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Diagonally and antidiagonally symmetric alternating sign matrices of odd order

Roger E. Behrend^{a,*}, Ilse Fischer^b, Matjaž Konvalinka^c

^a School of Mathematics, Cardiff University, Cardiff, CF24 4AG, UK

^b Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, 1090 Wien, Austria

^c Fakulteta za matematiko in fiziko, Univerza v Ljubljani in Inštitut za matematiko, fiziko in mehaniko, Jadranska 19, Ljubljana, Slovenia

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ABSTRACT

We study the enumeration of diagonally and antidiagonally symmetric alternating sign matrices (DASASMs) of fixed odd order by introducing a case of the six-vertex model whose configurations are in bijection with such matrices. The model involves a grid graph on a triangle, with bulk and boundary weights which satisfy the Yang–Baxter and reflection equations. We obtain a general expression for the partition function of this model as a sum of two determinantal terms, and show that at a certain point each of these terms reduces to a Schur function. We are then able to prove a conjecture of Robbins from the mid 1980's that the total number of $(2n+1) \times (2n+1)$ DASASMs is $\prod_{i=0}^n \frac{(3i)!}{(n+i)!}$, and a conjecture of Stroganov from 2008 that the ratio between the numbers of $(2n+1) \times (2n+1)$ DASASMs with central entry -1 and 1 is $n/(n+1)$. Among the several product formulae for the enumeration of symmetric alternating sign matrices which were conjectured in the 1980's, that for odd-order DASASMs is the last to have been proved.

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* Corresponding author.

E-mail addresses: behrendr@cardiff.ac.uk (R.E. Behrend), ilse.fischer@univie.ac.at (I. Fischer), matjaz.konvalinka@mf.uni-lj.si (M. Konvalinka).

1. Preliminaries

1.1. Introduction

An alternating sign matrix (ASM) is a square matrix in which each entry is 0, 1 or -1 , and along each row and column the nonzero entries alternate in sign and have a sum of 1. These matrices were introduced by Mills, Robbins and Rumsey in the early 1980s, accompanied by various conjectures concerning their enumeration [33, Conjs. 1 & 2], [34, Conjs. 1–7]. Shortly after this, as discussed by Robbins [42, p. 18], [43, p. 2], Richard Stanley made the important suggestion of systematically studying the enumeration of ASMs invariant under the action of subgroups of the symmetry group of a square. This suggestion led to numerous conjectures for the straight and weighted enumeration of such symmetric ASMs, with these conjectures being summarized by Robbins in a preprint written in the mid 1980s, and placed on the arXiv in 2000 [43]. Much of the content of this preprint also appeared in review papers in 1986 and 1991 by Stanley [46] and Robbins [42], and in 1999 in a book by Bressoud [13, pp. 201–202]. In the preprint, simple product formulae (or, more specifically, recursion relations which lead to such formulae) were conjectured for the straight enumeration of several symmetry classes of ASMs, and it was suggested that no such product formulae exist for the other nonempty classes. All except one of these conjectured product formulae had been proved by 2006. (See Section 1.2 for further details.) The single remaining case was that the number of $(2n + 1) \times (2n + 1)$ diagonally and antidiagonally symmetric ASMs (DASASMs) is $\prod_{i=0}^n \frac{(3i)!}{(n+i)!}$, and a primary purpose of this paper is to provide the first proof of this formula. In so doing, a further open conjecture of Stroganov [50, Conj. 2], that the ratio between the numbers of $(2n + 1) \times (2n + 1)$ DASASMs with central entry -1 and 1 is $n/(n + 1)$, will also be proved.

These results will be proved using a method involving the statistical mechanical six-vertex model. The structure of the proofs can be summarized as follows. First, a new case of the six-vertex model is introduced in Sections 1.6–1.7. The configurations of this case of the model are in bijection with DASASMs of fixed odd order, and consist of certain orientations of the edges of a grid graph on a triangle, (10). The associated partition function (17) consists of a sum, over all such configurations, of products of certain parameterized bulk and boundary weights, as given in Table 2, with these weights satisfying the Yang–Baxter and reflection equations, (47) and (48). Straight sums over all configurations, or over all configurations which correspond to DASASMs with a fixed central entry, are obtained for certain specializations, (21), (22) and (24), of the parameters in the partition function. By identifying particular properties which uniquely characterize the partition function (including symmetry with respect to certain parameters, and reduction to a lower order partition function at certain values of some of the parameters), it is shown in Section 3.5 that the partition function can be expressed as a sum of two determinantal terms, as given in Theorem 1. It is also shown, using a general determinantal identity (57), that at a certain point, each of these terms reduces, up to simple

factors, to a Schur function, as given in [Theorem 3](#). Finally, the main results for the enumeration of odd-order DASASMs, [Corollaries 5 and 9](#), are obtained by appropriately specializing the variables in the Schur functions, and using standard results for these functions and for numbers of semistandard Young tableaux.

The proofs given in this paper of enumeration results for odd-order DASASMs share several features with known proofs of enumeration formulae for other symmetry classes of ASMs, such as those of Kuperberg [[31,32](#)], Okada [[37](#)], and Razumov and Stroganov [[39,40](#)]. However, the proofs of this paper also contain various new and distinguishing characteristics, which will be outlined in [Section 1.3](#).

1.2. Symmetry classes of ASMs

Several enumerative aspects of standard symmetry classes, and some related classes, of ASMs will now be discussed in more detail.

The symmetry group of a square is the dihedral group $D_4 = \{\mathcal{I}, \mathcal{V}, \mathcal{H}, \mathcal{D}, \mathcal{A}, \mathcal{R}_{\pi/2}, \mathcal{R}_\pi, \mathcal{R}_{-\pi/2}\}$, where \mathcal{I} is the identity, \mathcal{V} , \mathcal{H} , \mathcal{D} and \mathcal{A} are reflections in vertical, horizontal, diagonal and antidiagonal axes, respectively, and \mathcal{R}_θ is counterclockwise rotation by θ . The group has a natural action on the set $\text{ASM}(n)$ of $n \times n$ ASMs, in which $(\mathcal{I}A)_{ij} = A_{ij}$, $(\mathcal{V}A)_{ij} = A_{i,n+1-j}$, $(\mathcal{H}A)_{ij} = A_{n+1-i,j}$, $(\mathcal{D}A)_{ij} = A_{ji}$, $(\mathcal{A}A)_{ij} = A_{n+1-j,n+1-i}$, $(\mathcal{R}_{\pi/2}A)_{ij} = A_{j,n+1-i}$, $(\mathcal{R}_\pi A)_{ij} = A_{n+1-i,n+1-j}$ and $(\mathcal{R}_{-\pi/2}A)_{ij} = A_{n+1-j,i}$, for each $A \in \text{ASM}(n)$. The group has ten subgroups: $\{\mathcal{I}\}$, $\{\mathcal{I}, \mathcal{V}\} \approx \{\mathcal{I}, \mathcal{H}\}$, $\{\mathcal{I}, \mathcal{V}, \mathcal{H}, \mathcal{R}_\pi\}$, $\{\mathcal{I}, \mathcal{D}\} \approx \{\mathcal{I}, \mathcal{A}\}$, $\{\mathcal{I}, \mathcal{D}, \mathcal{A}, \mathcal{R}_\pi\}$, $\{\mathcal{I}, \mathcal{R}_\pi\}$, $\{\mathcal{I}, \mathcal{R}_{\pi/2}, \mathcal{R}_\pi, \mathcal{R}_{-\pi/2}\}$ and D_4 , where \approx denotes conjugacy. In studying the enumeration of symmetric ASMs, the primary task is to obtain formulae for the cardinalities of each set $\text{ASM}(n, H)$ of $n \times n$ ASMs invariant under the action of subgroup H . Since this cardinality is the same for conjugate subgroups, there are eight inequivalent classes. The standard choices and names for these classes, together with information about empty subclasses, conjectures and proofs of straight enumeration formulae, and numerical data are as follows.

- $\text{ASM}(n) = \text{ASM}(n, \{\mathcal{I}\})$. Unrestricted ASMs. The formula $|\text{ASM}(n)| = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$ was conjectured by Mills, Robbins and Rumsey [[33, Conj. 1](#)], and first proved by Zeilberger [[53, p. 5](#)], with further proofs, involving different methods, subsequently being obtained by Kuperberg [[31](#)] and Fischer [[25](#)].
- $\text{ASM}(n, \{\mathcal{I}, \mathcal{V}\})$. Vertically symmetric ASMs (VSASMs). For n even, the set is empty. For n odd, a formula was conjectured by Robbins [[43, Sec. 4.2](#)], and proved by Kuperberg [[32, Thm. 2](#)].
- $\text{ASM}(n, \{\mathcal{I}, \mathcal{V}, \mathcal{H}, \mathcal{R}_\pi\})$. Vertically and horizontally symmetric ASMs (VHSASMs). For n even, the set is empty. For n odd, formulae were conjectured by Mills [[43, Sec. 4.2](#)], and proved by Okada [[37, Thm. 1.2 \(A5\) & \(A6\)](#)].
- $\text{ASM}(n, \{\mathcal{I}, \mathcal{R}_\pi\})$. Half-turn symmetric ASMs (HTSASMs). Formulae were conjectured by Mills, Robbins and Rumsey [[35, p. 285](#)], and proved for n even by Kuperberg [[32, Thm. 2](#)], and n odd by Razumov and Stroganov [[40, p. 1197](#)].

