You have ~~30~~ 25 minutes to complete the quiz.

Solutions

1. (10 points) Diagonalize this matrix. i.e. Write matrices $P$, $D$ such that $A = PDP^{-1}$.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 & 1 \\ 6 & -1-\lambda & 0 \\ -1 & -2 & -1-\lambda \end{vmatrix} = 6(-1)^{2+1}\begin{vmatrix} 2 & 1 \\ -2 & -1-\lambda \end{vmatrix} + (-1-\lambda)(-1)^{2+2}\begin{vmatrix} 1-\lambda & 1 \\ -1 & -1-\lambda \end{vmatrix}$$

$$= -6\left[2(-1-\lambda)+2\right] + (-1-\lambda)\left[(1-\lambda)(-1-\lambda)+1\right]$$

$$= 12\lambda + (-1-\lambda)\lambda^2 = -\lambda^3 - \lambda^2 + 12\lambda = 0$$

$\lambda = 0$

$-\lambda^2 - \lambda + 12 = 0$

$\lambda^2 + \lambda - 12 = 0$

$(\lambda+4)(\lambda-3) = 0$

$\lambda = -4$, $\lambda = 3$

$Eig_{\lambda=0} = Nul(A)$ $\quad A \sim \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -13 & -6 \\ 0 & 0 & 0 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & \frac{1}{13} \\ 0 & -13 & -6 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad x_1 = -\frac{1}{13}x_3$, $x_2 = -\frac{6}{13}x_3$, $x_3$ free $\quad x = x_3\begin{bmatrix} -\frac{1}{13} \\ -\frac{6}{13} \\ 1 \end{bmatrix}$

So $\begin{bmatrix} -1 \\ -6 \\ 13 \end{bmatrix}$ is an eigenvector

$Eig_{\lambda=3} = Nul(A - 3I)$ $\quad A - 3I \sim \begin{bmatrix} -2 & 2 & 1 \\ 6 & -4 & 0 \\ -1 & -2 & -4 \end{bmatrix} \sim \begin{bmatrix} +1 & +2 & +4 \\ 0 & 2 & 3 \\ 0 & 6 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad x_1 = -x_3$, $x_2 = -\frac{3}{2}x_3$, $x_3$ free

$x = x_3\begin{bmatrix} -1 \\ -\frac{3}{2} \\ 1 \end{bmatrix}$

$\begin{bmatrix} -2 \\ -3 \\ 2 \end{bmatrix}$ is an eigenvector

$Eig_{\lambda=-4} = Nul(A+4I)$

$$\begin{bmatrix} 5 & 2 & 1 \\ 6 & 3 & 0 \\ -1 & -2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & -9 & 18 \\ 0 & -8 & 16 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

So $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ is an eigenvector

$x_3$ free

$x_2 = 2x_3$

$x_{\cancel{3}} = -x_3$

$$x = x_3 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -4 \end{bmatrix} \qquad P = \begin{bmatrix} -1 & -2 & -1 \\ -6 & -3 & 2 \\ 13 & 2 & 1 \end{bmatrix}$$