec{\mu}^\times_{ij}}{}_\nu \qquad \text{with} \qquad \mathcal{X} = \mathrm{Gr}^{\vec{\mu}}{}_\nu$$

our transverse slice also becomes smooth,

since the single monopole of non-minuscule charge

splits back into a pair:



so we replace

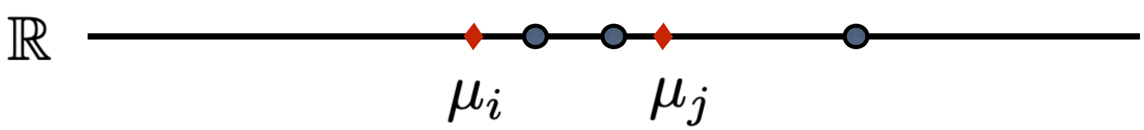
$$W^\times_{k_m} = \mathrm{Gr}^{\mu^\times_{ij}}{}_{\mu_{k_m}} \qquad \text{with} \qquad W_{k_m} = \mathrm{Gr}^{(\mu_i,\mu_j)}_{\mu_{k_m}}$$

The smooth transverse slice

$$W_{k_m} = \mathrm{Gr}^{(\mu_i,\mu_j)}_{\mu_{k_m}}$$

has a single Kahler modulus corresponding to the relative position of

the two singular monopoles:



It is a cotangent bundle to a holomorphic Lagrangian

$$W_{k_m} = T^* F_{k_m}$$

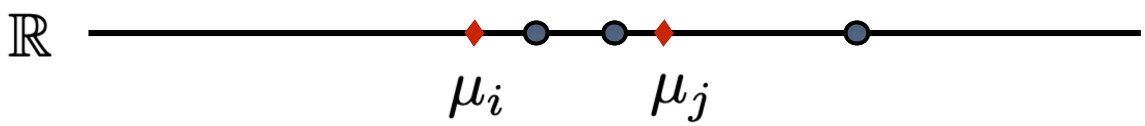
This holomorphic Lagrangian F_{k_m} turns out to be our vanishing cycle.

The dimension of the vanishing cycle

$$F_{k_m} \quad \leftrightarrow \quad V_{k_m}$$

is half the dimension of the monopole moduli space corresponding to

$$W_{k_m} = \mathrm{Gr}^{(\mu_i, \mu_j)}_{\mu_{k_m}} = T^* F_{k_m}$$



with total monopole charge

$$\mu_{k_m}$$

The complex dimension of the vanishing cycle

$$F_{k_m}$$

is the number of smooth monopoles which equals to

$$\dim_{\mathbb{C}} F_{k_m} = \langle \mu_i + \mu_j - \mu_{k_m}, \rho \rangle = D_m$$

the same integer D_m to governs the asymptotics of

central charges near the singularity

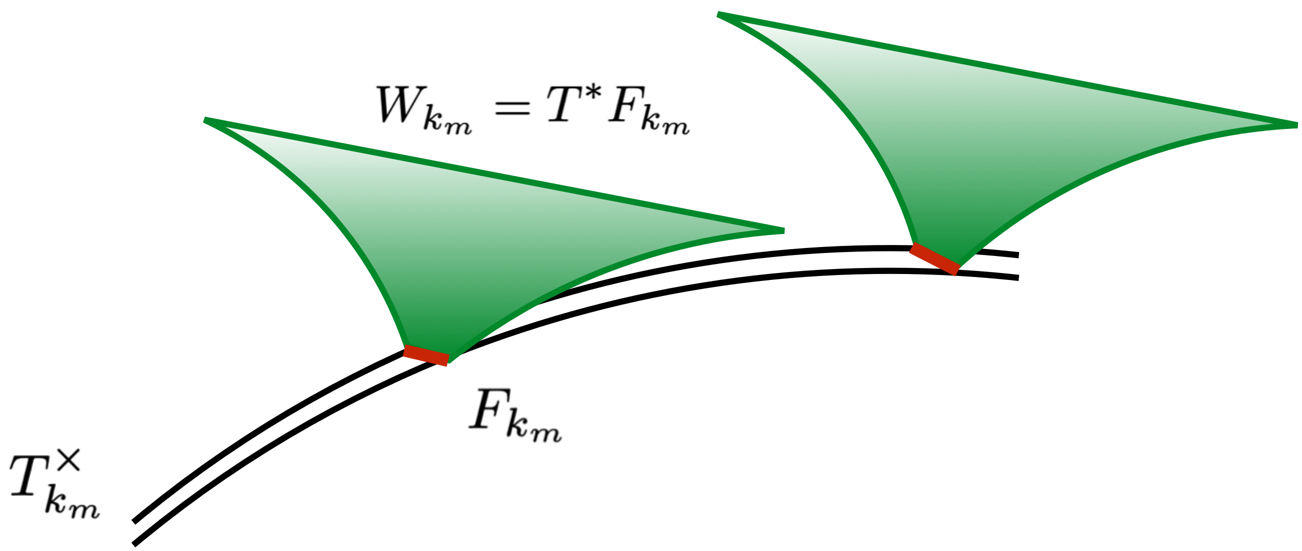
$$\mathcal{Z}^0_{k_m} = (a_i - a_j)^{D_m} \times \text{finite}$$

