Longest Increasing Subsequences and Oscillating Tableaux

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Abstract

These are an accompanying set of (incomplete) notes to an expository talk on longest increasing subsequences and analogues to oscillating tableaux. We start with a quick primer on the RSK correspondence and Berele insertion. This will lead us into longest increasing subsequences of permutations and eventually analogous statistics on oscillating tableaux. Along the way we will discuss computational methods, explicit formulas, and asymptotics, with surprising ventures into other areas. Most of this has been taken straight from [1, 11]. Any and all errors are completely my own.

1 Insertion Algorithms

We begin with a little fun numerology:[1]

1. Let's count the number of SYT on all shapes $\lambda \vdash 1, 2, 3, 4$. Let's then sum the squares in each case. The results are shown in Table 1.

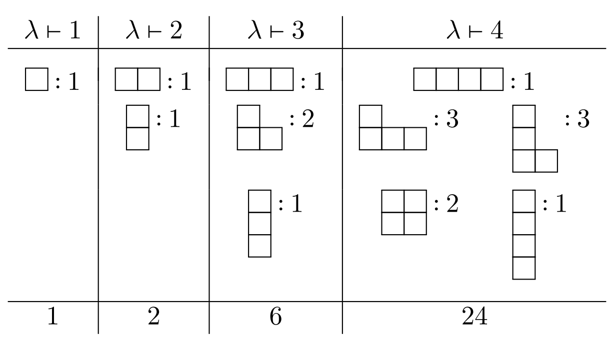


Table 1: Sums of squares of #SYT on shapes $\lambda \vdash 1, 2, 3, 4$.

[1] We refer the reader to [10] for the definitions and properties of standard Young tableaux (SYT) and semistandard Young tableaux (SSYT).

Note that summing the squares of each column gives us 1, 2, 6, 24, which we can recognize as 1!, 2!, 3!, 4!. We are then led to our first identity:

$$d! = \sum_{\lambda \vdash d} \left(f^{\lambda}\right)^2 \tag{1}$$

where f^{λ} denotes the number of SYT with shape λ.

2. Let's count the number of SSYT on all shapes $\lambda \vdash 3$ with entries $\{1\}, \{1,2\}, \{1,2,3\}$. Now, instead of summing the squares, let's take the sum of the entry-wise product between these values and the number of SYT on $\lambda \vdash 3$. The results are given in Table 2.

3. Let's again count the number of SSYT on all shapes $\lambda \vdash 3$ with entries $\{1\}, \{1,2\}, \{1,2,3\}$, but now look at the inner product between every pair of columns.

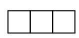
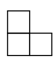
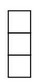
		#SSYT with entries		
$\lambda \vdash 3$	#SYT	$\{1\}$	$\{1,2\}$	$\{1,2,3\}$
	1	1	4	10
	2	0	2	8
	1	0	0	1
Inner products				
$1 = 1^3$	•	•		
$8 = 2^3$	•		•	
$27 = 3^3$	•			•
$1 = \binom{3}{3} = \left(\!\binom{1}{3}\!\right)$		••		
$4 = \binom{4}{3} = \left(\!\binom{2}{3}\!\right)$		•	•	
$10 = \binom{5}{3} = \left(\!\binom{3}{3}\!\right)$		•		•
$20 = \binom{6}{3} = \left(\!\binom{4}{3}\!\right)$			••	
$56 = \binom{8}{3} = \left(\!\binom{6}{3}\!\right)$			•	•
$165 = \binom{11}{3} = \left(\!\binom{9}{3}\!\right)$				••

Table 2: SSYT on shapes $\lambda \vdash 3$ with entries $\{1\}, \{1,2\}, \{1,2,3\}$. Inner products refers to choosing 2 columns, given by the bullets, and summing up the entry-wise products

After some careful inspection, we deduce the second and third identities

$$n^d = \sum_{\lambda \vdash d} d_\lambda(n) f^\lambda \tag{2}$$

$$\left(\!\binom{nm}{d}\!\right) = \sum_{\lambda \vdash d} d_\lambda(n) d_\lambda(m) \tag{3}$$

where $d_\lambda(n)$ denotes the number of SSYT of shape λ and entries in $[n]$.

