Knot Categorification
from Mirror Symmetry

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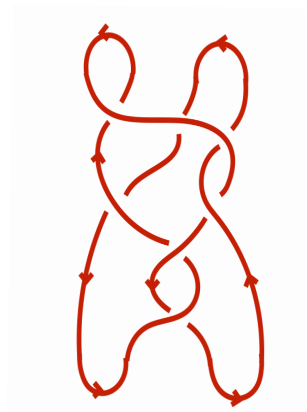
In 1985 Jones explained how to associate to a link K in $\mathbb{R}^3$



a polynomial in one variable

$$J_K(\mathfrak{q})$$

The polynomial is a link invariant,

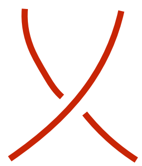
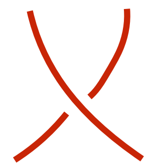
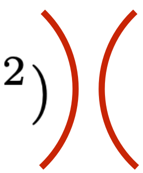
$$J_K(\mathfrak{q})$$

so links with different values of the polynomial

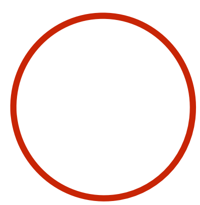
can not be smoothly deformed into each other.

The Jones polynomial is defined in a simple way,

by how it changes as one crosses a pair of strands,

$\mathbf{q}$  $-$ $\mathbf{q}^{-1}$  $= (\mathbf{q}^{1/2} - \mathbf{q}^{-1/2})$ 

and the value for the unknot,

 $= \mathbf{q}^{1/2} + \mathbf{q}^{-1/2}$

The coefficients of the polynomial are always integers.

The relation of Jones'

knot invariant to physics was explained by Witten in 1989.

Witten showed that the Jones polynomial

comes from Chern-Simons theory with gauge group

$$^{L}G = SU(2)$$

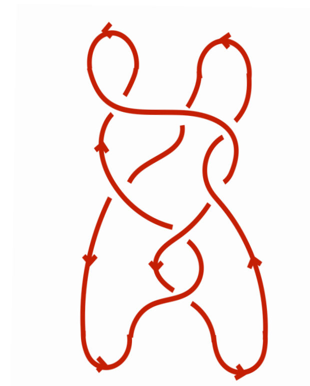
with links colored by its fundamental representation.

The parameter $\mathfrak{q}$ of the Jones polynomial

arizes from the level of Chern-Simons theory by

$$\mathfrak{q} = e^{\frac{2\pi i}{\kappa}}$$

This placed the Jones polynomial into a more general framework



which one gets by

considering Chern-Simons theory based on

different Lie algebras ${}^L\mathfrak{g}$ and by varying representations

coloring the knots.

Chern-Simons theory can be solved exactly.

The resulting link invariants are known as

$$U_{\mathfrak{q}}(^L\mathfrak{g})$$

quantum group invariants.

Formulation of Witten's link invariants

in terms of quantum groups

was developed by Reshetikhin and Turaev in '89.

Khovanov explained in '99 that one can recover the Jones polynomial

as a shadow of a cohomology theory

which assigns to a link a collection of vector spaces

$$\mathcal{H}_K^{i,j}$$

graded by a fermion number i and an "equivariant grading" j

whose graded Euler characteristic is the Jones polynomial

$$J_K(\mathfrak{q}) = \sum_{i,j\in\mathbb{Z}} (-1)^i \mathfrak{q}^{j/2} \dim_{\mathbb{C}} \mathcal{H}_K^{i,j}$$

In this way, the coefficients of the Jones polynomial

$$J_K(\mathfrak{q}) = \sum_{i,j\in\mathbb{Z}} (-1)^i \mathfrak{q}^{j/2} \dim_{\mathbb{C}} \mathcal{H}_K^{i,j}$$

get reinterpreted as dimensions of vector spaces

$$\mathcal{H}_K^{i,j}$$

which themselves are finer link invariants.

The problem Khovanov initiated is to find

a physical, or at least geometric, meaning of

of Khovanov homology,

one that works uniformly for all gauge groups.

I will explain that Khovanov's homologies also

have origin in physics,

which places them into a more general framework

in parallel to what Witten did in '88.

From string theory, one derives in fact two approaches,

related by a version of two dimensional mirror symmetry.

Two dimensional mirror symmetry

is a string duality with a precise mathematical description,

connecting algebraic and symplectic geometry.

We will discover here a new application of it,

to knot theory

and to representation theory.

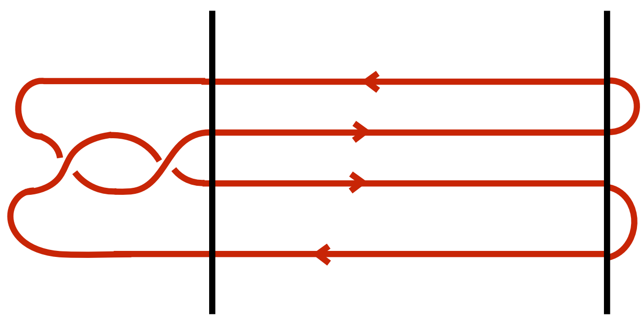
Two-dimensional physics enters here

because knot homologies will

come from two-dimensional theories associated to

$\text{link } K \times \text{time}$ in $\mathbb{R}^3 \times \text{time}$

as the spaces of supersymmetric ground states.



The theories come from two dimensional

defects in the six dimensional (0,2) conformal field theory

as anticipated from the works of Ooguri and Vafa in ’99,

and Gukov, Schwarz and Vafa in ’04.

The implication of their works was that, if one is able to understand how to extract spaces of BPS states from the (0,2) theory with defects, they will be Khovanov homology groups and their generalizations.

The 6d or 5d bulk approach to the problem is being developed by Witten.

The approaches I will describe are complementary

and solve the problem from the perspective

of two-dimensional theories on defects themselves.

Chern-Simons knot invariants are interesting,

not (only) because knots and their invariants are interesting,

but rather because of the wealth of mathematics and physics connections

one gets to discover once one understands them.

The fact that this structure arises from a deeper theory,

will no doubt lead to many more connections.

We will see one of them here,

to solvable, yet highly non-trivial examples

of homological mirror symmetry,

which connect it to representation theory.

In the same '89 paper, Witten also showed that

underlying Chern-Simons theory is a

two-dimensional conformal field theory associated to

$^L\mathfrak{g}$ and κ

We can take this, rather than Chern-Simons theory, as the starting point.

The conformal field theory one needs has an

affine Lie algebra symmetry

$$\widehat{{}^L\mathfrak{g}_\kappa}$$

obtained as the central extension of the loop algebra of ${}^L\mathfrak{g}$

where one fixes the central element to be κ

We will begin by reviewing the relation of

conformal field theory to quantum knot invariants.

Then, we will explain how the entire structure

and its categorification emerges

from geometry.

In this first lecture, I will explain the first approach.

The second approach,

and the string theory origins of these approaches

will be the topic of the second and third lectures.

Chapter I

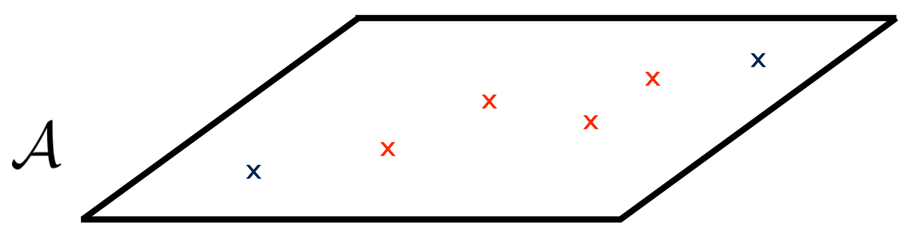
Conformal field theory origin of link invariants

To eventually get invariants of knots in $\mathbb{R}^3$ or S^3

we want to start with a Riemann surface

$$\mathcal{A}$$

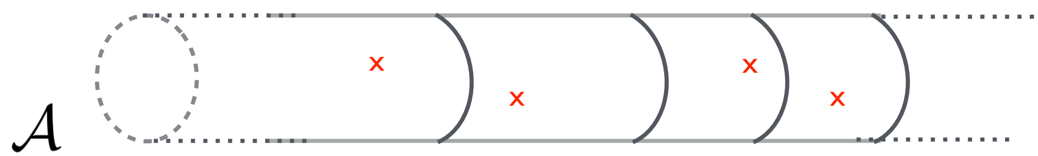
which is a complex plane with punctures.



It is equivalent, but better for our purpose,

to take

$\mathcal{A}$



to be a punctured infinite cylinder.