which identifies it with the mass of the brane supported on F_{k_m}

These vanishing cycle come in a family parameterized by

$$T^{\times}_{k_m} = \mathrm{Gr}^{\vec{\mu}^{\times}_{k_m}}{}_{\nu}$$

since $W^{\times}_{k_m}$ was the transverse slice to it.



A brane

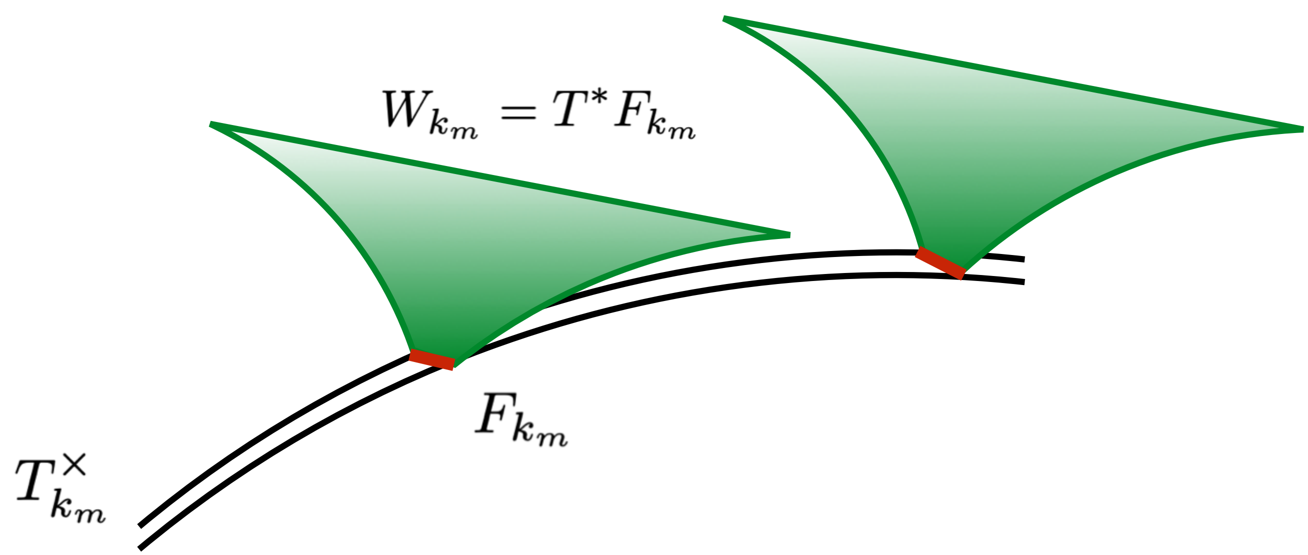
$$\mathcal{F}_{k_m} \in \mathscr{D}_{\mathcal{X}}$$

whose central charge vanishes at least as fast as

$$\mathcal{Z}^0[\mathcal{F}_{k_m}] \sim \quad \mathcal{Z}^0_{k_m} = (a_i - a_j)^{D_m} \times \text{finite}$$

may be obtained by pairing the structure sheaf of F_{k_m} in W_{k_m}

with any brane on $T^{\times}_{k_m}$



These branes are the objects of the derived category

$$\mathcal{F}_{k_m} \in \mathscr{D}_{\mathcal{X}}$$

whose vertex functions have the same leading behavior near

the singularity at

$$a_i \to a_j$$

as the conformal blocks in the fusion basis

$$\mathcal{V}[\mathcal{F}_{k_m}] \sim \mathcal{V}_{k_m} \quad = \quad (a_i - a_j)^{h_{k_m} - h_i - h_j} \times \text{finite}$$