The RSK correspondence gives combinatorial proofs of these 3 identities. Being a little cavalier with the details, it gives a bijection between

$$\text{pairs } (P,Q) \text{ where } \begin{cases} (1)\ P \in SYT, Q \in SYT \\ (2)\ P \in SSYT, Q \in SYT \\ (3)\ P \in SSYT, Q \in SSYT \end{cases} \quad \text{and} \quad \begin{cases} (1) \text{ permutations} \\ (2) \text{ words} \\ (3) \text{ 2-lined arrays} \end{cases}$$

For the versed reader, these identities have representation theoretic underpinnings and are known by other names: (2) is known as "Schur-Weyl duality" and (3) is known as "$GL_n - GL_m$ duality". (1) doesn't have a name persay, but follows from a general result in finite character theory. Their more general forms are

$$(x_1 + \cdots + x_n)^d = \sum_{\lambda \vdash d} s_\lambda(x_1, \ldots, x_n) f^\lambda$$

$$\prod_{i=1}^{n} \prod_{j=1}^{m} \frac{1}{1 - x_i y_j} = \sum_{\lambda} s_\lambda(x_1, \ldots, x_n) s_\lambda(y_1, \ldots, y_m)$$

We won't take the time here to go over RSK, but the interested reader should consult any of the excellent texts [9, 5, 10] for further study.

A goal of these notes is to come up with (bijective proofs) of identities $(\widetilde{1}), (\widetilde{2}), (\widetilde{3})$ for "type C", i.e. analogous identities involving oscillating tableaux and symplectic tableaux.

Let's recall these objects:

Definition 1.1. Let λ, μ be straight shapes. An *n-oscillating tableau* of shape λ/μ is a sequence

$$\mu = \nu^0, \nu^1, \nu^2, \ldots, \lambda$$

of partitions such that for each i,

(i) ν^i differs from ν^{i-1} by a single box.

(ii) $\ell(\nu^i) \le n$.

In the literature this is also known as an *n-symplectic up-down tableau.* When the length restriction is implicit or not imposed, we will drop the n and simply refer to this as an oscillating tableau or an up-down tableau.

Definition 1.2. A *symplectic tableau* T of shape λ is a filling of the Ferrers diagram of λ with the letters $1 < \overline{1} < 2 < \cdots < n < \overline{n}$ such that

(a) T is semistandard with respect to the above ordering

(b) The entries $\overline{i}$ must be in row $\leq i$.

Our first new identity can be proved via *Berele insertion.* The reader should consult [12] for another excellent exposition on Berele insertion. We will only state the result:

$$(x_1 + x_1^{-1} + \cdots + x_n + x_n^{-1})^d = \sum_{\lambda} sp_\lambda(x_1^{\pm 1}, \ldots, x_n^{\pm 1}) \widetilde{f_d^\lambda}(n) \tag{$\widetilde{2}$}$$

where sp_λ is the generating function for symplectic tableaux and $\widetilde{f_d^\lambda}(n)$ is the number of n-oscillating tableaux of shape λ with d steps.

2 Longest Increasing Subsequence

Given a permutation π in 1-line notation, an *increasing subsequence* $(i_1, \ldots, i_k)$ of π is a subsequence satisfying

$$i_1 < \ldots < i_k, \qquad \pi(i_1) < \ldots < \pi(i_k)$$

Define $\mathrm{is}(\pi)$ to be the length of the longest increasing subsequence of π.

The first question we can ask ourselves, as we do in all areas of our life: why? Why would one be interested in $\mathrm{is}(\pi)$? Why have we introduced $\mathrm{is}(\pi)$ after first discussing RSK? In an attempt to answer these questions, we consider the following scenarios:

Scenario 1: Imagine you have a standard 52 card deck that you would like to sort (or if you're teaching this semester, 52 midterms that need to be alphabetized.) What is the fastest way to do so?

We consider the following game, called *patience sorting*: Take a deck of cards labeled $1, \ldots, n$. The deck is shuffled and cards are turned over one at a time and dealt into piles on a table, according to the rule:

- A low card may be placed on a higher card (e.g. 2 may be placed on 7), or may be put into a new pile to the right of the existing piles.