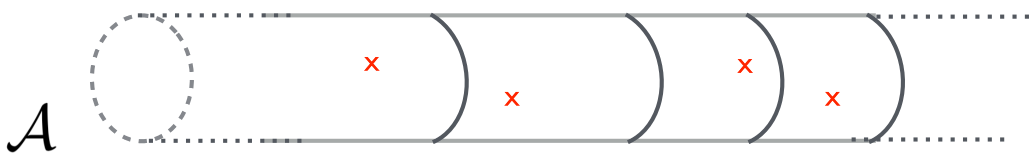
Punctures are labeled by representations of

$$^{L}\mathfrak{g}$$

Two types of representations will play a role for us.

To a puncture at a finite point

$$y = a_i$$



we will associate a finite dimensional representation

V_i of ${}^L\mathfrak{g}$,

which we will take to be minuscule.

To punctures at the two ends at infinity,

we will associate a pair of

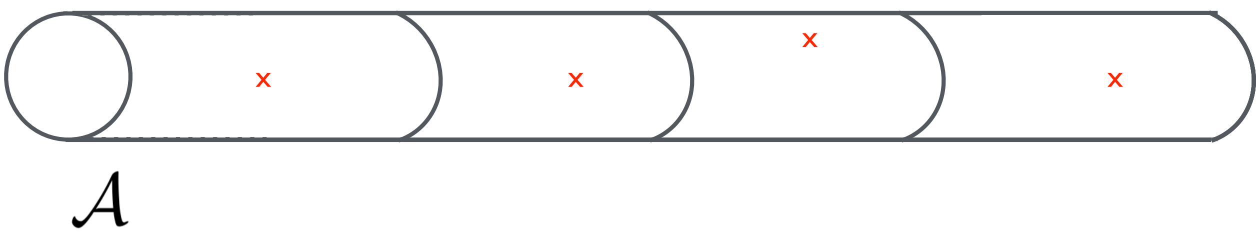
infinite dimensional, Verma module representations,

whose highest weight vectors

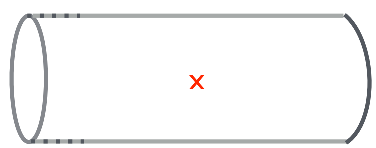
$$|\lambda\rangle \qquad |\lambda'\rangle$$

are given by generic weights of $^L\mathfrak{g}$.

One can view the Riemann surface



as obtained by sewing from



3-punctured spheres.

Conformal field theory associates to a 3-punctured sphere

$$\Phi_{V_i}(a_i)$$

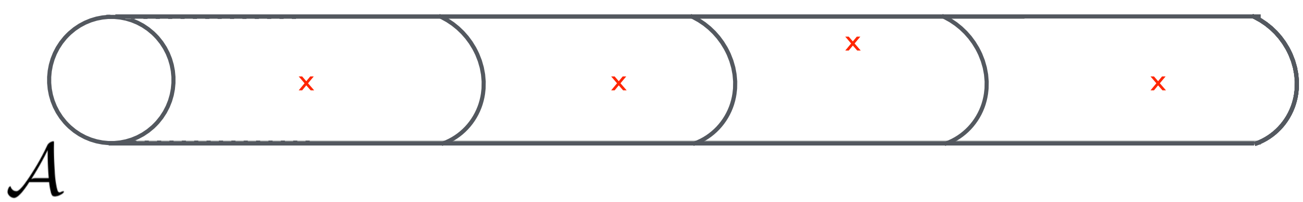
$$\langle \lambda_i | \qquad\qquad | \lambda_{i+1} \rangle$$

a chiral vertex operator

$$\Phi_{V_i}(a_i) : V_{\lambda_i} \;\rightarrow\; V_i(a_i) \otimes V_{\lambda_{i+1}}$$

which acts as intertwiner between pairs of Verma module representations.

To a Riemann surface with punctures, it associates a



``conformal block"

$$\langle\lambda| \, \Phi_{V_1}(a_1)\cdots\Phi_{V_\ell}(a_\ell)\cdots\Phi_{V_n}(a_n)\,|\lambda'\rangle$$

obtained by sewing chiral vertex operators.

More precisely, we get in this way not a single conformal block,

but a vector space of conformal blocks,

whose dimension turns out to be that of

the subspace of ${}^L\mathfrak{g}$ representation

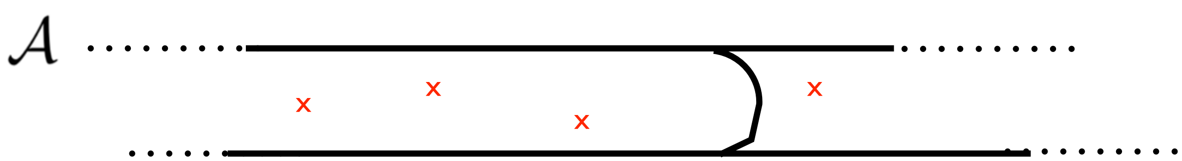
$$V = \bigotimes_{i=1}^{n} V_i$$

with fixed weight $\nu = \lambda - \lambda'$

Rather than characterizing conformal blocks

$$\langle\lambda| \; \Phi_{V_1}(a_1) \cdots \Phi_{V_\ell}(a_\ell) \cdots \Phi_{V_n}(a_n) \; |\lambda'\rangle$$

in terms of vertex operators and sewing,



one can describe them as solutions to a differential equation.

The equation solved by conformal blocks of $\widehat{{}^L\mathfrak{g}_\kappa}$ on $\mathcal{A}$

$$\mathcal{V}(a_1,\ldots,a_\ell,\ldots,a_n) = \langle\lambda|\; \Phi_{V_1}(a_1)\cdots\Phi_{V_\ell}(a_\ell)\cdots\Phi_{V_n}(a_n)\;|\lambda'\rangle$$

is the equation discovered by Knizhnik and Zamolodchikov in '84:

$$\kappa\, a_\ell \frac{\partial}{\partial a_\ell}\mathcal{V} = \sum_{j\neq\ell} r_{\ell i}(a_\ell/a_j)\,\mathcal{V}.$$

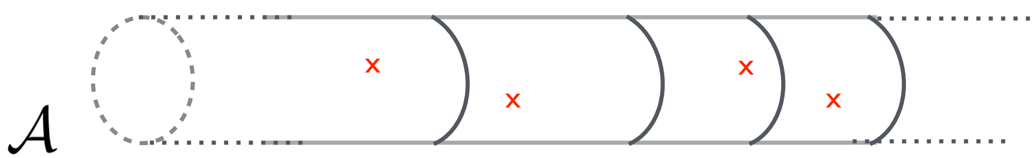
The specific flavor of the equation we need,

is known as the Knizhnik-Zamolodchikov equation

$$\kappa\, a_\ell \frac{\partial}{\partial a_\ell}\, \mathcal{V} = \sum_{j\neq\ell} r_{\ell i}(a_\ell/a_j)\, \mathcal{V}.$$

of trigonometric type since

we are thinking of the Riemann surface



as the infinite cylinder.

The coefficients on its right hand side

$$\kappa\, a_\ell \frac{\partial}{\partial a_\ell} \mathcal{V} = \sum_{j \neq \ell} r_{\ell i}(a_\ell / a_j)\, \mathcal{V}.$$

are the classical r-matrices of ${}^L\mathfrak{g}$:

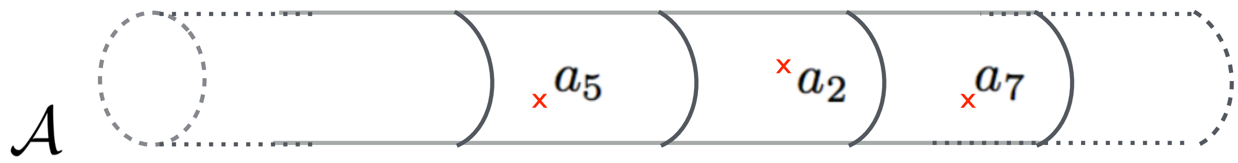
$$r_{ij}(a_i/a_j) = \frac{r_{ij} a_i + r_{ji} a_j}{a_i - a_j},$$

where

$$r = \frac{1}{2} \sum_a {}^L h_a \otimes {}^L h_a + \sum_{\alpha > 0} {}^L e_\alpha \otimes {}^L e_{-\alpha},$$

in the standard Lie theory notation.