Unlike for abstract conformal blocks,

in general one cannot find branes

$$\mathcal{F}_{k_m} \in \mathscr{D}_\mathcal{X}$$

for which the relation

$$\mathcal{V}[\mathcal{F}_{k_m}] \sim \mathcal{V}_{k_m} \quad = \quad (a_i - a_j)^{h_{k_m} - h_i - h_j} \times \text{finite}$$

becomes exact.

Conformal blocks which diagonalize braiding

do not in general come from of actual objects

of the derived category $\mathscr{D}_{\mathcal{X}} = D^b Coh_{\mathrm{T}}(\mathcal{X})$:

Eigensheaves of braiding $\mathcal{E} \subset \mathscr{D}_{\mathcal{X}}$ on which

the braiding functor acts as

$$\mathscr{B}\mathcal{E} = \mathcal{E}[-D_{\mathcal{E}}]\{C_{\mathcal{E}}\}$$

are rare.

For actual branes, $\mathcal{F}_{k_m} \in \mathscr{D}_{\mathcal{X}}$ the vertex functions have leading behavior

$$\mathcal{V}[\mathcal{F}_{k_m}] \sim \mathcal{V}_{k_m} \quad = \quad (a_i - a_j)^{h_{k_m} - h_i - h_j} \times \text{finite}$$

but, in general contain subleading terms:

$$\mathcal{V}[\mathcal{F}_{k_m}] = \mathcal{V}_{k_m} + \sum_{\mu_{k_\ell} < \mu_{k_m}} n_\ell \cdot \mathcal{V}_{k_\ell}$$

which vanish faster,

with coefficients n_ℓ which are rational functions of $\mathfrak{q}$.

What we get instead is a filtration

$$\mathscr{D}_{k_0} \subset \mathscr{D}_{k_1} \ldots \subset \mathscr{D}_{k_{max}} = \mathscr{D}_{\mathcal{X}}$$

on the derived category $\mathscr{D}_{\mathcal{X}} = D^b Coh_{\mathrm{T}}(\mathcal{X})$,

with terms in the filtration labeled by distinct representations

$$V_i \otimes V_j = \bigoplus_{m=0}^{max} V_{k_m}$$

in the tensor product

The m-th term in the filtration

$$\mathscr{D}_{k_0} \subset \mathscr{D}_{k_1} \ldots \subset \mathscr{D}_{k_{max}} = \mathscr{D}_{\mathcal{X}}$$

is a subcategory $\mathscr{D}_m$ of the derived category

generated by objects whose central charges vanish at least as fast as

$$\mathcal{Z}^0_{k_m} = (a_i - a_j)^{D_m} \times \text{finite}$$

where $D_m = \dim_{\mathbb{C}} F_{k_m}$, the dimension of the vanishing cycle

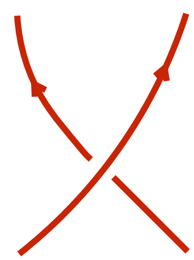
and where

$$D_m \geq D_{m+1}$$

the order of vanishing increases as m decreases.

Braiding which exchanges the pair of vertex operators

$$\Phi_{V_j}(a_j) \otimes \Phi_{V_i}(a_i)$$



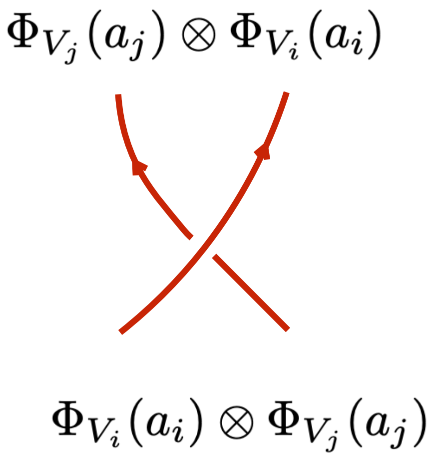
$$\Phi_{V_i}(a_i) \otimes \Phi_{V_j}(a_j)$$

corresponds to a generalized flop in the geometry.