- $\text{ASM}(n, \{\mathcal{I}, \mathcal{R}_{\pi/2}, \mathcal{R}_{\pi}, \mathcal{R}_{-\pi/2}\})$. Quarter-turn symmetric ASMs (QTSASMs). For $n \equiv 2 \pmod{4}$, the set is empty. For $n \not\equiv 2 \pmod{4}$, formulae were conjectured by Robbins [43, Sec. 4.2], and proved for $n \equiv 0 \pmod{4}$ by Kuperberg [32, Thm. 2], and n odd by Razumov and Stroganov [39, p. 1649].
- $\text{ASM}(n, \{\mathcal{I}, \mathcal{D}\})$. Diagonally symmetric ASMs (DSASMs). No formula is currently known or conjectured. Data for $n \leq 20$ is given by Bousquet-Mélou and Habsieger [12, Tab. 1].
- $\text{ASM}(n, \{\mathcal{I}, \mathcal{D}, \mathcal{A}, \mathcal{R}_{\pi}\})$. Diagonally and antidiagonally symmetric ASMs (DASASMs). For n even, no formula is currently known or conjectured. Data for $n \leq 24$ is given by Bousquet-Mélou and Habsieger [12, Tab. 1]. For n odd, a formula was conjectured by Robbins [43, Sec. 4.2], and is proved in this paper.
- $\text{ASM}(n, D_4)$. Totally symmetric ASMs (TSASMs). For n even, the set is empty. For n odd, no formula is currently known or conjectured. Data for $n \leq 27$ is given by Bousquet-Mélou and Habsieger [12, Tab. 1].

Alternative approaches to certain parts of some of the proofs cited in this list are also known. For example, such alternatives have been obtained for unrestricted ASMs by Colomo and Pronko [19, Sec. 5.3], [20, Sec. 4.2], Okada [37, Thm. 2.4(1)], Razumov and Stroganov [38, Sec. 2], [41, Sec. 2], and Stroganov [49, Sec. 4]; VSASMs by Okada [37, Thm. 2.4(3)], and Razumov and Stroganov [38, Sec. 3]; even-order HTSASMs by Okada [37, Thm. 2.4(2)], Razumov and Stroganov [40, Eq. (31)], and Stroganov [48, Eq. (11)]; and QTSASMs of order $0 \pmod{4}$ by Okada [37, Thm 2.5(1)].

In addition to the eight standard symmetry classes of ASMs, various closely related classes have also been studied. A few examples are as follows.

- Quasi quarter-turn symmetric ASMs (qQTSASMs). These are $(4n + 2) \times (4n + 2)$ ASMs A for which the four central entries $A_{2n+1, 2n+1}$, $A_{2n+1, 2n+2}$, $A_{2n+2, 2n+1}$ and $A_{2n+2, 2n+2}$ are either 1, 0, 0 and 1 respectively, or 0, -1 , -1 and 0 respectively, while the remaining entries satisfy invariance under quarter-turn rotation, i.e., $A_{ij} = A_{j, 4n+3-i}$ for all other i, j . They were introduced, and a product formula for their enumeration was obtained, by Aval and Duchon [1, 2].
- Off-diagonally symmetric ASMs (OSASMs). These are $2n \times 2n$ DSASMs in which each entry on the diagonal is 0. They were introduced, and a product formula for their straight enumeration (which is identical to that for $(2n+1) \times (2n+1)$ VSASMs) was obtained, by Kuperberg [32, Thm. 5].
- Off-diagonally and off-antidiagonally symmetric ASMs (OOSASMs). These are $4n \times 4n$ DASASMs in which each entry on the diagonal and antidiagonal is 0. They were introduced, and certain results were obtained, by Kuperberg [32], although no simple formula for their straight enumeration is currently known or conjectured.

In addition to results for the straight enumeration of all elements of standard or related symmetry classes of ASMs, various other enumeration results and conjectures are known for certain classes, some examples being as follows.

- Certain results and conjectures are known for the refined or weighted enumeration of classes of ASMs with respect to statistics from among the following: the positions of 1's in or near the outer rows or columns of an ASM; the number of -1 's in an ASM or part of an ASM; and the number of so-called inversions in an ASM. For the case of unrestricted ASMs see, for example, Ayer and Romik [5], Behrend [7], and references therein. For cases involving certain other classes of ASMs see, for example, Cantini [14], de Gier, Pyatov and Zinn-Justin [22], Fischer and Riegler [26], Hagendorf and Morin-Duchesne [28], Kuperberg [32], Okada [37], Razumov and Stroganov [38, 40], Robbins [43], and Stroganov [48].
- In addition to the formula (43) proved in this paper for the ratio between the numbers of odd-order DASASMs with central entry -1 and 1 , analogous formulae have been proved for HTSASMs by Razumov and Stroganov [40, Sec. 5.2] and for qQTSASMs by Aval and Duchon [2, Sec. 5], and have been conjectured for odd-order QTSASMs by Stroganov [50, Conjs. 1a & 1b].
- Various results and conjectures are known for the refined enumeration of several classes of ASMs with respect to so-called link patterns of associated fully packed loop configurations. For further information, see for example Cantini and Sportiello [15, 16], and de Gier [21].
- Various connections between the straight or refined enumeration of classes of ASMs and classes of certain plane partitions have been proved or conjectured. For further information see, for example, Behrend [7, Secs. 3.12, 3.13 & 3.15] and references therein.
- Relationships between the partition functions of cases of the six-vertex model associated with classes of ASMs and certain refined Cauchy and Littlewood identities have been obtained by Betea and Wheeler [10], Betea, Wheeler and Zinn-Justin [11], and Wheeler and Zinn-Justin [52].

1.3. *Proofs of enumeration results for symmetry classes of ASMs*

Most of the proofs mentioned in Section 1.2 use a method which directly involves the statistical mechanical six-vertex model. (A few exceptions are proofs of Cantini and Sportiello [15,16], Fischer [25], and Zeilberger [53].)

Although there are several variations on this method, the general features are often as follows. First, a case of the six-vertex model on a particular graph with certain boundary conditions is introduced, for which the configurations are in bijection with the ASMs under consideration. The associated partition function is then a sum, over all such configurations, of products of parameterized bulk weights, and possibly also boundary weights. By using certain local relations satisfied by these weights, such as the

Yang–Baxter and reflection equations, properties which uniquely determine the partition function are identified. These properties are then used to show that the partition function can be expressed in terms of one or more determinants or Pfaffians. Finally, enumeration formulae are obtained by suitably specializing the parameters in the partition function, and applying certain results for the transformation or evaluation of determinants or Pfaffians.

This general method was introduced in the proofs of Kuperberg [31,32], with certain steps in the unrestricted ASM and VSASM cases being based on previously-known results. For example, a bijection between $\text{ASM}(n)$ and configurations of the six-vertex model on an $n \times n$ square grid with domain-wall boundary conditions had been observed by Elkies, Kuperberg, Larsen and Propp [24, Sec. 7], and (using different terminology) by Robbins and Rumsey [44, pp. 179–180], properties which uniquely determine the partition function for $\text{ASM}(n)$ had been identified by Korepin [30], a determinantal expression for the partition function for $\text{ASM}(n)$ had been obtained by Izergin [29, Eq. (5)], and a determinantal expression closely related to the partition function for odd-order VSASMs had been obtained by Tsuchiya [51].

Following the pioneering proofs of Kuperberg [31,32], a further important development was the observation, made by Okada [37], Razumov and Stroganov [38,40] and Stroganov [48,49], that for certain combinatorially-relevant values of a particular parameter, the partition function can often be written in terms of determinants which are associated with characters of irreducible representations of classical groups. Expressing the partition function in this way, and using known product formulae for the dimensions of such representations, can then lead to simplifications in the proofs of enumeration results.

The approach used in this paper to prove enumeration formulae for odd-order DASASMs has already been summarized in Section 1.1, and largely follows the general method outlined in the current section. However, some specific comparisons between odd-order DASASMs and other ASM classes which have been studied using this method, are as follows.

- As will be discussed in Section 1.7, six-vertex model boundary weights depend on four possible local configurations at a degree-2 boundary vertex. Boundary weights were previously used by Kuperberg [32, Fig. 15] for classes including VSASMs, VHSASMs, OSASMs and OOSASMs, and in each of these cases, the boundary weights were identically zero for two of the four local configurations. However, the boundary weights used for odd-order DASASMs differ from those used previously for other ASM classes, and are not identically zero for any of the four local configurations. As will be discussed in Section 3.2, the boundary weights used by Kuperberg [32, Fig. 15] and those used in this paper constitute various special cases of the most general boundary weights which satisfy the reflection equation for the six-vertex model.