Conformal blocks obtained by sewing chiral vertex operators



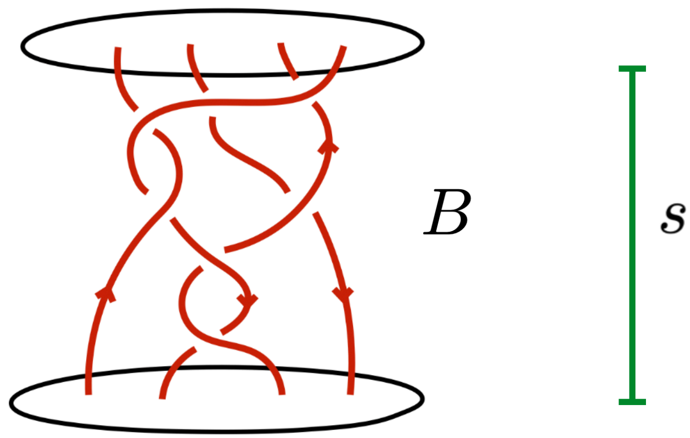
are solutions of the KZ equation

which are analytic in a chamber such as

$$|a_5| > |a_2| > |a_7| > \dots .$$

corresponding to the choice of ordering of vertex operators

By varying the positions of vertex operators on $\mathcal{A}$ as a function of "time" $s \in [0,1]$



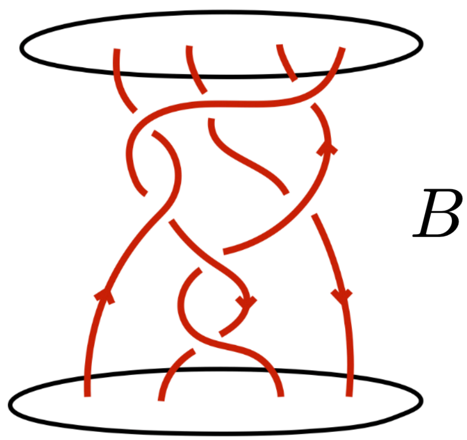
we get a colored braid in three dimensional space

$$\mathcal{A} \times [0,1]$$

This leads to a monodromy problem,

which is to analytically continue

the fundamental solution to the Knizhnik-Zamolodchikov equation



along the path described by the braid.

Monodromy along a path B depends only on its homotopy type,

so the resulting monodromy matrix

$$\mathfrak{B}$$

is an invariant of the colored braid.

The monodromy problem of the $\widehat{{}^L\mathfrak{g}}_\kappa$ Knizhnik-Zamolodchikov equation

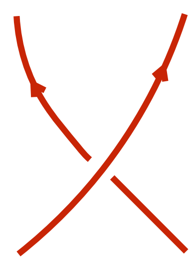
$$\kappa\, a_\ell \frac{\partial}{\partial a_\ell} \mathcal{V} = \sum_{j \neq \ell} r_{\ell i}(a_\ell / a_j)\, \mathcal{V}.$$

was solved by Tsuchia and Kanie in ’88 and by Drinfeld and Kohno in ’89.

They showed that monodromy matrix that reorders

a neighboring pair of vertex operators

$$\Phi_{V_j}(a_j) \otimes \Phi_{V_i}(a_i)$$



$$\Phi_{V_i}(a_i) \otimes \Phi_{V_j}(a_j)$$

is an R-matrix of the a quantum group

$$U_{\mathfrak{q}}({}^L\mathfrak{g})$$

corresponding to ${}^L\mathfrak{g}$

The quantum group symmetry

$$U_{\mathfrak{q}}({}^L\mathfrak{g})$$

is not manifest anywhere in our formulation of the problem,

starting with either conformal field theory,

or with Chern-Simons theory.

One discovers it only once one has solved the problem exactly.

This is a general feature of quantum symmetries.

Action by monodromies

turns the space of conformal blocks into a module for the

$$U_{\mathfrak{q}}({}^L\mathfrak{g})$$

quantum group in representation,

$$V = \bigotimes_{i=1}^{n} V_i$$

The representation V_i is viewed here as a representation of $U_{\mathfrak{q}}({}^L\mathfrak{g})$

and not of ${}^L\mathfrak{g}$, but we will denote by the same letter,

in particular since their dimensions are the same.

The monodromy action

is irreducible only in the subspace of

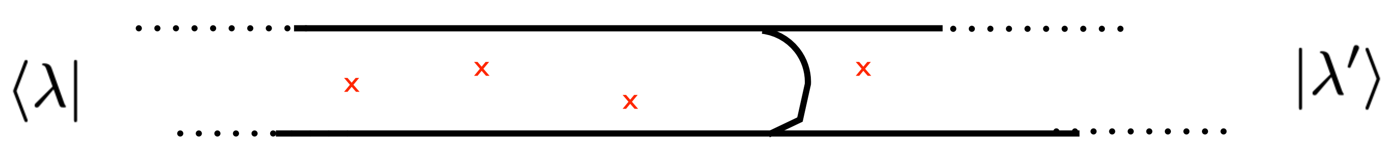
$$V = \bigotimes_{i=1}^{n} V_i$$

of fixed

$$\text{weight } \nu = \lambda - \lambda'$$

corresponding to conformal blocks of the form

$$\langle \lambda | \, \Phi_{V_1}(a_1) \cdots \Phi_{V_\ell}(a_\ell) \cdots \Phi_{V_n}(a_n) \, | \lambda' \rangle$$

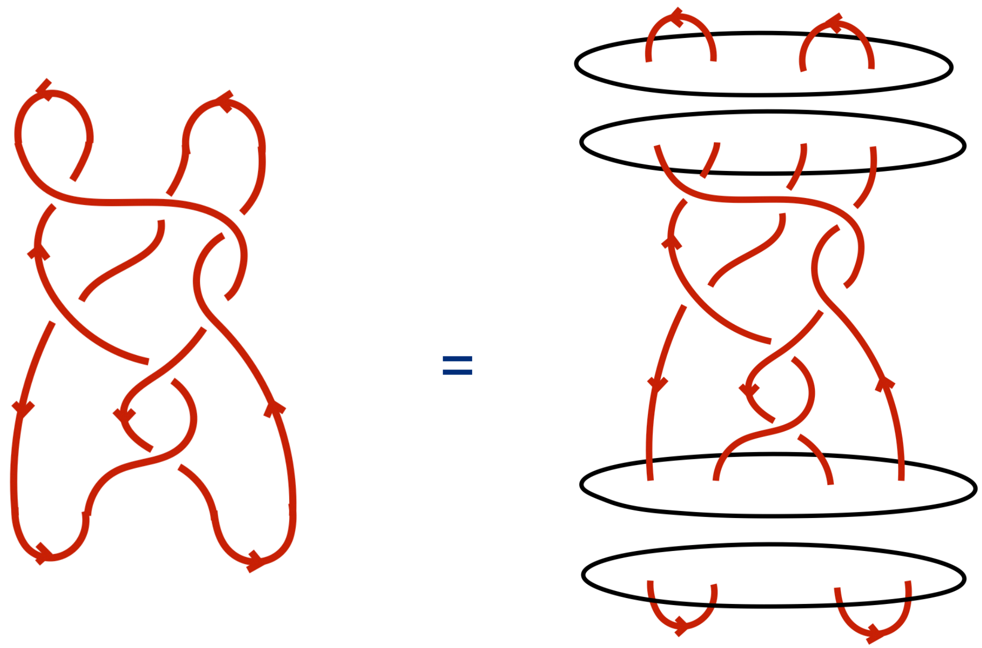


This perspective leads to

quantum invariants of not only braids

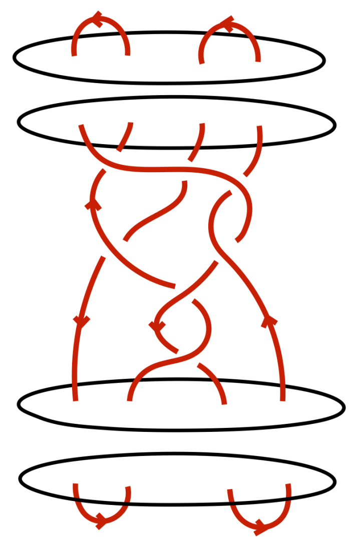
but knots and links as well.

Any link K can be represented as a



a closure of some braid.

The corresponding quantum link invariant is the matrix element



$$(\mathfrak{U}_1, \mathfrak{B}\,\mathfrak{U}_0)$$

of the braiding matrix,

taken between a pair of conformal blocks

which correspond to the top and the bottom of the picture.