The object of the game is to finish with as few piles as possible. To illustrate, suppose a shuffled deck of 10 cards is in the order

$$\pi \quad = \quad 7 \quad 2 \quad 8 \quad 1 \quad 3 \quad 4 \quad 10 \quad 6 \quad 9 \quad 5$$

If we play this game á la greedy strategy, the game unfolds as follows:

			1	1	1	1	1	1	1 **5**
	2	2	2	2 **3**	2 3	2 3	2 3 **6**	2 3 6	2 3 6
7	7	7 **8**	7 8	7 8	7 8 **4**	7 8 4 **10**	7 8 4 10	7 8 4 10 **9**	7 8 4 10 9

Note that to sort the cards, we simply take the lowest visible card, then the next lowest visible card, and so forth. Quoting [1], "Whether this is the fastest practical method for sorting real cards is an interesting topic for coffee-room conversation."

Observation 2.1. There are 5 piles at the end and $\mathrm{is}(\pi) = 5$.

As it turns out, this is no coincidence.

Lemma 2.1. *With deck π, patience sorting played with the greedy strategy ends with exactly $\mathrm{is}(\pi)$ piles. Furthermore, the game played with any legal strategy ends with at least $\mathrm{is}(\pi)$ piles, so that the greedy strategy is optimal.*

This game not only inspires the study of longest increasing subsequences, but as we see it also gives an algorithm for computing $\mathrm{is}(\pi)$.

Scenario 2: Now imagine you are playing bridge and you want to sort your hand of 13 cards. What is the minimum number of "sorts" needed to sort your hand?

Ulam thought hard about this and defined the following metric:

$$U(\pi_1, \pi_2) = \min\{d \mid \pi_1 g_1 g_2 \cdots g_d = \pi_2, \quad g_i \in G\}$$

where G consists of the generators $\{(i, i+1, \ldots, j)^{\pm 1} \mid 1 \le i < j \le n\}$. Ulam showed that

$$U(e, \pi) = n - \mathrm{is}(\pi)$$

Scenario 3: Let $U(k)$ denote the group of $k \times k$ unitary matrices. Recall that this is a compact group and so has an associated Haar measure dg. Given a function $f : U(k) \to \mathbb{C}$, we can then define the expected value of f with respect to this Haar measure:

$$E[f] = \int_{U(k)} f(g)\, dg$$

Diaconis, Shashahani first show that if $n \le k$, then

$$E[|\operatorname{Tr}^n|^2] = n!$$

where Tr is the usual trace of a matrix. Rains [7] extends this to the case $n > k$ and also to the other classical groups. For the unitary group, Rains shows that these higher moments of the trace of a random unitary matrix is related to longest increasing subsequences via

Proposition 2.1.

$$E[|\operatorname{Tr}^n|^2] = \#\{\pi \in S_n \mid \mathrm{is}(\pi) \le k\}$$

In other words, if we define the random variable $\mathrm{is}_n = \mathrm{is}(\pi_n)$ where π_n is chosen uniformly randomly from S_n, then

$$P(\mathrm{is}_n \le k) = \frac{1}{n!} \int_{U(k)} |\operatorname{tr} g|^{2n}\, dg$$

At first glance this seems like a rather surprising connection. However, this follows straightforwardly from (2), orthogonality of characters, and a property of RSK discussed later. Rains proves several other identities between combinatorial objects and various moments over the unitary group. For the symplectic and orthogonal groups, Rains shows

Theorem 2.1. *Let $\mathcal{I}_n^*$ denote the subset of fixed-point-free involutions of S_n. Let $O(k), Sp(2k)$ denote the group of linear transformations of $\mathbb{C}^k$ (resp. $\mathbb{C}^{2k}$) preserving a nondegenerate symmetric (resp. skew-symmetric) bilinear form. Then,*

$$E_{O(k)}[\mathrm{Tr}^n] = \#\{\pi \in \mathcal{I}_n^* \mid \mathrm{is}(\pi) \leq k\}$$
$$E_{Sp(2k)}[\mathrm{Tr}^n] = \#\{\pi \in \mathcal{I}_n^* \mid \mathrm{ds}(\pi) \leq 2k\}$$