The flop is generalized since more than one cycle vanishes,

and in general the vanishing cycles are not spherical.

Braiding preserves the filtrations one gets on the two sides of the flop,



$$\mathscr{D}'_{k_0} \subset \mathscr{D}'_{k_1} \ldots \subset \mathscr{D}'_{k_{max}} = \mathscr{D}'_{\mathcal{X}}$$

and

$$\mathscr{D}_{k_0} \subset \mathscr{D}_{k_1} \ldots \subset \mathscr{D}_{k_{max}} = \mathscr{D}_{\mathcal{X}}$$

since it has the effect of mixing up objects of a

given order of vanishing of central charge,

with those of that vanish faster, and which belong to lower orders

in the filtration.

Since along the path objects of given order vanishing mix up with those of the same or lower order of vanishing, on quotient category

$$\mathscr{B}:\ \mathscr{D}_m/\mathscr{D}_{m-1}\ \rightarrow\ \mathscr{D}'_m/\mathscr{D}'_{m-1}\ \cong\ \mathscr{D}_m/\mathscr{D}_{m-1}[-D_m]\{C_m\}$$

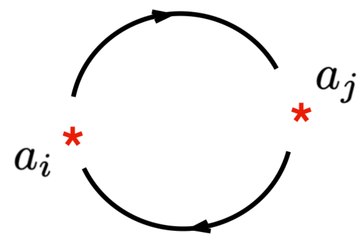
in which one treats all branes of coming from lower orders as zero the functor acts at most by degree shifts.

The degree shifts

$$\mathscr{B}: \mathscr{D}_m/\mathscr{D}_{m-1} \to \mathscr{D}'_m/\mathscr{D}'_{m-1} \cong \mathscr{D}_m/\mathscr{D}_{m-1}[-D_m]\{C_m\}$$

do not depend on the objects, but only on the order of the filtration

and on the path around the singularity



which one can read off from

$$\mathcal{Z}_{k_m} = (a_i - a_j)^{D_m - C_m/\kappa} \times \text{finite}$$

Derived equivalences of this type realize perverse equivalences

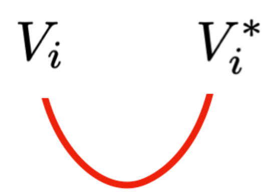
defined abstractly by

Rouquier and Chuang,

which comes from geometry and physics.

Chapter IV

Cups from geometry

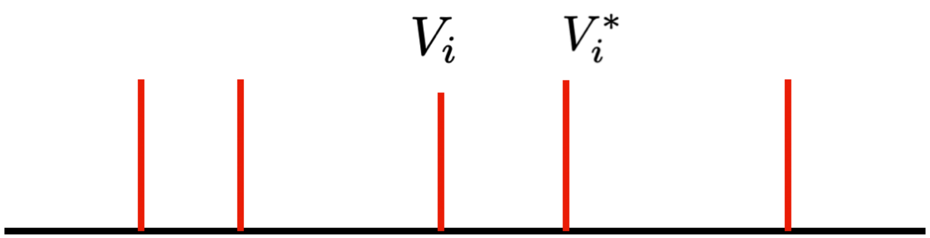


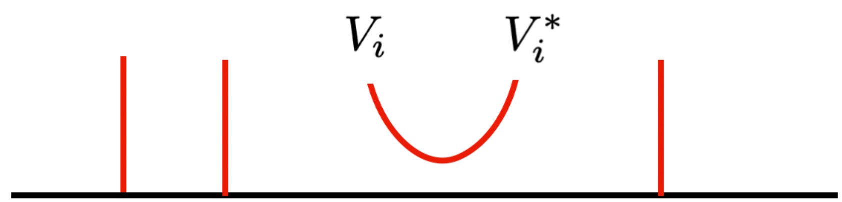
A very special case of fusion occurs when we bring together

a pair of vertex operators colored by conjugate representations

$$\Phi_{V_i}(a_i) \otimes \Phi_{V_i^*}(a_j) \quad \rightarrow \quad \mathbb{1}$$

which fuse to the identity and disappear:

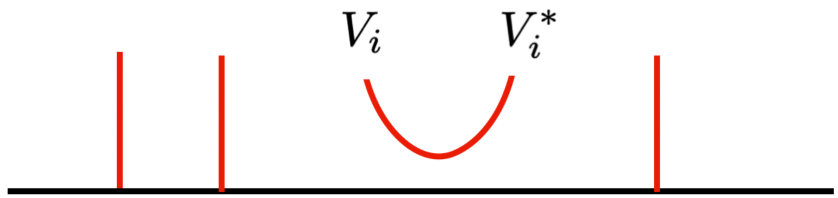




corresponding to the trivial representation in the tensor product:

$$V_i \otimes V_i^* = \mathbb{1} \oplus \ldots$$

In this case



the transversal slice $W_{k_m} = \mathrm{Gr}_{\mu_{k_m}}^{(\mu_i,\mu_j)}$ becomes

$$\mathrm{Gr}_0^{(\mu_i,\mu_i^*)} = T^*F_{k_0}$$

The corresponding vanishing cycle can be shown to be the

"minuscule Grassmannian"

$$F_{k_0} = G/P_i$$

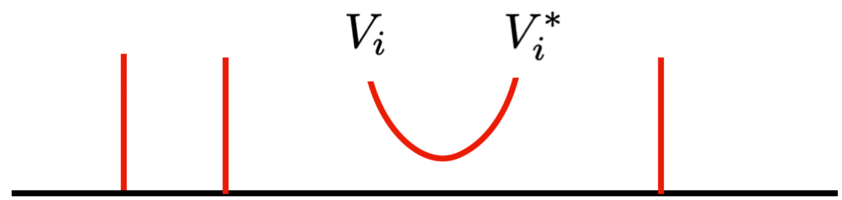
where P_i is a maximal parabolic subgroup of G

associated to the minuscule representation V_i

We will give it a special label

$$U_i \equiv F_{k_0}$$

The branes $\mathcal{U} \in \mathscr{D}_{\mathcal{X}}$ corresponding to such conformal blocks



belong to the lowest term of the filtration

$$\mathscr{D}_{k_0} \subset \mathscr{D}_{k_1} \ldots \subset \mathscr{D}_{k_{max}} = \mathscr{D}_{\mathcal{X}}$$

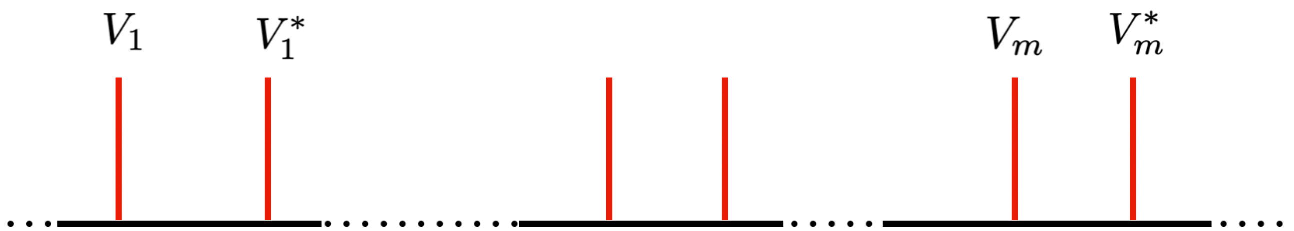
so they are necessarily eigensheaves the braiding functor

$$\mathscr{B}\mathcal{U} = \mathcal{U}[-D_0]\{C_0\}$$

Even then, they are extremely special ones,

for the same reason the identity representation is special.

When a collection of vertex operators come together in pairs of minuscule representations and their conjugates



our manifold has a local neighborhood where we can approximate it as

$$\mathcal{X} \sim T^*U$$

where

$$U = U_1 \times \ldots \times U_m = G/P_1 \times \ldots \times G/P_m$$

is a product of minuscule Grassmannians.