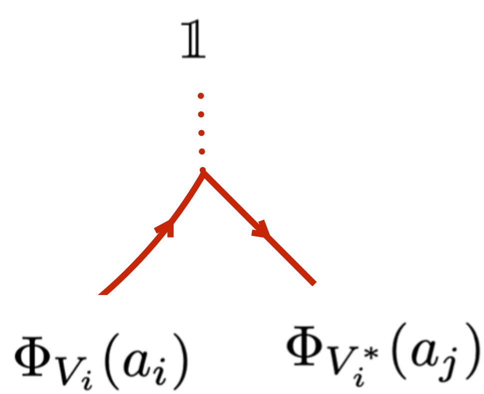
The conformal blocks

we need are very special solutions to KZ equations

which describe pairs of vertex operators,

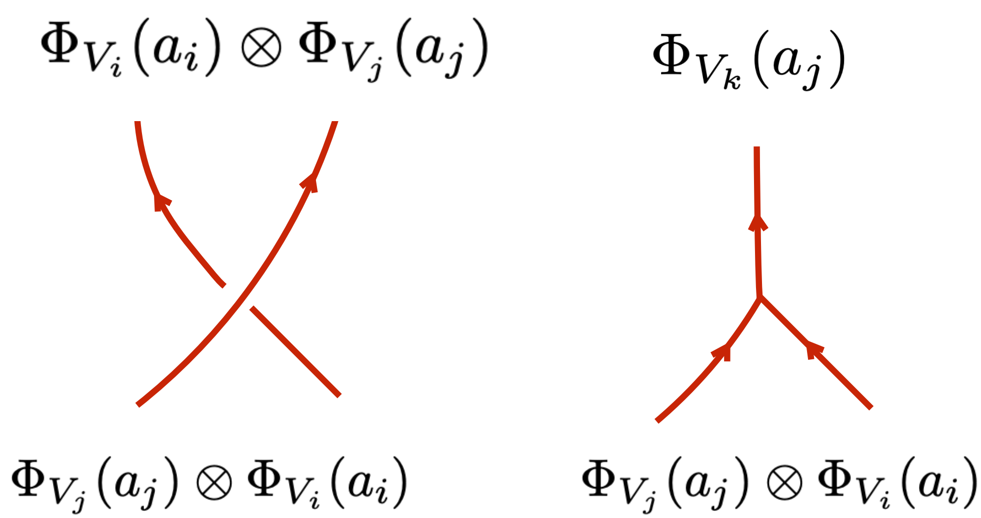
colored by complex conjugate representations

which come together and “fuse” to disappear.



This way,

both braiding and fusion of conformal field theory



play an important role in the story.

The starting point for us is

a geometric realization

Knizhnik-Zamolodchikov equation.

Chapter II

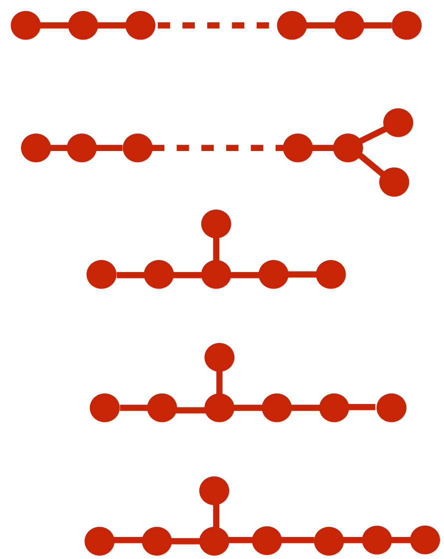
Conformal blocks from geometry

and a supersymmetric QFT

For the time being,

we will specialize ${}^L\mathfrak{g}$ to be a simply laced Lie algebra

so ${}^L\mathfrak{g} = \mathfrak{g}$ are one of the following types:



$\mathfrak{g} = A_n$

$\mathfrak{g} = D_n$

$\mathfrak{g} = E_6$

$\mathfrak{g} = E_7$

$\mathfrak{g} = E_8$

The generalization to non-simply laced Lie algebras

involves an extra step, which we will describe in the last lecture.

It turns out that Knizhnik-Zamolodchikov equation of

$$\widehat{{}^L\mathfrak{g}}$$

is the "quantum differential equation" of a

certain holomorphic symplectic manifold.

This result has been proven recently

by Ivan Danilenko, in his thesis.

Quantum differential equation of a Kahler manifold $\mathcal{X}$

is an equation for flat sections

$$a_i \frac{\partial}{\partial a_i} \mathcal{V} - C_i \star \mathcal{V} = 0$$

of a connection on a vector bundle

over its complexified Kahler moduli space,

with fibers $H^*(\mathcal{X})$.

The connection is defined in terms of "quantum multiplication" by divisors

$$C_i \in H^2(\mathcal{X})$$

Quantum multiplication on $H^*(\mathcal{X})$.

$$\langle \alpha \star \beta, \gamma \rangle = \sum_{d \geq 0, d \in H^2(\mathcal{X})} (\alpha, \beta, \gamma)_d \; a^d$$

is defined by Gromov-Witten theory,

or, the topological ``A-model" of

$\mathcal{X}$

The first, $d = 0$ term of the quantum multiplication

$$\langle \alpha \star \beta, \gamma \rangle = \sum_{d \geq 0, d \in H^2(\mathcal{X})} (\alpha, \beta, \gamma)_d \; a^d$$

is the classical product on $H^*(\mathcal{X})$:

$$(\alpha, \beta, \gamma)_0 = \int_{\mathcal{X}} \alpha \wedge \beta \wedge \gamma$$

subsequent $d > 0$ terms are quantum corrections.

Just as the Knizhnik-Zamolodchikov equation

is central for many questions in representation theory,

quantum differential equation

is central for many questions in

algebraic geometry and in mirror symmetry.

The story in these lectures follows from

the new connection between these two.

To get the quantum differential equation

$$a_i \frac{\partial}{\partial a_i} \mathcal{V} - C_i \star \mathcal{V} = 0$$

to coincide with the Knizhnik-Zamolodchikov equation

$$\kappa \, a_\ell \frac{\partial}{\partial a_\ell} \mathcal{V} = \sum_{j \neq \ell} r_{\ell i}(a_\ell / a_j) \, \mathcal{V}.$$

solved by conformal blocks of $\widehat{{}^L\mathfrak{g}_k}$,

$$\mathcal{V}(a_1, \ldots, a_\ell, \ldots, a_n) = \langle \lambda | \; \Phi_{V_1}(a_1) \cdots \Phi_{V_\ell}(a_\ell) \cdots \Phi_{V_n}(a_n) \; | \lambda' \rangle$$

one wants to take $\mathcal{X}$ to be a very special manifold.

The manifold

$$\mathcal{X}$$

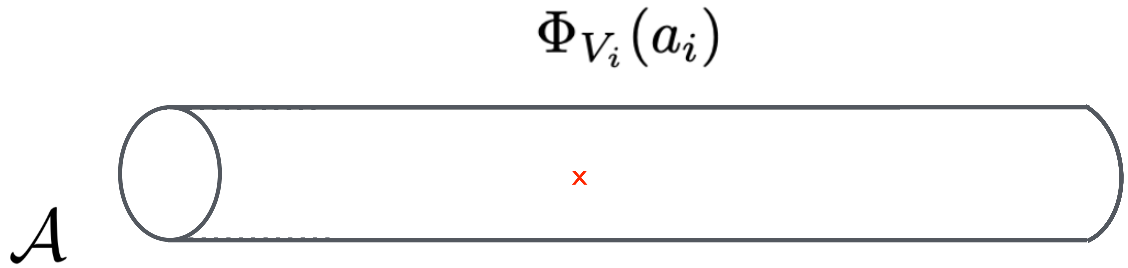
we need can be described as the moduli space of

singular G monopoles, with prescribed Dirac singularities, on

$$\mathbb{R}^3 = \mathbb{R} \times \mathbb{C}$$

where G is the Lie group of adjoint type with Lie algebra $\mathfrak{g}$

For every vertex operator



place a singular G monopole

at the origin of $\mathbb{C}$ and at the corresponding point on $\mathbb{R}$

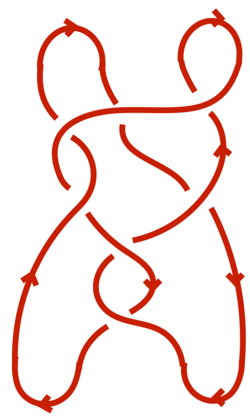
(obtained by forgetting the S^1 in $\mathcal{A}$).

The charge of a singular G monopole is identified via Langlands correspondence

with the highest weight μ_ℓ of an ${}^L G$ representation V_ℓ coloring a strand of a link.

In Chern-Simons theory,

view the knots in three dimensional space



as paths of heavy particles charged under

^{L}G

To obtain their homological link invariants,

we use a description

in which the same heavy particles

appear as Dirac monopoles of the Langlands dual group

G

In practice,

we loose nothing by taking the character lattice of

${}^L G$ to be as large as possible,

so it coincides with the weight lattice of ${}^L \mathfrak{g}$

Then, ${}^L G$ is simply connected, and of G is of adjoint type,

which is what we assumed.