We return now back to RSK to unveil its connection to longest increasing subsequences. Let's take

$$\pi \quad = \quad 7 \quad 2 \quad 8 \quad 1 \quad 3 \quad 4 \quad 10 \quad 6 \quad 9 \quad 5$$

and row insert. We get

$$\varnothing \leftarrow \pi = \begin{array}{|c|c|c|c|c|} \hline 7 & 8 \\ \cline{1-2} 2 & 6 & 10 \\ \cline{1-3} 1 & 3 & 4 & 5 & 9 \\ \hline \end{array}$$

Observation 2.2. The size of the first part of the tableau above is 5.

Again, this is no coincidence.

Theorem 2.2. *Let π be a permutation and let λ be the shape of the tableaux corresponding to π under the RSK correspondence. Then,*

$$\mathrm{is}(\pi) = \lambda_1$$

Moreover, if we define $\mathrm{is}(\pi, k)$ *to be the length of the largest union of k disjoint increasing subsequences, then*

$$\mathrm{is}(\pi, k) = \lambda_1 + \cdots + \lambda_k$$

Hence

Corollary 2.1. *Fix integers $j, k, n \geq 0$. Then,*

$$\#\{\pi \in S_n \mid \mathrm{is}(\pi) \leq k\} = \sum_{\substack{\lambda \vdash n \\ \lambda_1 \leq k}} (f^\lambda)^2$$
$$\#\{\pi \in S_n \mid \mathrm{is}(\pi) \leq k, \mathrm{ds}(\pi) \leq j\} = \sum_{\substack{\lambda \vdash n \\ \lambda_1 \leq k, \ell(\lambda) \leq j}} (f^\lambda)^2$$

This connection between $\mathrm{is}(\pi)$ and RSK allows us to prove a theorem dating all the way back to Erdös (who else?)

Corollary 2.2 (Erdös, Szekeres)**.** *Let* $\pi \in S_n$*. Write* $n = pq+1$ *for some* $p, q \geq 1$*. Then, either* $\mathrm{is}(\pi) > p$ *or* $\mathrm{ds}(\pi) > q$*.*

In fact, this is strongest possible: If $\pi \in S_{pq}$, then there are permutations with $\mathrm{is}(\pi) \leq p$ and $\mathrm{ds}(\pi) \leq q$. Indeed, it is not hard to see that

$$\#\{\pi \in S_{pq} \mid \mathrm{is}(\pi) = p, \mathrm{ds}(\pi) = q\} = (f^{(p^q)})^2$$

3 Asymptotics

We are interested in the "typical" values of $\mathrm{is}(\pi)$. More specifically, let's take a random permutation π of S_n, sampled uniformly, and consider the random variable $\mathrm{is}_n := \mathrm{is}(\pi)$. What are the expectation $E[\mathrm{is}_n]$ and standard deviation $\sigma(\mathrm{is}_n)$ for large n? What about the entire distribution?

We've seen that RSK gives us a formula for $P(\mathrm{is}_n = k)$, so that

$$\frac{1}{n!}\sum_{\pi \in S_n} \mathrm{is}(\pi) = E[\mathrm{is}_n] = \sum_k kP(\mathrm{is}_n = k) = \sum_k k\frac{1}{n!}\sum_{\substack{\lambda \vdash n \\ \lambda_1 = k}} (f^\lambda)^2 = \frac{1}{n!}\sum_{\lambda \vdash n} \lambda_1 \left(f^\lambda\right)^2$$

If we let $\lambda^{\max}$ denote the partition which maximizes f^λ, then we have the close approximation $E[\mathrm{is}_n] \approx \frac{1}{n!}\lambda_1^{\max}\left(f^{\lambda^{\max}}\right)^2$. In turn though, we can approximate $f^{\lambda^{\max}}$ via (1), which we recall says that

$$n! = \sum_{\lambda \vdash n} \left(f^\lambda\right)^2$$

We note that there are far fewer than $n!$ terms on the right hand side, so that $f^{\lambda^{\max}} \approx \sqrt{n!}$. Thus, we have the approximation

$$E[\mathrm{is}_n] \approx \frac{1}{n!}\lambda_1^{\max}\left(f^{\lambda^{\max}}\right)^2 \approx \lambda_1^{\max}$$