- The natural decomposition of the partition function into a sum of two terms has previously only been observed for one case, that of odd-order HTSASMs, for which a decomposition in which each term involves a product of two determinants, of matrices whose sizes differ by 1, was obtained by Razumov and Stroganov [40, Thm. 1]. Odd-order DASASMs now provide a further example in which the partition function is expressed, in Theorem 1, as a sum of two terms, but in this case each term involves only a single determinant. A further difference is that the matrices for odd-order HTSASMs have a uniform structure in all rows and columns, whereas the matrices for odd-order DASASMs have a special structure in the last row, and a uniform structure elsewhere. For odd-order HTSASMs, the decomposition of the partition function into two terms is closely related to the behavior of parameters associated with the central row and column of the HTSASMs, and similarly, for odd-order DASASMs, it is related to the behavior of a parameter associated with the central column of the DASASMs. However, when this parameter is set to 1, the odd-order DASASM partition function reduces to a single determinantal term, as given in Corollary 2.
- The previously-studied ASM classes can be broadly divided into the following types: (i) those (including unrestricted ASMs, VSASMs, VHSASMs and HTSASMs) in which one set of parameters is associated with the horizontal edges of a grid graph, and another set of parameters is associated with the vertical edges; (ii) those (including QTSASMs, qQTSASMs, OSASMs and OOSASMs) in which a single set of parameters is associated with both horizontal and vertical edges of a grid graph. For the classes of type (i), the partition function is naturally expressed in terms of determinants of matrices whose rows are associated with the horizontal parameters, and columns are associated with the vertical parameters. For the classes of type (ii), the partition function is naturally expressed in terms of Pfaffians. For odd-order DASASMs, there is a single set of parameters associated with both horizontal and vertical edges of a grid graph, and hence this class can be regarded as belonging to type (ii). However, in contrast to the previously-studied classes of this type, the partition function is naturally expressed in terms of determinants.

1.4. DASASMs

The set $\text{ASM}(n, \{\mathcal{I}, \mathcal{D}, \mathcal{A}, \mathcal{R}_\pi\})$ of all $n \times n$ DASASMs will be denoted in the rest of this paper as $\text{DASASM}(n)$. Hence,

$$\text{DASASM}(n) = \{A \in \text{ASM}(n) \mid A_{ij} = A_{ji} = A_{n+1-j, n+1-i}, \text{ for all } 1 \leq i, j \leq n\}. \quad (1)$$

Note that each $A \in \text{DASASM}(n)$ also satisfies $A_{ij} = A_{n+1-i, n+1-j}$ for all $1 \leq i, j \leq n$, i.e., A is half-turn symmetric.

Table 1
Numbers of odd-order DASASMs.

n	0	1	2	3	4	5	6	7
$ \text{DASASM}(2n + 1) $	1	3	15	126	1782	42471	1706562	115640460
$ \text{DASASM}_+(2n + 1) $	1	2	9	72	990	23166	918918	61674912
$ \text{DASASM}_-(2n + 1) $	0	1	6	54	792	19305	787644	53965548

For example,

$$\text{DASASM}(3) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\}, \tag{2}$$

and an element of $\text{DASASM}(7)$ is

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}. \tag{3}$$

Consider any $A \in \text{DASASM}(2n + 1)$. The central column (or row) of A is invariant under reversal of the order of its entries, and its nonzero entries alternate in sign and have a sum of 1. Therefore, the central entry $A_{n+1,n+1}$ is nonzero (since either $A_{n+1,n+1} = 1$ and all other entries of the central column are 0, or else the two nearest nonzero entries to $A_{n+1,n+1}$ in the central column are identical), and it is 1 or -1 according to whether the number of nonzero entries in the first n rows of the central column is even or odd, respectively (since the first nonzero entry in the central column is a 1).

These observations can be summarized as

$$A_{n+1,n+1} = (-1)^{n+N(A)}, \quad \text{for each } A \in \text{DASASM}(2n + 1), \tag{4}$$

where $N(A)$ is the number of 0’s in the first n entries of the central column of A .

The sets of all $(2n + 1) \times (2n + 1)$ DASASMs with fixed central entry 1 and -1 will be denoted as $\text{DASASM}_+(2n + 1)$ and $\text{DASASM}_-(2n + 1)$, respectively, i.e.,

$$\text{DASASM}_\pm(2n + 1) = \{A \in \text{DASASM}(2n + 1) \mid A_{n+1,n+1} = \pm 1\}. \tag{5}$$

The rest of this paper will be primarily focused on obtaining results which lead to formulae (given in [Corollaries 5 and 9](#)) for $|\text{DASASM}(2n + 1)|$ and $|\text{DASASM}_\pm(2n + 1)|$. For reference, these cardinalities for $n = 0, \dots, 7$ are given in [Table 1](#).

1.5. Odd DASASM triangles

Let an odd DASASM triangle A of order n be a triangular array

$$\begin{array}{cccccccc}
 A_{11} & A_{12} & A_{13} & \dots & A_{1,n+1} & \dots & A_{1,2n-1} & A_{1,2n} & A_{1,2n+1} \\
 & A_{22} & A_{23} & \dots & A_{2,n+1} & \dots & A_{2,2n-1} & A_{2,2n} & \\
 & & \ddots & & \vdots & & \ddots & & \\
 & & & & A_{nn} & A_{n,n+1} & A_{n,n+2} & & \\
 & & & & & & A_{n+1,n+1}, & &
 \end{array} \tag{6}$$

such that each entry is 0, 1 or -1 and, for each $i = 1, \dots, n + 1$, the nonzero entries along the sequence

$$\begin{array}{ccccccc}
 A_{1i} & & & & & & A_{1,2n+2-i} \\
 A_{2i} & & & & & & A_{2,2n+2-i} \\
 \vdots & & & & & & \vdots \\
 A_{i-1,i} & & & & & & A_{i-1,2n+2-i} \\
 A_{ii} & A_{i,i+1} & \dots & A_{i,2n+1-i} & & & A_{i,2n+2-i}
 \end{array} \tag{7}$$

alternate in sign and have a sum of 1, where the sequence is read downward from A_{1i} to A_{ii} , then rightward to $A_{i,2n+2-i}$, and then upward to $A_{1,2n+2-i}$ (and for $i = n + 1$, the sequence is taken to be $A_{1,n+1}, \dots, A_{n,n+1}, A_{n+1,n+1}, A_{n,n+1}, \dots, A_{1,n+1}$).

It can be seen that there is a bijection from $\text{DASASM}(2n + 1)$ to the set of odd DASASM triangles of order n , in which the entries A_{ij} of $A \in \text{DASASM}(2n + 1)$ are simply restricted to $i = 1, \dots, n + 1$ and $j = i, \dots, 2n + 2 - i$.

As examples, the set of odd DASASM triangles of order 1 is

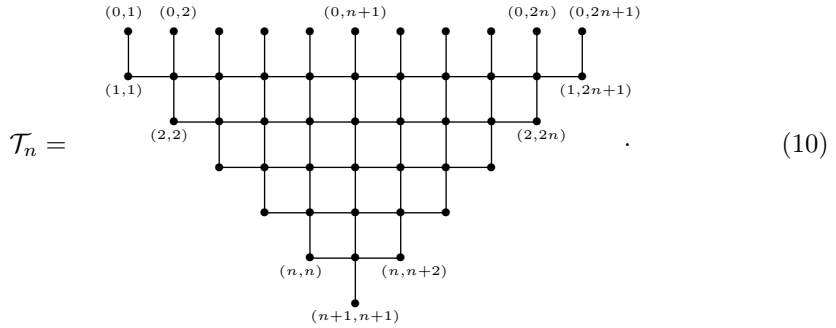
$$\left\{ \begin{array}{cccccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ & 1 & & & -1 & & & 1 & \end{array} \right\} \tag{8}$$

(where the elements correspond, in order, to the DASASMs in (2)), and the odd DASASM triangle which corresponds to the DASASM in (3) is

$$\begin{array}{cccccc}
 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
 & 1 & -1 & 0 & 1 & 0 & \\
 & & 0 & 1 & -1 & & \\
 & & & & -1 & &
 \end{array} . \tag{9}$$

1.6. Six-vertex model configurations

Define a grid graph on a triangle as



Note that \mathcal{T}_n can be regarded as an odd-order analog of the graph introduced by Kuperberg [32, Fig. 13] for OOSASMs.

The vertices of \mathcal{T}_n consist of top vertices $(0, j)$, $j = 0, \dots, 2n + 1$, of degree 1, left boundary vertices (i, i) , $i = 1, \dots, n$, of degree 2, bulk vertices (i, j) , $i = 1, \dots, n$, $j = i + 1, \dots, 2n + 1 - i$, of degree 4, right boundary vertices $(i, 2n + 2 - i)$, $i = 1, \dots, n$, of degree 2, and a bottom vertex $(n + 1, n + 1)$ of degree 1. The edges incident with the top vertices will be referred to as top edges.

Now define a configuration of the six-vertex model on \mathcal{T}_n to be an orientation of the edges of \mathcal{T}_n , such that each top edge is directed upwards, and among the four edges incident to each bulk vertex, two are directed into and two are directed out of the vertex, i.e., the so-called six-vertex rule is satisfied.

For such a configuration C , and a vertex (i, j) of \mathcal{T}_n , define the local configuration C_{ij} at (i, j) to be the orientation of the edges incident to (i, j) . Hence, the possible local configurations are \uparrow at a top vertex, $\uparrow\rightarrow, \downarrow\leftarrow, \uparrow\leftarrow$ or $\downarrow\rightarrow$ at a left boundary vertex, $\uparrow\leftrightarrow, \downarrow\leftrightarrow, \uparrow\leftrightarrow, \downarrow\leftrightarrow$ or $\uparrow\leftrightarrow, \downarrow\leftrightarrow$ at a bulk vertex, $\leftarrow\uparrow, \rightarrow\downarrow, \rightarrow\uparrow$ or $\leftarrow\downarrow$ at a right boundary vertex, and \uparrow or \downarrow at the bottom vertex.