For example, for ${}^L \mathfrak{g} = \mathfrak{su}_2$ we would take:

$${}^L G = SU(2) \quad \text{and} \quad G = SU(2)/\mathbb{Z}_2 = SO(3)$$

We will see in the last lecture

that the fact Langlands correspondence enters this story

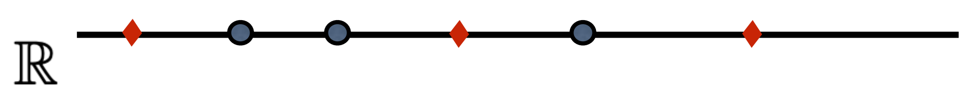
is not an accident.

The choice of weight ν of the subspace of representation

$$V = \bigotimes_{i=1}^{n} V_i$$

which the conformal blocks transform in

determines the total monopole charge,



including that of smooth monopoles.

$$\text{weight } \nu = \underbrace{\text{highest weight}}_{\text{singular}} \mu - \sum_{a=1}^{\mathrm{rk}} \underbrace{d_a \, {}^L e_a}_{\text{smooth}} \geq 0$$

The manifold

$$\mathcal{X}$$

is holomorphic symplectic, so it has hyper-Kahler structure.

A choice of complex structure on $\mathcal{X}$

splits $\mathbb{R}^3$ as

$$\mathbb{R}^3 = \mathbb{R} \times \mathbb{C}$$

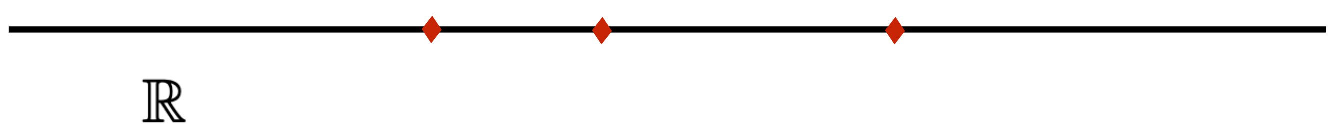
the positions of singular monopoles on $\mathbb{R}$

$\mathbb{R}$

are the real Kahler moduli of $\mathcal{X}$,

and their positions on $\mathbb{C}$ the complex structure moduli.

Since the position of monopoles on



lead to Kahler moduli of $\mathcal{X}$,

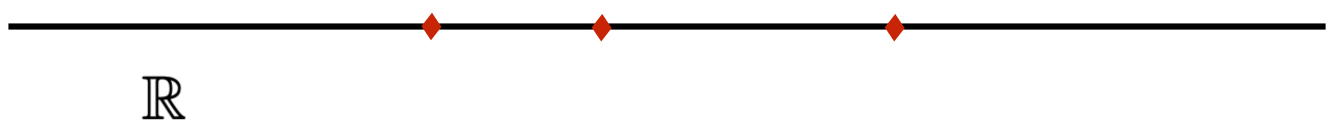
the choice of ordering of monopoles is a chamber in Kahler moduli,

We will record it in a vector

$$\vec{\mu} = (\mu_5, \mu_2, \mu_7, \ldots)$$

For $\mathcal{X}$ to be smooth, it is not enough

for singular monopoles to be at distinct points on



since we can always interpret the picture as

obtained by colliding a larger number of singular monopoles.

It turns out we need, in addition,

the charge μ_ℓ of every singular monopole in

$$\vec{\mu} = (\mu_5, \mu_2, \mu_7, \ldots)$$

to be the highest weight of a minuscule representation

V_ℓ of ${}^L G$.

If all the representations V_i are minuscule,

and the positions of singular monopoles on

$\mathbb{R}$

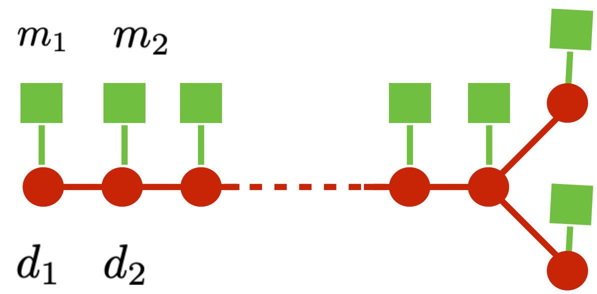
are generic, $\mathcal{X}$ is smooth.

To physicists,

$$\mathcal{X}$$

is the Coulomb branch of a certain three dimensional quiver gauge theory

$\mathfrak{g} = D_n$



where singular and smooth monopole charges

determine the ranks of gauge and flavor symmetry groups.

$$\text{highest weight } \mu = \sum_{i=1}^{n} \mu_i = \sum_{a=1}^{\text{rk}} m_a \, {}^L w_a$$

$$\text{weight } \nu = \text{ highest weight } \mu - \sum_{a=1}^{\text{rk}} d_a \, {}^L e_a \geq 0$$

Our manifold $\mathcal{X}$ has several other useful descriptions.

It is also known as a resolution of

$$\mathcal{X} = \mathrm{Gr}^{\vec{\mu}}{}_{\nu}$$

a certain slice in affine Grassmannian of G

$$\mathrm{Gr}_G = G((z))/G[[z]]$$

Here, the vector $\vec{\mu}$

encodes the singular monopole charges in order they appear

$\mathbb{R}$

and ν is the total monopole charge.

The description in terms of

$$\mathcal{X} = \mathrm{Gr}^{\vec{\mu}}{}_{\nu}$$

arises by thinking about singular G monopoles on

$$\mathbb{R} \times \mathbb{C}$$

as a sequence of Hecke modifications of holomorphic G-bundles on $\mathbb{C}$ parameterized by $\mathbb{R}$.

The loop variable z of the affine Grassmannian

$$\mathrm{Gr}_G = G((z))/G[[z]]$$

is the coordinate on $\mathbb{C}$

Since $\mathcal{X}$ is holomorphic symplectic,

its quantum cohomology is trivial,

unless we work equivariantly with respect to a torus action

that scales the holomorphic symplectic form

$$\omega^{2,0} \to \mathfrak{q}\,\omega^{2,0}$$

We chose all the singular monopoles to be at the origin of

$\mathbb{C}$ in $\mathbb{R} \times \mathbb{C}$

in order for this to be symmetry.

To get the quantum differential equation to coincide

with the Knizhnik-Zamolodchikov equation solved by

$$\mathcal{V}(a_1, \ldots, a_\ell, \ldots, a_n) = \langle\lambda|\, \Phi_{V_1}(a_1) \cdots \Phi_{V_\ell}(a_\ell) \cdots \Phi_{V_n}(a_n)\, |\lambda'\rangle$$

one needs to work equivariantly with respect to a larger torus of symmetries

$$\mathrm{T} = \Lambda \times \mathbb{C}^\times_{\mathbf{q}}$$

The symmetry corresponding to

$$\Lambda \subset \mathrm{T}$$

preserves the holomorphic symplectic form,

and comes from the maximal torus of G .

Its equivariant parameters determine the highest weight vector

of the Verma module $\langle\lambda|$

$$\mathcal{V}(a_1, \ldots, a_\ell, \ldots, a_n) = \langle\lambda|\ \Phi_{V_1}(a_1) \cdots \Phi_{V_\ell}(a_\ell) \cdots \Phi_{V_n}(a_n)\ |\lambda'\rangle$$

which is not fixed by the weight condition weight $\nu = \lambda - \lambda'$

All the ingredients in

$$\mathcal{V}(a_1, \ldots, a_m) = \langle\lambda|\, \Phi_{V_1}(a_1) \cdots \Phi_{V_m}(a_m)\, |\lambda'\rangle$$

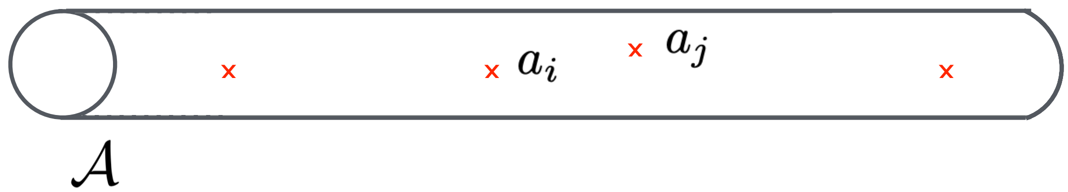
have a geometric interpretation in terms of $\mathcal{X}$,

starting with the (relative) positions of punctures on $\mathcal{A}$

$\mathcal{A}$ 

which are the complexified Kahler moduli of $\mathcal{X}$.

We took the Riemann surface $\mathcal{A}$ to be a cylinder rather than a plane,



because the B-fields

that pair with the real Kahler moduli to get the complex ones,

are periodic.

The fact that Knizhnik-Zamolodchikov equation solved by

$$\mathcal{V}(a_1,\ldots,a_\ell,\ldots,a_n) = \langle\lambda|\ \Phi_{V_1}(a_1)\cdots\Phi_{V_\ell}(a_\ell)\cdots\Phi_{V_n}(a_n)\ |\lambda'\rangle$$

has a geometric interpretation as the

quantum differential equation of

$$\mathcal{X}$$

computed by T- equivariant Gromov-Witten theory,

implies the conformal blocks too have a geometric interpretation.