We normalize the partitions to have area one, so that each box has side length $1/\sqrt{n}$. As $n \to \infty$, one might expect the shape of this normalized $\lambda^{\max}$ to approach some limiting curve. Indeed, furthering this analysis, Vershik and Kerov [13] and Logan and Shepp [6] independently find this curve and show

Proposition 3.1. *As* $n \to \infty$*,* $E[\mathrm{is}_n] \sim 2\sqrt{n}$ *and* $\sigma(\mathrm{is}_n) = o(n^{1/6})$*.*

The curve for reference is $y = \Phi(x)$ given parametrically by

$$\begin{cases} x = y + 2\cos\theta \\ y = \frac{2}{\pi}(\sin\theta - \theta\cos\theta) \end{cases}$$

for $\theta \in [0, \pi]$, shown below in Figure 1. We note that they obtain this limiting curve by solving a variational problem involving the hook length formula. A different proof was given by Aldous, Diaconis known in the language of statistical

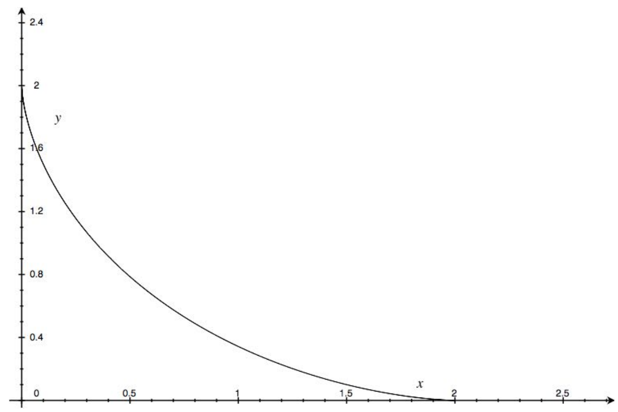


Figure 1: The limiting curve $\Phi(x)$ for the typical shape of a permutation under RSK.

physics as the *hydrodynamic limit* argument. Refinements given by Kerov and Borodin, Okounkov, Olshanski. A good survey can be found in Dan Romik's book [8].

The next breakthrough in these analyses came from Baik-Deift-Johansson [2] who gave the full distribution of is_n. To state their results, we define the *Tracy-Widom distribution* to be

$$F(t) = \exp\left(-\int_t^\infty (x-t)u(x)^2\,dx\right)$$

where $u(x)$ is a solution to a certain nonlinear second order differential equation known as the Painlevé II equation. Baik-Deift-Johansson prove the following result

Theorem 3.1. *We have for random (uniform) $\pi \in S_n$ and all $t \in \mathbb{R}$ that*

$$\lim_{n\to\infty} P\left(\frac{\text{is}_n - 2\sqrt{n}}{n^{1/6}} \le t\right) = F(t)$$

and give the corollary

Corollary 3.1.

$$E(\text{is}_n) = 2\sqrt{n} + \alpha n^{1/6} + o(n^{1/6})$$

where $\alpha = -1.7710868074\ldots$

You might wonder why this distribution does not share the names of the authors cited above. That's because the Tracy-Widom distribution has shown up before, surprisingly in the theory of random hermitian matrices. More specifically, let a random $n\times n$ hermitian matrix M have eigenvalues $\alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_n$. Then, Tracy and Widom showed that

$$\lim_{n\to\infty} P\left(\left(\alpha_1 - \sqrt{2n}\right)\sqrt{2}n^{1/6} \le t\right) = F(t)$$

Thus, as $n \to \infty$, is_n and α_1 have the same distribution, after scaling. Tracy and Widom also gave a generalization of this to the other eigenvalues α_k. Borodin, Okounkov, and Olshanski [3] later show that if $\lambda = sh(\pi)$, then as $n \to \infty$, λ_k and α_k have the same distribution. In other words, the difference $\mathrm{is}_n(\pi, k) - \mathrm{is}_n(\pi, k-1)$ and α_k have the same distribution.