There is a natural bijection from the set of configurations of the six-vertex model on \mathcal{T}_n to the set of odd DASASM triangles of order n , in which a configuration C is mapped to an odd DASASM triangle A given by

$$A_{ij} = \begin{cases} 1, & C_{ij} = \uparrow\leftrightarrow, \uparrow\leftarrow, \rightarrow\uparrow \text{ or } \uparrow, \\ -1, & C_{ij} = \downarrow\leftrightarrow, \downarrow\rightarrow, \leftarrow\downarrow \text{ or } \downarrow, \\ 0, & C_{ij} = \uparrow\leftrightarrow, \downarrow\leftrightarrow, \uparrow\leftrightarrow, \downarrow\leftrightarrow, \uparrow\leftrightarrow, \downarrow\leftrightarrow, \leftarrow\uparrow, \rightarrow\downarrow, \leftarrow\downarrow \text{ or } \rightarrow\uparrow, \end{cases} \tag{11}$$

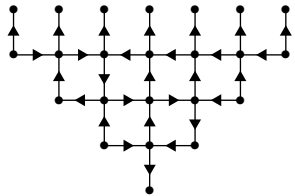
for $i = 1, \dots, n + 1$ and $j = i, \dots, 2n + 2 - i$. Note that the (fixed) local configurations at the top vertices are not associated with entries of A . Note also that the cases of (11) can be summarized as follows: $A_{ij} = 1$ if C_{ij} is $\uparrow\leftrightarrow$ or a restriction of that (to the upper

and right, upper and left, or upper edges), $A_{ij} = -1$ if C_{ij} is $\leftarrow\updownarrow\rightarrow$ or a restriction of that (again to the upper and right, upper and left, or upper edges), and $A_{ij} = 0$ otherwise.

As examples, the set of configurations of the six-vertex model on \mathcal{T}_1 is

$$\left\{ \begin{array}{c} \uparrow \uparrow \uparrow \\ \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet \\ \uparrow \uparrow \uparrow \end{array} , \begin{array}{c} \uparrow \uparrow \uparrow \\ \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \\ \uparrow \uparrow \uparrow \end{array} , \begin{array}{c} \uparrow \uparrow \uparrow \\ \leftarrow \bullet \rightarrow \bullet \leftarrow \bullet \\ \uparrow \uparrow \uparrow \end{array} \right\} \tag{12}$$

(where the elements correspond, in order, to the odd DASASM triangles in (8)), and the configuration which corresponds to the odd DASASM triangle in (9) is



$$\tag{13}$$

The fact that (11) is a well-defined bijection can be verified by considering the standard bijection between $\text{ASM}(2n + 1)$ and the set of configurations of the six-vertex model on a $(2n + 1) \times (2n + 1)$ square grid \mathcal{S}_n with domain-wall boundary conditions, and then restricting from $\text{ASM}(2n + 1)$ to $\text{DASASM}(2n + 1)$. Some of the details of this bijection and the restriction are as follows. The grid \mathcal{S}_n , which contains the grid \mathcal{T}_n as a subgraph, consists of bulk vertices (i, j) , of degree 4, together with top vertices $(0, j)$, right vertices $(i, 2n + 2)$, bottom vertices $(2n + 2, j)$ and left vertices $(i, 0)$, all of degree 1, for $i, j = 1, \dots, 2n + 1$, where (i, j) appears in row i and column j . The configurations are orientations of the edges of \mathcal{S}_n , such that each edge incident to a top, right, bottom or left vertex is directed upwards, leftwards, downwards or rightwards, respectively, and the six-vertex rule is satisfied at each bulk vertex. The ASM A which corresponds to a configuration C is given by $A_{ij} = 1$ or $A_{ij} = -1$ if the local configuration of C at (i, j) is $\rightarrow\updownarrow\leftarrow$ or $\leftarrow\updownarrow\rightarrow$, respectively, and $A_{ij} = 0$, otherwise. If A is a DASASM, then the symmetry conditions for A imply that C is uniquely determined by its restriction to the edges of \mathcal{T}_n . Note also that in this case, within \mathcal{S}_n , only $\rightarrow\updownarrow\leftarrow$, $\leftarrow\updownarrow\rightarrow$, $\rightarrow\updownarrow\rightarrow$ and $\leftarrow\updownarrow\leftarrow$ can occur at a vertex (i, i) on the diagonal, only $\rightarrow\updownarrow\leftarrow$, $\leftarrow\updownarrow\rightarrow$, $\rightarrow\updownarrow\leftarrow$ and $\leftarrow\updownarrow\rightarrow$ can occur at a vertex $(i, 2n + 2 - i)$ on the antidiagonal, and hence only $\rightarrow\updownarrow\leftarrow$ or $\leftarrow\updownarrow\rightarrow$ can occur at the central vertex $(n + 1, n + 1)$.

1.7. Weights and the partition function

Throughout the rest of this paper, the notation

Table 2

Bulk and boundary weights.

Bulk weights	Left boundary weights	Right boundary weights
$W(\begin{smallmatrix} \blacktriangleleft & \blacktriangleright \\ \blacktriangleright & \blacktriangleleft \end{smallmatrix}, u) = W(\begin{smallmatrix} \blacktriangleright & \blacktriangleleft \\ \blacktriangleleft & \blacktriangleright \end{smallmatrix}, u) = 1$	$W(\begin{smallmatrix} \blacktriangleleft \\ \blacktriangleright \end{smallmatrix}, u) = W(\begin{smallmatrix} \blacktriangleright \\ \blacktriangleleft \end{smallmatrix}, u) = 1$	$W(\begin{smallmatrix} \blacktriangleleft \\ \blacktriangleright \end{smallmatrix}, u) = W(\begin{smallmatrix} \blacktriangleright \\ \blacktriangleleft \end{smallmatrix}, u) = 1$
$W(\begin{smallmatrix} \blacktriangleleft & \blacktriangleright \\ \blacktriangleright & \blacktriangleleft \end{smallmatrix}, u) = W(\begin{smallmatrix} \blacktriangleright & \blacktriangleleft \\ \blacktriangleleft & \blacktriangleright \end{smallmatrix}, u) = \frac{\sigma(q^2 u)}{\sigma(q^2)}$	$W(\begin{smallmatrix} \blacktriangleleft \\ \blacktriangleright \end{smallmatrix}, u) = W(\begin{smallmatrix} \blacktriangleright \\ \blacktriangleleft \end{smallmatrix}, u) = \frac{\sigma(qu)}{\sigma(q)}$	
$W(\begin{smallmatrix} \blacktriangleleft & \blacktriangleright \\ \blacktriangleright & \blacktriangleleft \end{smallmatrix}, u) = W(\begin{smallmatrix} \blacktriangleright & \blacktriangleleft \\ \blacktriangleleft & \blacktriangleright \end{smallmatrix}, u) = \frac{\sigma(q^2 \bar{u})}{\sigma(q^2)}$		$W(\begin{smallmatrix} \blacktriangleleft \\ \blacktriangleright \end{smallmatrix}, u) = W(\begin{smallmatrix} \blacktriangleright \\ \blacktriangleleft \end{smallmatrix}, u) = \frac{\sigma(q\bar{u})}{\sigma(q)}$

$$\bar{x} = x^{-1} \quad \text{and} \quad \sigma(x) = x - \bar{x} \tag{14}$$

will be used.

For each possible local configuration c at a bulk or boundary vertex, and for parameters q and u , assign a weight $W(c, u)$, as given in Table 2 (where each boundary weight $W(c, u)$ appears in the same row as bulk weights whose local configurations restrict to c).

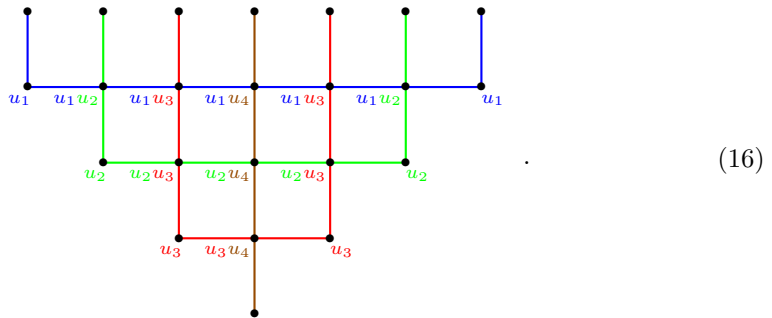
It will be convenient for the dependence on q not to be indicated explicitly by the notation $W(c, u)$, or by the notation for further q -dependent quantities in this paper. Note also that the corresponding parameter which is used for such weights in the literature is often the square of the q used in this paper.