Solutions of the quantum differential equation are

computed by T -equivariant Gromov-Witten theory of

$$\mathcal{X}$$

as Givental's J-functions, or "vertex functions"

They are equivariant counts of holomorphic maps of all degrees

from a domain curve D which is best thought of an infinite cigar

with an S^1 boundary at infinity.

This has more information than the conformal blocks themselves,

because underlying the Gromov-Witten theory

is a two-dimensional supersymmetric sigma model on

D

with $\mathcal{X}$ as a target space.

The geometric interpretation of conformal blocks of

$$\widehat{{}^L\mathfrak{g}}$$

in terms of

$$\mathcal{X}$$

has far more information than the conformal blocks themselves.

Underlying the Gromov-Witten theory of

$$\mathcal{X}$$

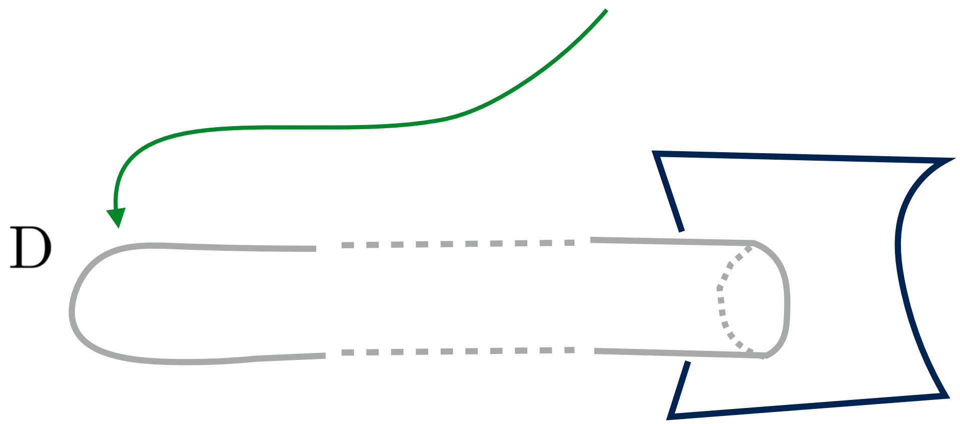
is a two-dimensional supersymmetric "sigma model"

with $\mathcal{X}$ as a target space.

The sigma model describes all maps

not only holomorphic ones.

In the interior of D, supersymmetry is preserved by an A-type topological twist,



so the partition function is computed by Gromov-Witten theory of $\mathcal{X}$.

The J-function is a vector

$$\mathcal{V}_\alpha[\mathcal{F}]$$

due to insertions of $\alpha \in \mathrm{H}^*_{\mathrm{T}}(\mathcal{X})$ classes at the origin of D.

Geometric Satake correspondence,

identifies $H^*_{\mathrm{T}}(\mathcal{X})$ with the weight ν subspace of

representation $\bigotimes_{i=1}^{n} V_i$ of ${}^L\mathfrak{g}$.

Crucially, to get the J-function,

at the S^1 boundary at infinity, one places a B-type boundary condition.

The infinite length of the cigar makes the A-type supersymmetry preserved by the interior compatible with any supersymmetry on the boundary, even of B-type.

This A/B type mix is characteristic of central charges of branes, as we will elaborate on momentarily.

Boundary conditions form a category, and

the category of boundary conditions of the sigma model on $\mathcal{X}$,

preserving a B-type supersymmetry and

working equivariantly with respect to T. is known as its

$$\mathscr{D}_{\mathcal{X}} = D^b Coh_{\mathrm{T}}(\mathcal{X})$$

the derived category of $\mathrm{T}-$ equivariant coherent sheaves.

Picking a B-type brane

$$\mathcal{F} \in \mathscr{D}_{\mathcal{X}}$$

as the boundary condition at infinity,

the supersymmetric partition function is the Givental's J-function,

$$\mathcal{V}[\mathcal{F}]$$

.

which depends on the brane only through its K-theory class

$$[\mathcal{F}] \in K_{\mathrm{T}}(\mathcal{X})$$

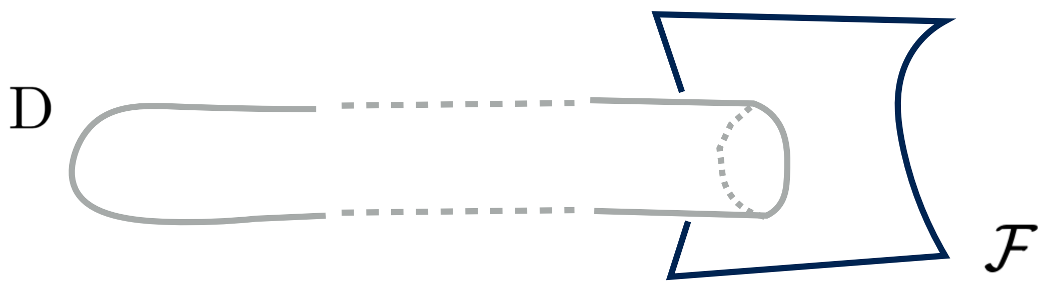
which is the charge of the brane.

While the partition function

$$\mathcal{V}[\mathcal{F}] = \mathsf{Vertex}[\mathcal{F}]$$

depends on the choice of the brane $\mathcal{F}$ only through its K-theory class,

$$[\mathcal{F}] \in K_{\mathrm{T}}(\mathcal{X})$$

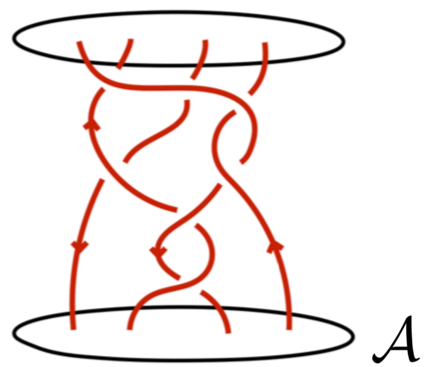


the underlying sigma model needs an actual object of the derived category

$$\mathcal{F} \in D^b Coh_{\mathrm{T}}(\mathcal{X})$$

to serve as the boundary condition.

A braid B has a geometric interpretation as a path in complexified Kahler moduli that avoids singularities,



since the complexified Kahler moduli of

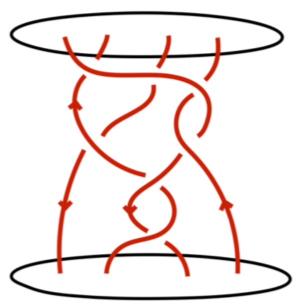
$$\mathcal{X} = \mathrm{Gr}^{\vec{\mu}}{}_{\nu}$$

are the relative positions of punctures on $\mathcal{A}$

Monodromy of the quantum differential equation of

$$\mathcal{X}$$

along the path in its Kahler moduli corresponding to the braid,

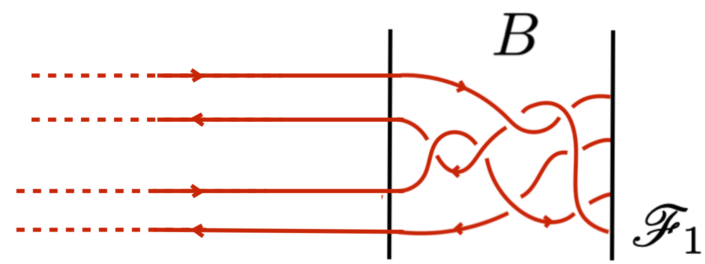


gives the geometric realization of the corresponding

$$U_{\mathfrak{q}}({}^L\mathfrak{g})$$

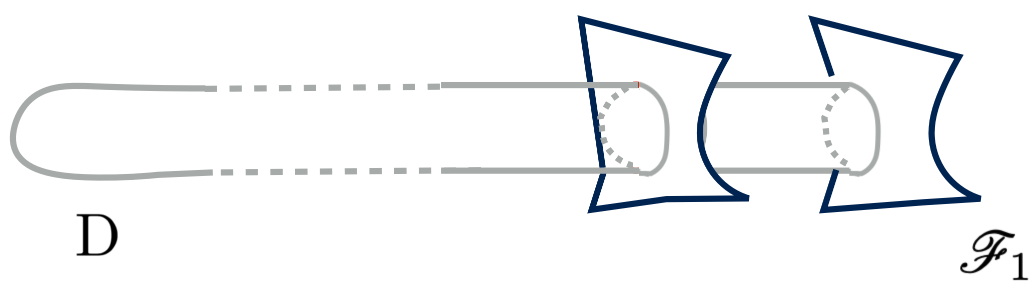
action on the space of $\widehat{{}^L\mathfrak{g}}$ conformal blocks

