1) $\lim_{x\to 0^-} \sin\left(\frac{|\pi x|}{2x}\right) = 1.$

Given $\epsilon > 0$, there exists $\delta > 0$ such that $-\delta < x - a < 0$ implies that $|f(x) - L| < \epsilon$.

Despite having a sine in it, this problem is actually quite easy. The trick is to think about what the given function looks like for $x < 0$. Up to a change of sign, the x's will cancel out, and we'll just have a constant function! That means that any δ we pick will work (think about the picture of a $\delta - \epsilon$ proof for a constant function).

Proof. Fix $\epsilon > 0$. Let $\delta = 1$. Then

$$\begin{aligned} -\delta < x - 0 < 0 \\ -1 < x - 0 < 0 \\ -1 < x < 0. \end{aligned}$$

Then,

$$\begin{aligned} |x| &= -x \\ \frac{|\pi x|}{2x} &= -\frac{\pi}{2} \\ \sin\left(\frac{|\pi x|}{2x}\right) &= \sin\left(\frac{-\pi}{2}\right) = -1 \\ \sin\left(\frac{|\pi x|}{2x}\right) - (-1) &= 0 \\ \left|\sin\left(\frac{|\pi x|}{2x}\right) - (-1)\right| &< \epsilon \end{aligned}$$

□

2) $\lim_{x\to -\infty} 2^x = 0.$

Given $\epsilon > 0$, there exists N such that $x < N$ implies that $|x - a| < \epsilon$.

Let's try some computations.

$$\begin{aligned} |2^x - 0| &< \epsilon \\ 2^x &< \epsilon \\ \ln(2^x) &< \ln(\epsilon) \\ x\ln(2) &< \ln(\epsilon) \\ x\ln(2) &< \ln(\epsilon) \\ x &< \frac{\ln(\epsilon)}{\ln(2)} \end{aligned}$$

It looks like $N = \frac{\ln(\epsilon)}{\ln(2)}$ will do the trick. Let's try.

Proof. Fix $M > 0$. Let $N = \frac{\ln(\epsilon)}{\ln(2)}$. Then

$$\begin{aligned} x &< N \\ x &< \frac{\ln(\epsilon)}{\ln(2)} \\ x\ln(2) &< \ln(\epsilon) \\ \ln(2^x) &< \ln(\epsilon) \\ 2^x &< \epsilon \\ |2^x - 0| &< \epsilon \end{aligned}$$

Where the second to last step comes from exponentiating. □

You might be tempted to say "But Chris! We're dealing with x going towards negative infinity, and your value of N looks positive!" The answer is that my value of N *can* be negative, because natural log of small numbers returns arbitrarily small values.

3) $\lim_{x\to-\infty} \frac{x^4+1}{x^2} = \infty$.

Given $M > 0$, there exists N such that $x < N$ implies that $f(x) > M$.

I'd like to point out that $\frac{x^4+1}{x^2} = x^2 + \frac{1}{x^2}$ for all x on which the function is defined. So let's work with $x^2 + \frac{1}{x^2}$, instead.

Computation time. Remember that we are working *backwards*.

$$\begin{aligned} x^2 + \frac{1}{x^2} &> M \\ x^2 &> M \\ x &> \sqrt{M} \text{ or} \\ x &< -\sqrt{M} \end{aligned}$$

We need x to be less than N, so let's say $N = -\sqrt{M}$.

Proof. Given M greater than 0, let $N = -\sqrt{M}$. Then

$$\begin{aligned} x &< N \\ x &< -\sqrt{M} \\ x^2 &> M \\ x^2 + \frac{1}{x^2} &> M \end{aligned}$$

□