Below is an excerpt taken from [11]:

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Okounkov provides a direct connection, via the topology of random
surfaces, between the two seemingly unrelated appearances of the
Tracy-Widom distribution in the theories of random matrices and
increasing subsequences. A very brief explanation of this connection
is the following: a surface can be described either by gluing together
polygons along their edges or by a ramified covering of a sphere. The
former description is related to random matrices via the theory of
quantum gravity, while the latter can be formulated in terms of the
combinatorics of permutations.
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Apparently the Tracy-Widom distribution is also related to the ASEP but I don't know anything about this...

4 Generalizations (to type C)

Those familiar with Lie theory might recognize GL_n and S_n as "type A" objects. Is there a theory of RSK and longest increasing subsequences to "other Lie types"?

For example, instead of considering longest increasing subsequences, we could look at *pattern avoidance*. Given $v = b_1 \cdots b_k \in S_k$, we say that a permutation $\pi = a_1 \cdots a_n \in S_n$ avoids v if it contains no subsequence $a_{i_1} \cdots a_{i_k}$ in the same relative order as v. Another way to say that $\mathrm{is}(\pi) < k$ is that it is $12\cdots k$-avoiding, and similarly $\mathrm{ds}(w) < k$ iff it is $k(k-1)\cdots 1$-avoiding. Pattern avoidance appears in Schubert calculus, among other areas, and extending these asymptotic results has been an object of study.

Another possible generalization is to consider longest increasing subsequences when restricted to subsets of the whole permutation group. For example, we could consider the subset consisting of involutions, i.e. $w \in S_n$ with $w^2 = 1$. We saw these objects show up in Rains' work on moments of the trace of matrices in the orthogonal or symplectic groups. Fortunately, the RSK correspondence can still give a formula for the number of involutions with longest increasing subsequence at most k, since

$$w \xleftrightarrow{RSK} (P, Q) \iff w^{-1} \xleftrightarrow{RSK} (Q, P)$$

Thus, involutions correspond to pairs of standard Young tableau (P, Q) with $P = Q$, and hence if we let $\mathcal{I}_n$ denote the subgroup of involutions in S_n, then

$$\#\{w \in \mathcal{I}_n \mid \mathrm{is}(w) \le k\} = \sum_{\substack{\lambda \vdash n \\ \lambda_1 \le k}} f^\lambda$$

Note that since $f^\lambda = f^{\lambda'}$ (as the transpose of a SYT is still a SYT), we have

$$\#\{w \in \mathcal{I}_n \mid \mathrm{ds}(w) \le k\} = \sum_{\substack{\lambda \vdash n \\ \lambda_1 \le k}} f^{\lambda'} = \sum_{\substack{\lambda \vdash n \\ \lambda_1 \le k}} f^{\lambda} = \#\{w \in \mathcal{I}_n \mid \mathrm{is}(w) \le k\}$$

For the subset $\mathcal{I}_n^*$ of fixed-point-free involutions, again the RSK correspondence gives us a counting tool via the property

$$w \xleftrightarrow{RSK} (P,P) \implies \#\{\text{fixed points of } w\} = \#\{\text{columns of } P \text{ of odd length}\}$$

Hence,

$$\#\{w \in \mathcal{I}_n^* \mid \mathrm{is}(w) \le k\} = \sum_{\substack{\lambda \vdash n \\ \lambda_1' \le k}} \left(f^{2\lambda'}\right)^2 \qquad \#\{w \in \mathcal{I}_n^* \mid \mathrm{ds}(w) \le 2k\} = \sum_{\substack{\lambda \vdash n \\ \lambda_1 \le k}} \left(f^{2\lambda'}\right)^2$$

where $2\lambda' = (2\lambda_1', 2\lambda_2', \ldots)$, the general partition with no columns of odd length.