From the sigma model perspective, monodromy is realized



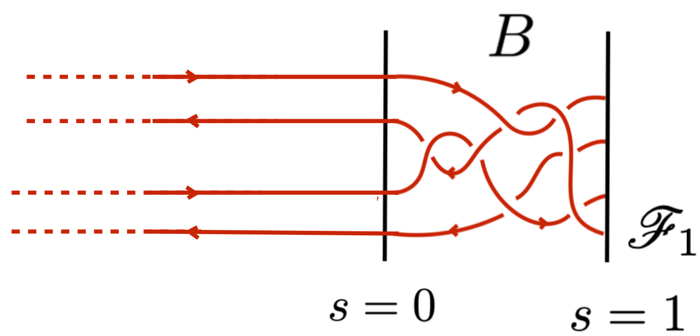
by letting the moduli of the theory vary according to the braid,

in the neighborhood of the boundary at infinity,



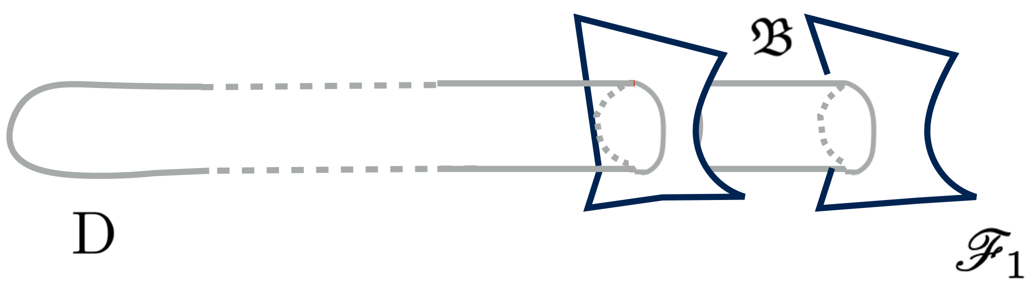
where the direction along the cigar coincides with the "time" along the braid.

By asking how monodromy



acts on the quantum state produced at $s = 0$

by the path integral over the cigar,



one gets a Berry phase type problem

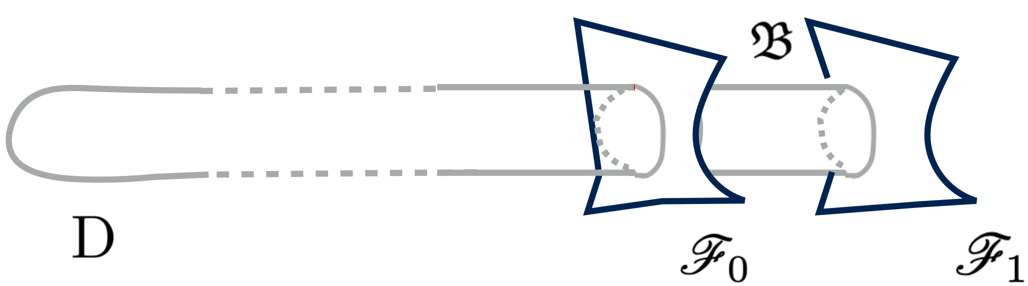
studied twenty years ago by Cecotti and Vafa, The solution of the problem is the linear map

$$\mathfrak{B} : K_{\mathrm{T}}(\mathcal{X}) \to K_{\mathrm{T}}(\mathcal{X})$$

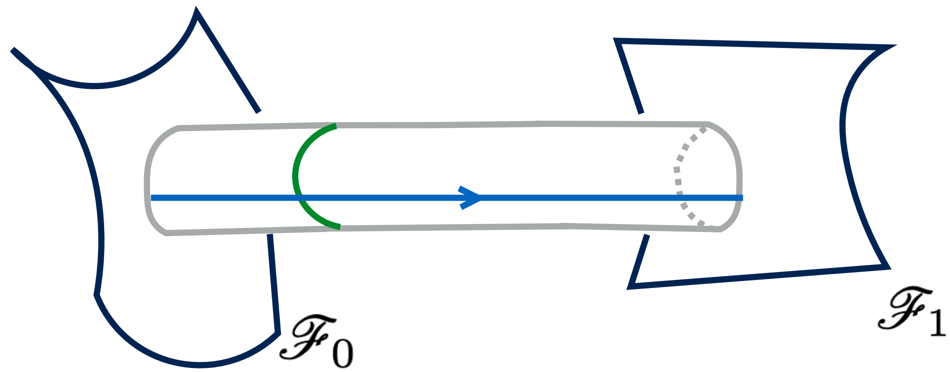
the monodromy of the quantum differential equation,

which acts on the K-theory class of the brane

$$[\mathcal{F}_0] \to \mathfrak{B}[\mathcal{F}_0]$$



It follows that the sigma model on the annulus



where time runs along the annulus

and moduli vary according to the braid,

computes the matrix element of the monodromy

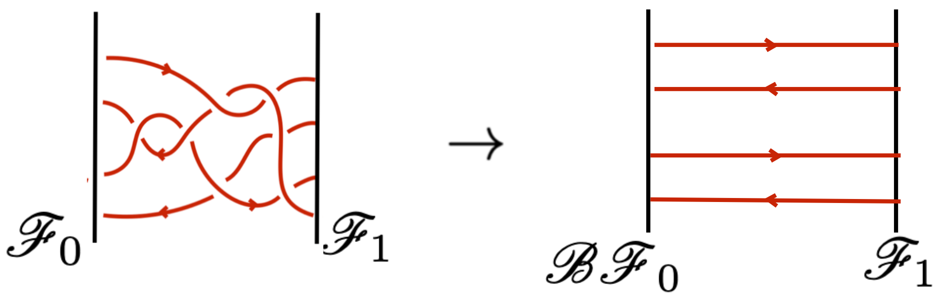
$\mathfrak{B}$

between pairs of conformal blocks picked out by the

B-branes at the two boundaries.

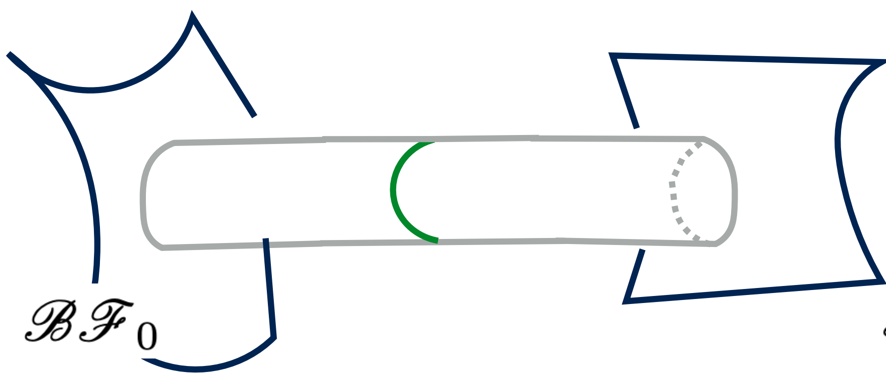
We can in fact take all the variation of the moduli

to happen near one of two boundaries,



at the expense of changing the boundary condition,

$$\mathscr{F}_0 \to \mathscr{B}\mathcal{F}_0$$



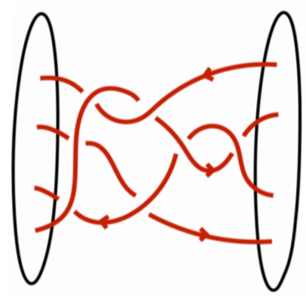
The braid group acts on branes

$$\mathcal{F} \to \mathscr{B}\mathcal{F}$$

by auto-equivalences $\mathscr{B}$ of the derived category,

$$\mathscr{B} : D^b Coh_{\mathrm{T}}(\mathcal{X}) \to D^b Coh_{\mathrm{T}}(\mathcal{X})$$

since along a path in Kahler moduli



B

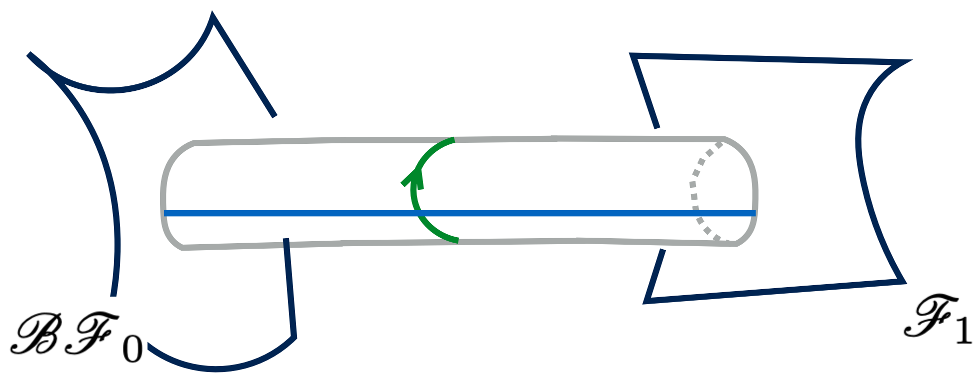
the category of B-type branes stays the same,

since the B-model does not depend on Kahler moduli.