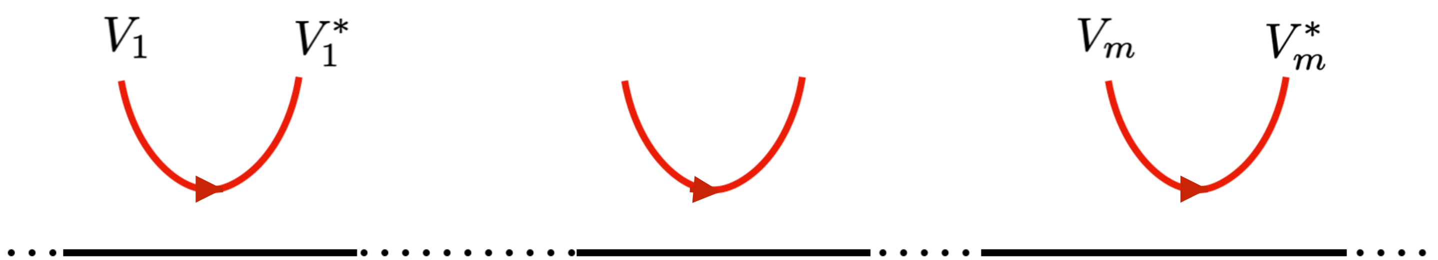
We get a very special B-type brane

$$\mathcal{U} \in \mathscr{D}_{\mathcal{X}}$$

which is the structure sheaf of this vanishing cycle,

$$\mathcal{U} = \mathcal{O}_U$$

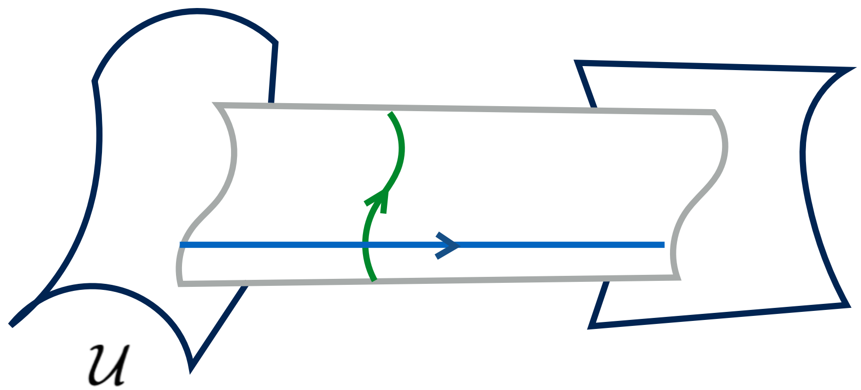
Among other things, the vertex function of this brane is the conformal block



which we will denote by

$$\mathfrak{U} = \mathcal{V}[\mathcal{U}]$$

Take the theory on an interval, with a pair of such branes as boundary conditions.

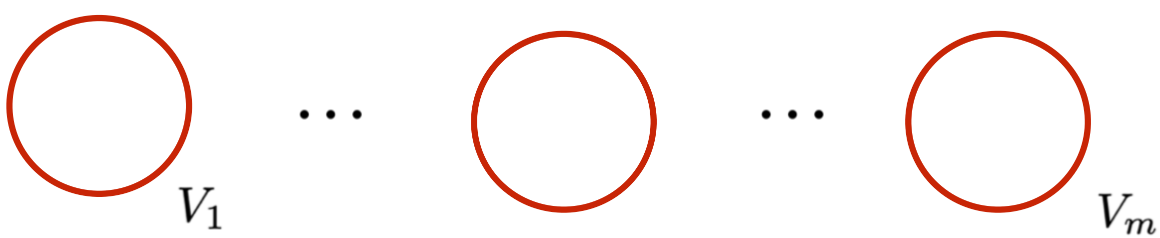


The cohomology of the supercharge Q,

the space of "endomorphisms" of $\mathcal{U} \in \mathscr{D}_{\mathcal{X}}$

$$\mathrm{Hom}^{*,*}_{\mathscr{D}_{\mathcal{X}}}(\mathcal{U}, \mathcal{U})$$

should categorify the quantum group invariant of a product of unknots:



For a brane

$$\mathcal{U} = \mathcal{O}_U$$

which is a "structure sheaf" of a cycle

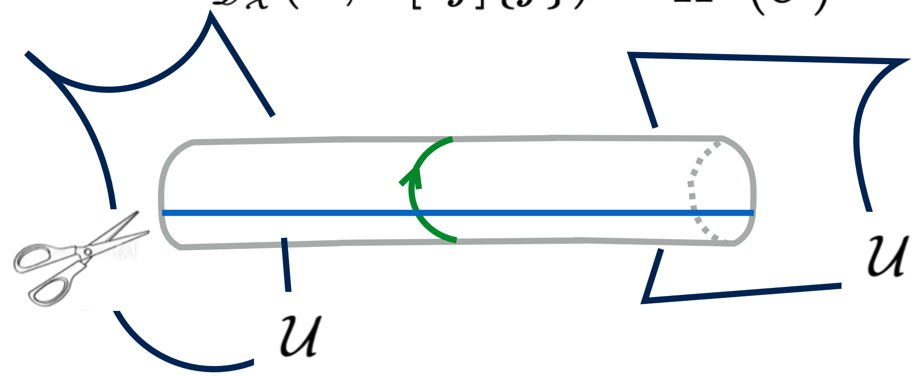
$$\mathcal{X} \sim T^*U$$

the non-vanishing homology groups are:

$$Hom_{\mathscr{D}_\mathcal{X}}(\mathcal{U}, \mathcal{U}[2j]\{j\}) = H^{j,j}(U) = H^j(U)$$

The Euler character of

$$Hom_{\mathscr{D}_{\mathcal{X}}}(\mathcal{U},\mathcal{U}[2j]\{j\}) = H^j(U)$$



$$\chi(\mathcal{U},\mathcal{U}) = \sum_{j,k}(-1)^j \mathfrak{q}^{k-D/2} Hom_{\mathscr{D}_{\mathcal{X}}}(\mathcal{U},\mathcal{U}[j]\{k\}).$$

computes the Poincare polynomial

$$\chi(\mathcal{U},\mathcal{U}) = \mathfrak{q}^{-D/2} \sum_{j=0}^{D} \mathfrak{q}^j \dim H^j(U)$$

of $\quad H^*(U) = \bigotimes_{i=1}^{m} H^*(G/P_i)$

The Poincare polynomial can be computed by Morse theory:

Pick a generic one parameter subgroup

$$\mathbb{C}^\times \subset \mathrm{T}.$$

of the torus action on $\mathcal{X}$

The Hamiltonian that generates it is a Morse function,

whose critical points are the torus fixed points.

Given a fixed point, the number of negative eigenvalues of the Hessian

is the degree with which the fixed point contributes to

$$H^*(U) = \bigotimes_{i=1}^{m} H^*(G/P_i)$$

The fixed points of torus action on any

$$\mathcal{X} = \mathrm{Gr}^{\vec{\mu}}{}_{\nu}$$

are in one to one correspondence with the weights

of the corresponding representation

$$(V_1 \otimes \ldots \otimes V_m)_\nu$$

In the present case, this translates into generators of

$$H^*(G/P_i)$$

being in the weights of the corresponding

representation

$$V_i$$

and its Poincare polynomial turns into a trace in this representation

$$\chi(\mathcal{U}_i, \mathcal{U}_i) = \mathfrak{q}^{-\dim U_i} P_{U_i}(\mathfrak{q}) = \mathrm{tr}_{V_i}\, \mathfrak{q}^{\rho}$$

where is the Weyl vector of $\mathfrak{g}$ (and not of ${}^L\mathfrak{g}$).

When,

$$^{L}\mathfrak{g} = \mathfrak{g}$$

the right hand side is

$$\chi(\mathcal{U}_i, \mathcal{U}_i) = \mathrm{tr}_{V_i}\, \mathfrak{q}^{\rho}$$

is the "quantum dimension" of representation V_i ,

which is the $U_{\mathfrak{q}}(^{L}\mathfrak{g})$ quantum group invariant of