Remark 4.1. Note that for the entire class of permutations, there is a symmetry between $\mathrm{is}(w)$ and $\mathrm{ds}(w)$, since reversing the permutation exchanges increasing subsequences with decreasing subsequences and also has a simple image under RSK. As noted above, we again have a symmetry between $\mathrm{is}(w)$ and $\mathrm{ds}(w)$ for the subset of involutions. However, this symmetry is broken for fixed-point free involutions; after all, for a fixed-point free involution, $\mathrm{ds}(w)$ can only be even, whereas $\mathrm{is}(w)$ can have any parity.

Surprisingly enough, in trying to fix this broken symmetry, fixed-point free involutions will be our way to generalizing to type C combinatorics. We first need to take a detour through (complete) *matchings*. A matching $M = \{B_1, \ldots, B_n\}$ of $[2n]$ is a partition of $[2n]$ into 2-element subsets. It can be visualized as a diagram of arcs, see Figure 2.

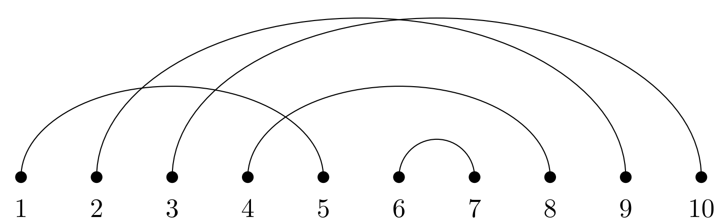

Figure 2: The matching $M = \{\{1,5\},\{2,9\},\{3,10\},\{4,8\},\{6,7\}\}$ on $[10]$

Note that fixed point free involutions in S_{2n} are in 1-1 correspondence with complete matchings on $[2n]$. Let $\mathcal{M}_n$ denote the set of matchings on $[2n]$ and let w_M denote the fixed-point free involution corresponding to M. We introduce two statistics on $\mathcal{M}_n$ that will replace $\mathrm{is}(w)$ and $\mathrm{ds}(w)$:

Definition 4.1. Let $M \in \mathcal{M}_n$. A *crossing* M consists of two arcs $\{i,j\}$ and $\{k,l\}$ with $i < k < j < l$. A *nesting* of M consists of two arcs $\{i,j\}$ and $\{k,l\}$

with $i < k < l < j$. The maximum number of mutually crossing arcs of M is called the crossing number of M, denoted $\mathrm{cr}(M)$. Similarly the nesting number $\mathrm{ne}(M)$ is the maximum number of mutually nesting arcs.

For the matching M of Figure 2, we have $\mathrm{cr}(M) = 3$ (corresponding to the arcs $\{1,5\},\{2,9\}$ and $\{3,10\}$), while also $\mathrm{ne}(M) = 3$ (corresponding to $\{2,9\},\{4,8\}$, and $\{6,7\}$).

It is easy to see that $\mathrm{ds}(w_M) = 2\,\mathrm{ne}(M)$. However, it is not so clear whether $\mathrm{cr}(M)$ is connected with increasing/decreasing subsequences. To this end, define

$$f_n(i,j) = \#\{M \in \mathcal{M}_n \mid \mathrm{cr}(M) = i, \mathrm{ne}(M) = j\}$$

It is well-known that

$$\sum_j f_n(0,j) = \sum_i f_n(i,0) = C_n$$

the n^{th} Catalan number. See [10, Exer. 6.19(n,o)]. The following generalization was given by Chen et al [4].

Theorem 4.1. *For all i,j,n we have $f_n(i,j) = f_n(j,i)$.*

To prove this, we first prove the following theorem

Theorem 4.2. *There is a bijection $\Phi : \mathcal{M}_n \to \widetilde{F}^{2n}_{\varnothing}$ from the set of matchings on $[2n]$ to the set of oscillating tableaux of shape $\varnothing$ and $[2n]$ steps.*

Before we prove this, we note the following consequence

Corollary 4.1.
$$\widetilde{f}^{2n}_{\varnothing} = (2n-1)!!$$

as the right hand side above counts the number of matchings on $[2n]$.