It can be seen that the weights of Table 2 satisfy

$$\begin{aligned} W(c, 1) \Big|_{q=e^{i\pi/6}} &= 1, \quad \text{for each local configuration } c \text{ at a bulk vertex,} \\ \text{and } W(c, 1) &= 1, \quad \text{for each local configuration } c \text{ at a boundary vertex.} \end{aligned} \tag{15}$$

For a configuration C of the six-vertex model on \mathcal{T}_n , and parameters q and u_1, \dots, u_{n+1} , define the weight of left boundary vertex (i, i) to be $W(C_{ii}, u_i)$, the weight of bulk vertex (i, j) to be $W(C_{ij}, u_i u_{\min(j, 2n+2-j)})$, and the weight of right boundary vertex $(i, 2n + 2 - i)$ to be $W(C_{i, 2n+2-i}, u_i)$, where, as before, C_{ij} is the local configuration of C at (i, j) . Note that q is an overall constant, which is the same in all of these weights.

The assignment of u_1, \dots, u_{n+1} in these weights can be illustrated, for $n = 3$, as¹



¹ For interpretation of the colors in this article, the reader is referred to the web version of this article.

The color coding in (16) indicates that for $i = 1, \dots, n+1$, u_i can be naturally associated with the edges in column i , row i and column $2n + 2 - i$ of \mathcal{T}_n , such that the parameter for the weight of a boundary vertex is the single parameter associated with the incident edges, and the parameter for the weight of a bulk vertex is the product of the two different parameters associated with the incident edges.

Now define the odd-order DASASM partition function $Z(u_1, \dots, u_{n+1})$ to be the sum of products of bulk and boundary vertex weights, over all configurations C of the six-vertex model on \mathcal{T}_n , i.e.,

$$Z(u_1, \dots, u_{n+1}) = \sum_C \prod_{i=1}^n W(C_{ii}, u_i) \left(\prod_{j=i+1}^{2n+1-i} W(C_{ij}, u_i u_{\min(j, 2n+2-j)}) \right) W(C_{i, 2n+2-i}, u_i). \tag{17}$$

Note that the top vertices and bottom vertex can be regarded as each having weight 1. For example,

$$Z(u_1, u_2) = \frac{\sigma(q^2 \bar{u}_1 \bar{u}_2) \sigma(q \bar{u}_1)}{\sigma(q^4) \sigma(q)} + \frac{\sigma(qu_1) \sigma(q \bar{u}_1)}{\sigma(q)^2} + \frac{\sigma(qu_1) \sigma(q^2 u_1 u_2)}{\sigma(q) \sigma(q^4)} \tag{18}$$

(where the terms are written in an order which corresponds to that used in (12)), and the term of $Z(u_1, \dots, u_4)$ which corresponds to the configuration in (13) is

$$\frac{\sigma(qu_1) \sigma(q^2 u_1 u_2) \sigma(q^2 \bar{u}_1 \bar{u}_4) \sigma(q^2 \bar{u}_1 \bar{u}_3) \sigma(q^2 \bar{u}_1 \bar{u}_2) \sigma(q \bar{u}_1) \sigma(q^2 u_2 u_4) \sigma(q \bar{u}_2) \sigma(qu_3)}{\sigma(q)^4 \sigma(q^4)^5}. \tag{19}$$

Similarly, define the odd-order DASASM₊ and DASASM₋ partition functions $Z_+(u_1, \dots, u_{n+1})$ and $Z_-(u_1, \dots, u_{n+1})$ to be the sum of products of bulk and boundary vertex weights, over all configurations C of the six-vertex model on \mathcal{T}_n in which the edge incident to the bottom vertex is directed upwards or downwards, respectively. Hence,

$$Z(u_1, \dots, u_{n+1}) = Z_+(u_1, \dots, u_{n+1}) + Z_-(u_1, \dots, u_{n+1}). \tag{20}$$

It follows from (15), and the bijections among the set of configurations of the six-vertex model on \mathcal{T}_n , the set of odd DASASM triangles of order n and DASASM($2n + 1$), that

$$|\text{DASASM}(2n + 1)| = Z(\underbrace{1, \dots, 1}_{n+1}) \Big|_{q=e^{i\pi/6}} \tag{21}$$

and

$$|\text{DASASM}_{\pm}(2n + 1)| = Z_{\pm}(\underbrace{1, \dots, 1}_{n+1}) \Big|_{q=e^{i\pi/6}}. \tag{22}$$

Now consider the replacement of u_{n+1} with $-u_{n+1}$ in the sum of (17). Then the bulk weights $W(\uparrow\downarrow, u_i u_{n+1})$, $W(\downarrow\uparrow, u_i u_{n+1})$, $W(\leftarrow\rightarrow, u_i u_{n+1})$ and $W(\rightarrow\leftarrow, u_i u_{n+1})$, whose local configurations are associated with the 0's in the first n rows of the central column of the corresponding DASASMs, change signs under this replacement, while all other bulk weights and all boundary weights are unchanged.

It follows, using (4), that

$$(-1)^n Z(u_1, \dots, u_n, -u_{n+1}) = Z_+(u_1, \dots, u_{n+1}) - Z_-(u_1, \dots, u_{n+1}), \tag{23}$$

and so, using (20), that

$$Z_{\pm}(u_1, \dots, u_{n+1}) = \frac{1}{2}(Z(u_1, \dots, u_n, u_{n+1}) \pm (-1)^n Z(u_1, \dots, u_n, -u_{n+1})). \tag{24}$$

1.8. Schur functions and semistandard Young tableaux

The notation and results regarding Schur functions and semistandard Young tableaux which will be used in this paper are as follows. (For further information, see for example Stanley [47, Ch. 7].)

For a partition λ of length $\ell(\lambda) \leq k$, and variables x_1, \dots, x_k , let $s_{\lambda}(x_1, \dots, x_k)$ be the Schur function (or Schur polynomial) indexed by λ , and let $\text{SSYT}_{\lambda}(k)$ be the set of semistandard Young tableaux of shape λ with entries from $\{1, \dots, k\}$.

A determinantal formula for Schur functions is

$$s_{\lambda}(x_1, \dots, x_k) = \frac{\det_{1 \leq i, j \leq k} (x_i^{\lambda_j + k - j})}{\prod_{1 \leq i < j \leq k} (x_i - x_j)}, \tag{25}$$

a formula for Schur functions involving a sum over semistandard Young tableaux is

$$s_{\lambda}(x_1, \dots, x_k) = \sum_{T \in \text{SSYT}_{\lambda}(k)} x_1^{\#(1, T)} \dots x_k^{\#(k, T)}, \tag{26}$$

and a product formula for the number of semistandard Young tableaux is

$$|\text{SSYT}_{\lambda}(k)| = \frac{\prod_{1 \leq i < j \leq k} (\lambda_i - \lambda_j - i + j)}{\prod_{i=1}^{k-1} i!}, \tag{27}$$

where $\lambda_i = 0$ for $i = \ell(\lambda) + 1, \dots, k$, and $\#(i, T)$ denotes the number of occurrences of i in T .

It follows immediately from (25) that $s_{\lambda}(x_1, \dots, x_k)$ is symmetric in x_1, \dots, x_k , and immediately from (26) that

$$s_{\lambda}(\underbrace{1, \dots, 1}_k) = |\text{SSYT}_{\lambda}(k)|. \tag{28}$$

A further identity is

$$\frac{d}{dx} s_\lambda(\underbrace{1, \dots, 1}_{k-1}, x) \Big|_{x=1} = \frac{|\lambda|}{k} |\text{SSYT}_\lambda(k)|, \tag{29}$$

which can be proved by using symmetry and (26) to give

$$\begin{aligned} k s_\lambda(\underbrace{1, \dots, 1}_{k-1}, x) &= s_\lambda(x, \underbrace{1, \dots, 1}_{k-1}) + \dots + s_\lambda(\underbrace{1, \dots, 1}_{k-1}, x) \\ &= \sum_{T \in \text{SSYT}_\lambda(k)} (x^{\#(1,T)} + \dots + x^{\#(k,T)}), \end{aligned}$$

and hence

$$\begin{aligned} k \frac{d}{dx} s_\lambda(\underbrace{1, \dots, 1}_{k-1}, x) \Big|_{x=1} &= \sum_{T \in \text{SSYT}_\lambda(k)} (\#(1,T) + \dots + \#(k,T)) \\ &= \sum_{T \in \text{SSYT}_\lambda(k)} |\lambda| = |\text{SSYT}_\lambda(k)| |\lambda|. \end{aligned}$$

2. Main results

In this section, the main results of the paper are presented, with the cases of odd-order DASASMs whose central entry is arbitrary or fixed being considered in separate subsections. The proofs of Theorems 1 and 3 will be given in Section 3. All of the other results are corollaries of these theorems, and their proofs from the theorems are given in this section. The notation of (14) is used in all of the results.

2.1. Results for odd-order DASASMs with arbitrary central entry

The main results of this paper involving odd-order DASASMs whose central entry is arbitrary are as follows.