The branes themselves are not invariant, as mirror symmetry will make manifest.

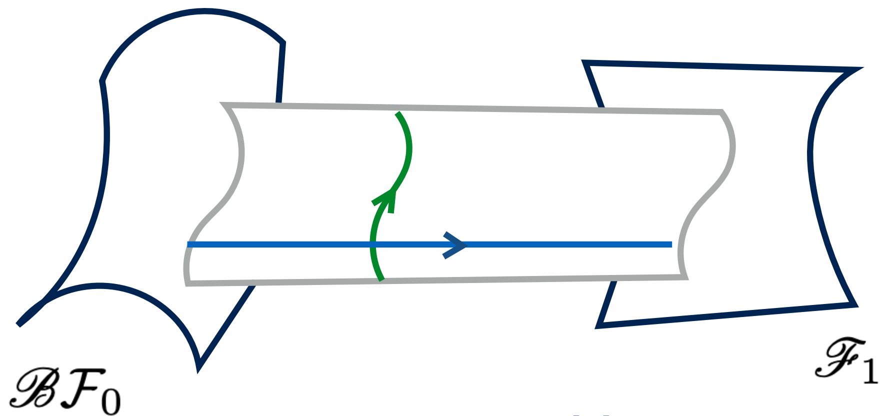
Sigma model on the same annulus

where we take the time that runs around the S^1



computes the index of the supercharge Q preserved by the two branes.

The cohomology of the supercharge Q



is computed by

$$\mathscr{D}_{\mathcal{X}} = D^b Coh_{\mathrm{T}}(\mathcal{X})$$

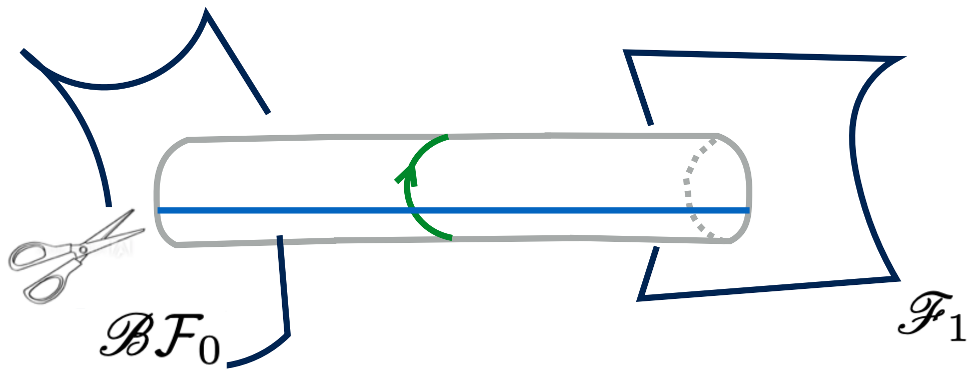
as its basic ingredient,

$$Hom^{*,*}_{\mathscr{D}_{\mathcal{X}}}(\mathscr{B}\mathcal{F}_0, \mathcal{F}_1)$$

the space of morphisms between a pair of branes.

The Euler characteristic of the homology theory

$$\chi(\mathscr{B}\mathcal{F}_0,\mathcal{F}_1) = \sum_{k\in\mathbb{Z},J\in\mathbb{Z}^{\mathrm{rkT}}} (-1)^k \mathfrak{q}^{J/2} \dim_{\mathbb{C}} Hom(\mathscr{B}\mathcal{F}_0,\mathcal{F}_1[k]\{J\})$$



manifestly computes the monodromy matrix element

$$\chi(\mathscr{B}\mathcal{F}_0,\mathcal{F}_1) = (\mathfrak{B}\mathcal{V}_0|\mathcal{V}_1)$$

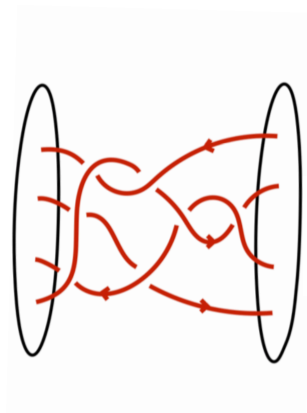
since we are free to think of either direction as time.

We have learned that derived equivalence

$$\mathscr{B} : D^b Coh_{\mathrm{T}}(\mathcal{X}) \to D^b Coh_{\mathrm{T}}(\mathcal{X})$$

categorifies

the monodromy matrix $\mathfrak{B}$ of the Knizhnik-Zamolodchikov equation.



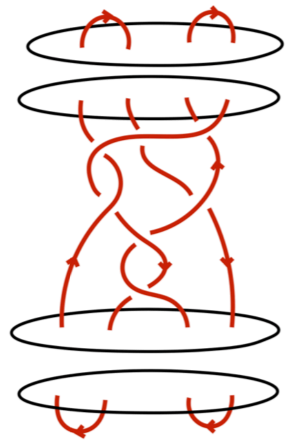
B

This explains a very difficult theorem of Bezrukavnikov and Okounkov,

which uses quantization of $\mathcal{X}$ in characteristic p.

The quantum invariants of links should be categorified by

$$\mathscr{D}_{\mathcal{X}} = D^b Coh_{\mathrm{T}}(\mathcal{X})$$



since they too can be expressed as matrix elements of the braiding matrix

$$(\mathfrak{U}_1, \mathfrak{B}\, \mathfrak{U}_0)$$

between pairs of conformal blocks.

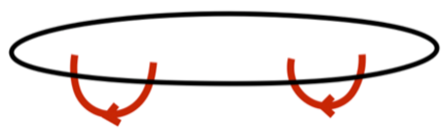
The first step is to find objects of

$$\mathscr{D}_{\mathcal{X}} = D^b Coh_{\mathrm{T}}(\mathcal{X})$$

whose vertex functions are conformal blocks

in which pairs of vertex operators fuse to trivial representation.

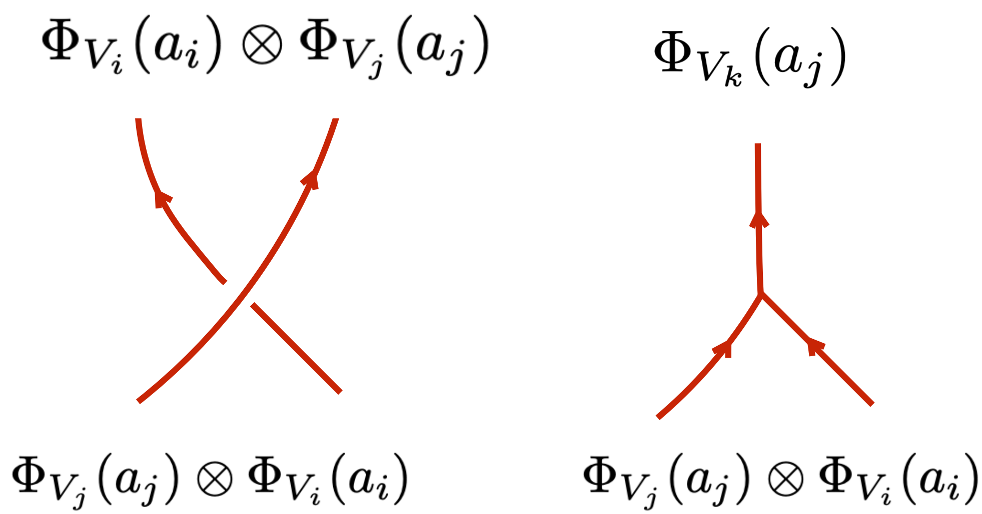
One way to characterize conformal blocks
where a pair of vertex operators come together



and fuse to a trivial representation,
is as specific eigenvectors of braiding.

This makes use of a classic result in conformal field theory, which is that

fusion diagonalizes braiding.



In looking for objects of

$$\mathscr{D}_{\mathcal{X}} = D^b Coh_{\mathrm{T}}(\mathcal{X})$$

whose vertex functions are conformal blocks corresponding to:

we will discover that not only braiding,

but also fusion has a geometric interpretation in terms of

$$\mathscr{D}_{\mathcal{X}} = D^b Coh_{\mathrm{T}}(\mathcal{X})$$