Proof of Thm 4.2. Given a matching $M \in \mathcal{M}_n$, define an oscillating tableau $\Phi(M) = (\varnothing, \lambda^1, \dots, \lambda^{2n} = \varnothing)$ as follows: Label the right endpoints of M with the numbers $\{\overline{1}, \dots, \overline{n}\}$ in order from right to left. Label each left endpoint with the unbarred number of its corresponding right endpoint. Let $a_1, \dots, a_{2n}$ be the labels of the endpoints from left to right.

We recursively construct a sequence of tableaux $\varnothing = T^0, T^1, \dots, T^{2n} = \varnothing$ for which we define $\lambda^i := sh(T^i)$. Once T^{i-1} has been obtained, let T^i be the tableau obtained by either

- Row bumping a_i into T^{i-1} if a_i is unbarred.
- Deleting the entry a_i from T^{i-1} and performing jeu-de-taquin at the hole if a_i is barred.

It is easy to see that $\Phi(M)$ is an oscillating tableau of shape $\varnothing$ and $[2n]$ steps. We leave it up to the reader to construct the inverse map. $\square$

To prove Theorem 4.1, we simply note the following

Proposition 4.1. *Let $M \in \mathcal{M}_n$ and let $\Phi(M) = (\lambda^0, \dots, \lambda^{2n})$. Then* $\mathrm{ne}(M)$ *equals the most number of columns of any λ^i and* $\mathrm{cr}(M)$ *equals the most number of rows of any λ^i.*

Composing conjugation with the bijection Φ gives a desired bijection on $\mathcal{M}_n$ that interchanges crossings with nestings, and hence is an analogue to reversing the permutation. We remark that if $M \in \mathcal{M}_n$ maps to $\Phi(M)$, and M' is the matching that corresponds to $\Phi(M)'$, it is not easy to describe a direct map from M to M'.

Via this bijection between oscillating tableaux and matchings, we can also derive a generalization of the hook length formula for oscillating tableaux of shape λ and $[2n]$ steps. Namely, any oscillating tableau $(\varnothing, \lambda^1, \dots, \lambda^{2n} = \varnothing)$ can be thought of as a pair of oscillating tableaux of shape $\lambda = \lambda^n$, the first being $(\varnothing, \lambda^1, \dots, \lambda^n)$, the second being $(\lambda^{2n}, \dots, \lambda^n)$. Thus, we get

$$\sum_{\lambda} \left(\widetilde{f_n^\lambda}\right)^2 = (2n-1)!! \tag{$\widetilde{1}$}$$

Just as in type A, this suggests a lurking representation theory. In fact, there is a $\mathbb{C}$-algebra $\mathcal{B}_n(x)$, where x is a real number, and which is semisimple for all but finitely many x (and such that these exceptional x are all integers), called the *Brauer algebra*. The Brauer algebra has a basis that is indexed in a natural way by matchings $M \in \mathcal{M}_n$, so that $\dim \mathcal{B}_n(x) = (2n-1)!!$. Just as S_n appears in Schur-Weyl duality, Brauer showed that $\mathcal{B}_n(x)$ is the centralizer algebra of the action of the orthogonal group $O(V)$ on $V^{\otimes n}$ (for $x = k$) as well as the action of $Sp(V)$ on $V^{\otimes n}$ (for $x = -2k$). When $\mathcal{B}_n(x)$ is semisimple, its irreducible representations have dimension $\widetilde{f_n^\lambda}$, so we obtain a representation-theoretic explanation of equation ($\widetilde{2}$).

5 Further Work

- We have the analogues ($\widetilde{1}$), ($\widetilde{2}$). What about ($\widetilde{3}$)? As it turns out, we do have an analogue:

$$\prod_{i=1}^{n}\prod_{j=1}^{m}(x_i + x_i^{-1} + y_j + y_j^{-1}) = \sum_{\lambda \subseteq (m^n)} sp_\lambda(x_1^{\pm 1}, \dots, x_n^{\pm 1}) sp_{(\lambda')^c}(y_1^{\pm 1}, \dots, y_m^{\pm 1}) \tag{$\widetilde{3}$}$$

- Type B: orthogonal tableaux
- Set partitions: vacillating tableaux and partition algebra.
- Skew symplectic tableaux identities
- q-analogues?

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