Theorem 1. *The odd-order DASASM partition function is given by*

$$\begin{aligned} Z(u_1, \dots, u_{n+1}) &= \\ &= \frac{\sigma(q^2)^n}{\sigma(q)^{2n} \sigma(q^4)^{n^2}} \prod_{i=1}^n \frac{\sigma(u_i) \sigma(qu_i) \sigma(q\bar{u}_i) \sigma(q^2 u_i u_{n+1}) \sigma(q^2 \bar{u}_i \bar{u}_{n+1})}{\sigma(u_i \bar{u}_{n+1})} \\ &\times \prod_{1 \leq i < j \leq n} \left(\frac{\sigma(q^2 u_i u_j) \sigma(q^2 \bar{u}_i \bar{u}_j)}{\sigma(u_i \bar{u}_j)} \right)^2 \end{aligned}$$

$$\begin{aligned} &\times \left(\det_{1 \leq i, j \leq n+1} \left(\begin{cases} \frac{q^2 + \bar{q}^2 + u_i^2 + \bar{u}_j^2}{\sigma(q^2 u_i u_j) \sigma(q^2 \bar{u}_i \bar{u}_j)}, & i \leq n \\ \frac{u_{n+1} - 1}{\bar{u}_j^2 - 1}, & i = n + 1 \end{cases} \right) \right. \\ &\quad \left. + \det_{1 \leq i, j \leq n+1} \left(\begin{cases} \frac{q^2 + \bar{q}^2 + \bar{u}_i^2 + u_j^2}{\sigma(q^2 u_i u_j) \sigma(q^2 \bar{u}_i \bar{u}_j)}, & i \leq n \\ \frac{\bar{u}_{n+1} - 1}{\bar{u}_j^2 - 1}, & i = n + 1 \end{cases} \right) \right). \quad (30) \end{aligned}$$

This result will be proved in Section 3.5, and an alternative proof will be sketched in Section 3.6.

Note that the two determinants on the RHS of (30) are related to each other by replacement of u_1, \dots, u_{n+1} by $\bar{u}_1, \dots, \bar{u}_{n+1}$, and that the prefactor is unchanged under this transformation.

Corollary 2. *The odd-order DASASM partition function at $u_{n+1} = 1$ is given by*

$$\begin{aligned} Z(u_1, \dots, u_n, 1) = & \frac{\sigma(q^2)^n}{\sigma(q)^{2n} \sigma(q^4)^{n^2}} \prod_{i=1}^n \sigma(q u_i) \sigma(q \bar{u}_i) \sigma(q^2 u_i) \sigma(q^2 \bar{u}_i) \prod_{1 \leq i < j \leq n} \left(\frac{\sigma(q^2 u_i u_j) \sigma(q^2 \bar{u}_i \bar{u}_j)}{\sigma(u_i \bar{u}_j)} \right)^2 \\ & \times \det_{1 \leq i, j \leq n} \left(\frac{q^2 + \bar{q}^2 + u_i^2 + \bar{u}_j^2}{\sigma(q^2 u_i u_j) \sigma(q^2 \bar{u}_i \bar{u}_j)} \right). \quad (31) \end{aligned}$$

Proof. Taking $u_{n+1} \rightarrow 1$ in (30), the last row of each matrix becomes $(0, \dots, 0, \frac{1}{2})$, and the result then follows. \square

An alternative proof of Corollary 2, which does not use Theorem 1, will be outlined in Section 3.6.

Note that, due to the factor $\prod_{i=1}^n \sigma(q^2 u_i) \sigma(q^2 \bar{u}_i)$ on the RHS of (31) (which, unlike other parts of the prefactor, does not cancel with terms from the determinant), $Z(u_1, \dots, u_n, 1)$ is zero at $u_i = q^{\pm 2}$ and $u_i = -q^{\pm 2}$, for each $i = 1, \dots, n$. This property will be explained further by the proof of Proposition 16.

Theorem 3. *The odd-order DASASM partition function at $q = e^{i\pi/6}$ is given by*

$$\begin{aligned} Z(u_1, \dots, u_{n+1}) \Big|_{q=e^{i\pi/6}} = & 3^{-n(n-1)/2} \left(\frac{u_{n+1}^n}{u_{n+1}+1} s_{(n, n-1, n-1, \dots, 2, 2, 1, 1)}(u_1^2, \bar{u}_1^2, \dots, u_n^2, \bar{u}_n^2, \bar{u}_{n+1}^2) \right. \\ & \left. + \frac{\bar{u}_{n+1}^n}{\bar{u}_{n+1}+1} s_{(n, n-1, n-1, \dots, 2, 2, 1, 1)}(u_1^2, \bar{u}_1^2, \dots, u_n^2, \bar{u}_n^2, u_{n+1}^2) \right). \quad (32) \end{aligned}$$

This result will be proved in Section 3.7, using Theorem 1.

Note that, by using the standard formula for taking the reciprocals of all variables in a Schur function (e.g., Stanley [47, Ex. 7.41]), the Schur functions in (32) could be written

instead as $s_{(n,n-1,n-1,\dots,2,2,1,1)}(u_1^2, \bar{u}_1^2, \dots, u_n^2, \bar{u}_n^2, u_{n+1}^{\mp 2}) = u_{n+1}^{\mp 2n} s_{(n,n,\dots,2,2,1,1)}(u_1^2, \bar{u}_1^2, \dots, u_n^2, \bar{u}_n^2, u_{n+1}^{\pm 2})$.

Corollary 4. *The odd-order DASASM partition function at $u_{n+1} = 1$ and $q = e^{i\pi/6}$ is given by*

$$Z(u_1, \dots, u_n, 1) \Big|_{q=e^{i\pi/6}} = 3^{-n(n-1)/2} s_{(n,n-1,n-1,\dots,2,2,1,1)}(u_1^2, \bar{u}_1^2, \dots, u_n^2, \bar{u}_n^2, 1). \tag{33}$$

Proof. Set $u_{n+1} = 1$ in (32). \square

A factorization of the Schur function in (33), involving odd orthogonal and symplectic characters, will be given in (63).

Note also that the factor $\prod_{i=1}^n \sigma(q^2 u_i) \sigma(q^2 \bar{u}_i)$ on the RHS of (31) leads, at $q = e^{i\pi/6}$, to a factor $\prod_{i=1}^n (u_i^2 + 1 + \bar{u}_i^2)$ in $s_{(n,n-1,n-1,\dots,2,2,1,1)}(u_1^2, \bar{u}_1^2, \dots, u_n^2, \bar{u}_n^2, 1)$ on the RHS of (33). This will appear explicitly in the factorization in (63).

Corollary 5. *The number of $(2n + 1) \times (2n + 1)$ DASASMs is given by*

$$\begin{aligned} |\text{DASASM}(2n + 1)| &= 3^{-n(n-1)/2} |\text{SSYT}_{(n,n-1,n-1,\dots,2,2,1,1)}(2n + 1)| \\ &= \prod_{i=0}^n \frac{(3i)!}{(n + i)!}. \end{aligned} \tag{34}$$

Proof. The first equality follows immediately by setting $u_1 = \dots = u_n = 1$ in (33), and using (21) and (28). The second equality (which was previously obtained by Okada [37, Conj. 5.1(2)]) then follows by applying (27), and simplifying the resulting expression. \square

As indicated in Sections 1.1–1.2, a recursion relation for $|\text{DASASM}(2n + 1)|$ which gives the product formula in (34) was conjectured by Robbins [43, Sec. 4.2].

Note also that, due to the comment after Theorem 3, the partition $(n, n - 1, n - 1, \dots, 2, 2, 1, 1)$ in (33) and (34) could be replaced by $(n, n, \dots, 2, 2, 1, 1)$.

2.2. Results for odd-order DASASMs with fixed central entry

The results of the previous section have certain corollaries for odd-order DASASMs whose central entry is fixed, as follows.

Corollary 6. *The odd-order DASASM $_{\pm}$ partition functions are given by*

$$Z_{\pm}(u_1, \dots, u_{n+1}) = \frac{\sigma(q^2)^n}{\sigma(q)^{2n} \sigma(q^4)^{n^2}} \prod_{i=1}^n \frac{\sigma(u_i) \sigma(qu_i) \sigma(q\bar{u}_i) \sigma(q^2 u_i u_{n+1}) \sigma(q^2 \bar{u}_i \bar{u}_{n+1})}{\sigma(u_i \bar{u}_{n+1})}$$

$$\begin{aligned} & \times \prod_{1 \leq i < j \leq n} \left(\frac{\sigma(q^2 u_i u_j) \sigma(q^2 \bar{u}_i \bar{u}_j)}{\sigma(u_i \bar{u}_j)} \right)^2 \\ & \times \left(\det_{1 \leq i, j \leq n+1} \left(\begin{cases} \frac{q^2 + \bar{q}^2 + u_i^2 + \bar{u}_j^2}{\sigma(q^2 u_i u_j) \sigma(q^2 \bar{u}_i \bar{u}_j)}, & i \leq n \\ \frac{1}{1 - u_j^2}, & i = n + 1 \end{cases} \right) \right. \\ & \quad \left. + \det_{1 \leq i, j \leq n+1} \left(\begin{cases} \frac{q^2 + \bar{q}^2 + \bar{u}_i^2 + u_j^2}{\sigma(q^2 u_i u_j) \sigma(q^2 \bar{u}_i \bar{u}_j)}, & i \leq n \\ \frac{1}{1 - \bar{u}_j^2}, & i = n + 1 \end{cases} \right) \right) \end{aligned} \tag{35}$$

and

$$\begin{aligned} Z_-(u_1, \dots, u_{n+1}) &= \\ & \frac{\sigma(q^2)^n}{\sigma(q)^{2n} \sigma(q^4)^{n^2}} \prod_{i=1}^n \frac{\sigma(u_i) \sigma(qu_i) \sigma(q\bar{u}_i) \sigma(q^2 u_i u_{n+1}) \sigma(q^2 \bar{u}_i \bar{u}_{n+1})}{\sigma(u_i \bar{u}_{n+1})} \\ & \times \prod_{1 \leq i < j \leq n} \left(\frac{\sigma(q^2 u_i u_j) \sigma(q^2 \bar{u}_i \bar{u}_j)}{\sigma(u_i \bar{u}_j)} \right)^2 \\ & \times \left(\det_{1 \leq i, j \leq n+1} \left(\begin{cases} \frac{q^2 + \bar{q}^2 + u_i^2 + \bar{u}_j^2}{\sigma(q^2 u_i u_j) \sigma(q^2 \bar{u}_i \bar{u}_j)}, & i \leq n \\ \frac{u_{n+1}}{u_j^2 - 1}, & i = n + 1 \end{cases} \right) \right. \\ & \quad \left. + \det_{1 \leq i, j \leq n+1} \left(\begin{cases} \frac{q^2 + \bar{q}^2 + \bar{u}_i^2 + u_j^2}{\sigma(q^2 u_i u_j) \sigma(q^2 \bar{u}_i \bar{u}_j)}, & i \leq n \\ \frac{\bar{u}_{n+1}}{\bar{u}_j^2 - 1}, & i = n + 1 \end{cases} \right) \right). \end{aligned} \tag{36}$$

Proof. Apply (24) to (30). \square

Note that, in contrast to the determinants in (30), each of the determinants in (35) and (36) is singular at $u_{n+1} \rightarrow 1$ (due to the form of the bottom right entry of each of the matrices).

Corollary 7. The odd-order DASASM $_{\pm}$ partition functions at $q = e^{i\pi/6}$ are given by

$$\begin{aligned} Z_+(u_1, \dots, u_{n+1}) \Big|_{q=e^{i\pi/6}} &= \\ & 3^{-n(n-1)/2} \left(\frac{u_{n+1}^n}{1 - u_{n+1}^2} s_{(n, n-1, n-1, \dots, 2, 2, 1, 1)}(u_1^2, \bar{u}_1^2, \dots, u_n^2, \bar{u}_n^2, \bar{u}_{n+1}^2) \right. \\ & \quad \left. + \frac{\bar{u}_{n+1}^n}{1 - \bar{u}_{n+1}^2} s_{(n, n-1, n-1, \dots, 2, 2, 1, 1)}(u_1^2, \bar{u}_1^2, \dots, u_n^2, \bar{u}_n^2, u_{n+1}^2) \right) \end{aligned} \tag{37}$$

and

$$Z_-(u_1, \dots, u_{n+1}) \Big|_{q=e^{i\pi/6}} =$$

$$3^{-n(n-1)/2} \left(\frac{u_{n+1}^{n+1}}{u_{n+1}^2 - 1} s_{(n,n-1,n-1,\dots,2,2,1,1)}(u_1^2, \bar{u}_1^2, \dots, u_n^2, \bar{u}_n^2, \bar{u}_{n+1}^2) + \frac{\bar{u}_{n+1}^{n+1}}{\bar{u}_{n+1}^2 - 1} s_{(n,n-1,n-1,\dots,2,2,1,1)}(u_1^2, \bar{u}_1^2, \dots, u_n^2, \bar{u}_n^2, u_{n+1}^2) \right). \tag{38}$$

Proof. Apply (24) to (32). \square

Corollary 8. The odd-order DASASM $_{\pm}$ partition functions at $u_{n+1} = 1$ and $q = e^{i\pi/6}$ are given by

$$\begin{aligned} Z_+(u_1, \dots, u_n, 1) \Big|_{q=e^{i\pi/6}} = & \\ 3^{-n(n-1)/2} \left(2 \frac{d}{dx} s_{(n,n-1,n-1,\dots,2,2,1,1)}(u_1^2, \bar{u}_1^2, \dots, u_n^2, \bar{u}_n^2, x) \Big|_{x=1} \right. & \\ \left. - (n-1) s_{(n,n-1,n-1,\dots,2,2,1,1)}(u_1^2, \bar{u}_1^2, \dots, u_n^2, \bar{u}_n^2, 1) \right) & \tag{39} \end{aligned}$$

and

$$\begin{aligned} Z_-(u_1, \dots, u_n, 1) \Big|_{q=e^{i\pi/6}} = & \\ 3^{-n(n-1)/2} \left(n s_{(n,n-1,n-1,\dots,2,2,1,1)}(u_1^2, \bar{u}_1^2, \dots, u_n^2, \bar{u}_n^2, 1) \right. & \\ \left. - 2 \frac{d}{dx} s_{(n,n-1,n-1,\dots,2,2,1,1)}(u_1^2, \bar{u}_1^2, \dots, u_n^2, \bar{u}_n^2, x) \Big|_{x=1} \right). & \tag{40} \end{aligned}$$

Proof. Take $u_{n+1} \rightarrow 1$ in (37) and (38), and apply L’Hôpital’s rule. \square

Corollary 9. The numbers of $(2n + 1) \times (2n + 1)$ DASASMs with fixed central entry 1 and -1 are given by

$$|\text{DASASM}_+(2n + 1)| = \frac{n + 1}{2n + 1} \prod_{i=0}^n \frac{(3i)!}{(n + i)!} \tag{41}$$

and

$$|\text{DASASM}_-(2n + 1)| = \frac{n}{2n + 1} \prod_{i=0}^n \frac{(3i)!}{(n + i)!}, \tag{42}$$

and hence satisfy

$$\frac{|\text{DASASM}_-(2n + 1)|}{|\text{DASASM}_+(2n + 1)|} = \frac{n}{n + 1}. \tag{43}$$

Proof. Set $u_1 = \dots = u_n = 1$ in (39) and (40), and apply (22), (28), (29) and the second equality of (34). \square

As indicated in Section 1.1, the relation (43) was conjectured by Stroganov [50, Conj. 2].

3. Proofs

In this section, full proofs of [Theorems 1 and 3](#), and sketches of alternative proofs of [Theorem 1](#) and [Corollary 2](#), are given. Preliminary results are stated or obtained in [Sections 3.1–3.4](#), while the main steps of the proofs appear in [Sections 3.5–3.7](#).

3.1. Simple properties of bulk and boundary weights

The bulk and boundary weights, as given in [Table 2](#), can immediately be seen to satisfy certain simple properties. Some examples are as follows, where the notation will be explained at the end of the list.

- Invariance under diagonal reflection or arrow reversal,

$$\begin{aligned}
 W\left(a \begin{array}{c} d \\ \uparrow \\ c \end{array} b, u\right) &= W\left(d \begin{array}{c} a \\ \uparrow \\ c \end{array} b, u\right) = W\left(b \begin{array}{c} c \\ \uparrow \\ a \end{array} d, u\right) = W\left(\tilde{a} \begin{array}{c} \tilde{d} \\ \uparrow \\ \tilde{c} \end{array} u\right), \\
 W\left(\begin{array}{c} a \\ \uparrow \\ b \end{array}, u\right) &= W\left(\begin{array}{c} b \\ \uparrow \\ a \end{array}, u\right) = W\left(\tilde{a} \begin{array}{c} \tilde{b} \\ \uparrow \\ \tilde{c} \end{array}, u\right), \quad W\left(\begin{array}{c} b \\ \uparrow \\ a \end{array}, u\right) = W\left(\begin{array}{c} a \\ \uparrow \\ b \end{array}, u\right) = W\left(\tilde{a} \begin{array}{c} \tilde{b} \\ \uparrow \\ \tilde{c} \end{array}, u\right). \quad (44)
 \end{aligned}$$

- Invariance under simultaneous vertical reflection and replacement of u with \bar{u} ,

$$W\left(a \begin{array}{c} d \\ \uparrow \\ c \end{array} b, u\right) = W\left(c \begin{array}{c} d \\ \uparrow \\ a \end{array} b, \bar{u}\right), \quad W\left(\begin{array}{c} a \\ \uparrow \\ b \end{array}, u\right) = W\left(\begin{array}{c} a \\ \uparrow \\ b \end{array}, \bar{u}\right). \quad (45)$$

- Reduction of the bulk weights at $q^{\pm 2}$ or boundary weights at $q^{\pm 1}$,

$$W\left(a \begin{array}{c} d \\ \uparrow \\ c \end{array} b, q^2\right) = W\left(c \begin{array}{c} d \\ \uparrow \\ a \end{array} b, \bar{q}^2\right) = \delta_{a\tilde{a}} \delta_{b\tilde{c}}, \quad W\left(\begin{array}{c} a \\ \uparrow \\ b \end{array}, \bar{q}\right) = W\left(\begin{array}{c} a \\ \uparrow \\ b \end{array}, q\right) = \delta_{a\tilde{b}}. \quad (46)$$

These equations are satisfied for all edge orientations a, b, c and d such that the six-vertex rule is satisfied at degree 4 vertices, with an orientation being taken as in or out, with respect to the indicated endpoint of the edge. Also, \tilde{a} denotes the reversal of edge orientation a , and δ is the Kronecker delta.

As examples, the $a = b = \text{in}$ and $c = d = \text{out}$ cases of [\(45\)](#) and [\(46\)](#) are $W\left(\begin{array}{c} \uparrow \\ \uparrow \\ \uparrow \end{array}, u\right) = W\left(\begin{array}{c} \uparrow \\ \uparrow \\ \uparrow \end{array}, \bar{u}\right)$, $W\left(\begin{array}{c} \uparrow \\ \uparrow \\ \uparrow \end{array}, u\right) = W\left(\begin{array}{c} \uparrow \\ \uparrow \\ \uparrow \end{array}, \bar{u}\right)$, $W\left(\begin{array}{c} \uparrow \\ \uparrow \\ \uparrow \end{array}, q^2\right) = W\left(\begin{array}{c} \uparrow \\ \uparrow \\ \uparrow \end{array}, \bar{q}^2\right) = 1$ and $W\left(\begin{array}{c} \uparrow \\ \uparrow \\ \uparrow \end{array}, \bar{q}\right) = W\left(\begin{array}{c} \uparrow \\ \uparrow \\ \uparrow \end{array}, q\right) = 0$.

3.2. Local relations for bulk and boundary weights

The bulk and boundary weights, as given in [Table 2](#), also satisfy certain local relations. The relations relevant to this paper are as follows, where the notation will again be explained at the end of the list.

- Vertical and horizontal forms of the Yang–Baxter equation (VYBE and HYBE),

and

(47)

- Left and right forms of the reflection (or boundary Yang–Baxter) equation (LRE and RRE),

and

(48)

- Left and right forms of the boundary unitarity equation (LBUE and RBUE),

$$= \frac{\sigma(qu)\sigma(q\bar{u})}{\sigma(q)^2} \delta_{a\bar{b}}$$

and

$$= \frac{\sigma(qu)\sigma(q\bar{u})}{\sigma(q)^2} \delta_{a\bar{b}}$$

(49)

In these equations, each graph contains external edges, for which only one of the endpoints is indicated, and internal edges, for which both endpoints are indicated. Each equation holds for all orientations, a_1, b_1, \dots , of the external edges, with (as in Section 3.1) an orientation being taken as in or out, with respect to the indicated endpoint, and \bar{a} denoting the reversal of orientation a . For a particular orientation of the external edges, each graph denotes a sum, over all orientations of the internal edges which satisfy the six-vertex rule at each degree 4 vertex, of products of weights for each degree 2 and degree 4 vertex shown. If the edges incident to a degree 4 vertex appear horizontally and vertically, with an associated parameter u to the left of and below the vertex, then the weight of the vertex is $W\left(\begin{smallmatrix} d \\ \leftarrow \uparrow \\ a \downarrow \\ b \end{smallmatrix} c, u\right)$, for orientations a, b, c and d of the edges incident left, below, right and above the vertex, respectively. If the edges incident to a degree 4 vertex appear diagonally, then these edges and the associated parameter should be rotated so that the parameter appears to the left of and below the vertex, with the weight then being determined as previously. For degree 2 vertices, the incident edges always appear in the same configurations as used in the notation for the boundary weights, with the weights being determined accordingly. The color coding indicates that the parameters u, v and w can be naturally associated with certain edges.

As an example, the $a_1 = a_2 = b_1 = \text{in}$ and $b_2 = \text{out}$ case of the right form of the reflection equation (48) is

$$W(\begin{smallmatrix} \leftarrow \uparrow \\ \leftarrow \downarrow \end{smallmatrix}, q^2 u \bar{v}) W(\begin{smallmatrix} \leftarrow \uparrow \\ \leftarrow \downarrow \end{smallmatrix}, u) W(\begin{smallmatrix} \leftarrow \uparrow \\ \leftarrow \downarrow \end{smallmatrix}, uv) W(\begin{smallmatrix} \leftarrow \uparrow \\ \leftarrow \downarrow \end{smallmatrix}, v) = W(\begin{smallmatrix} \leftarrow \uparrow \\ \leftarrow \downarrow \end{smallmatrix}, v) W(\begin{smallmatrix} \leftarrow \uparrow \\ \leftarrow \downarrow \end{smallmatrix}, uv) W(\begin{smallmatrix} \leftarrow \uparrow \\ \leftarrow \downarrow \end{smallmatrix}, u) W(\begin{smallmatrix} \leftarrow \uparrow \\ \leftarrow \downarrow \end{smallmatrix}, q^2 u \bar{v}) + \\ W(\begin{smallmatrix} \leftarrow \uparrow \\ \leftarrow \downarrow \end{smallmatrix}, v) W(\begin{smallmatrix} \leftarrow \uparrow \\ \leftarrow \downarrow \end{smallmatrix}, uv) W(\begin{smallmatrix} \leftarrow \uparrow \\ \leftarrow \downarrow \end{smallmatrix}, u) W(\begin{smallmatrix} \leftarrow \uparrow \\ \leftarrow \downarrow \end{smallmatrix}, q^2 u \bar{v}) + W(\begin{smallmatrix} \leftarrow \uparrow \\ \leftarrow \downarrow \end{smallmatrix}, v) W(\begin{smallmatrix} \leftarrow \uparrow \\ \leftarrow \downarrow \end{smallmatrix}, uv) W(\begin{smallmatrix} \leftarrow \uparrow \\ \leftarrow \downarrow \end{smallmatrix}, u) W(\begin{smallmatrix} \leftarrow \uparrow \\ \leftarrow \downarrow \end{smallmatrix}, q^2 u \bar{v}).$$

The local relations (47)–(49) can be proved by directly verifying that each equation, for each orientation of external edges, is either trivial or reduces to a simple identity satisfied by the rational functions of Table 2. Due to the symmetry properties of the weights identified in Section 3.1, many of the different cases in this verification are equivalent. For example, invariance under arrow reversal (44) implies that cases of an equation which are related by reversal of all external edge orientations are equivalent, invariance under diagonal reflection (44) implies that the vertical and horizontal forms of the Yang–Baxter equation (47) are equivalent, and the properties of vertical reflection (45) imply that the left and right forms of the reflection equation (48) are equivalent.

There is an extensive literature regarding the local relations (47)–(49). For some general information regarding the Yang–Baxter equation, as applied to the six-vertex model, see, for example, Baxter [6, pp. 187–190]. The reflection equation was introduced (and applied to six-vertex model bulk weights) by Cherednik [17, Eq. (10)], with important further results being obtained by Sklyanin [45]. The most general boundary weights which satisfy the reflection equation for standard six-vertex model bulk weights were obtained, independently, by de Vega and González-Ruiz [23, Eq. (15)], and Ghoshal and Zamolodchikov [27, Eq. (5.12)]. The boundary weights used in this paper are a special case of these general weights, which were chosen for their property of all having value 1

at $u = 1$, as in (15), thereby enabling the straight enumeration of odd-order DASASMs, as in (21) and (22). It can be shown that, up to unimportant normalization, these are the only case of the general boundary weights which have this property. Finally, note that two other special cases of the general six-vertex model boundary weights were used by Kuperberg [32, Fig. 15], to study classes of ASMs including VSASMs, VHSASMs, OSASMs and OOSASMs.

3.3. Properties of the odd-order DASASM partition function

Some important properties of the odd-order DASASM partition function $Z(u_1, \dots, u_{n+1})$ will now be identified.

Proposition 10. *The odd-order DASASM partition function satisfies*

$$Z(\bar{u}_1, \dots, \bar{u}_{n+1}) = Z(u_1, \dots, u_{n+1}). \tag{50}$$

Proof. First observe that an involution on the set of configurations of the six-vertex model on \mathcal{T}_n is provided by reflection of each configuration in the central vertical line of \mathcal{T}_n . Applying this involution to each configuration in the sum (17) for $Z(u_1, \dots, u_{n+1})$, and using the properties (45) of the bulk and boundary weights under vertical reflection, leads to the required result. \square

Proposition 11. *The odd-order DASASM partition function $Z(u_1, \dots, u_{n+1})$ is an even Laurent polynomial in u_i of lower degree at least $-2n$ and upper degree at most $2n$, for each $i = 1, \dots, n$, and a Laurent polynomial in u_{n+1} of lower degree at least $-n$ and upper degree at most n .*

Note that the definitions of degrees being used in this paper are that a Laurent polynomial $\sum_{i=m}^n a_i x^i$ in x , with $m \leq n$ and $a_m, a_n \neq 0$, has lower and upper degrees m and n , respectively.

Proof. Consider a configuration C of the six-vertex model on \mathcal{T}_n , and $i \in \{1, \dots, n\}$. The C -dependent term in the sum (17) for $Z(u_1, \dots, u_{n+1})$ consists of a product of $n(n+2)$ weights, among which there are $2n - 1$ bulk weights, one left boundary weight and one right boundary weight that depend on u_i . Under the bijection (11) from C to an odd DASASM triangle, the local configurations which determine these $2n + 1$ u_i -dependent weights correspond to the entries of the triangle in the sequence (7). Also, it follows from the bijection (11), the explicit weights in Table 2, and the form of the C -dependent term in (17), that each nonzero entry in (7) is associated with a weight of 1, and each zero entry in (7) is associated with a weight which is an odd Laurent polynomial in u_i of lower degree -1 and upper degree 1. The properties of the sequence (7) imply that its number of zero entries is even and at most $2n$. Therefore, the C -dependent term in (17)

is an even Laurent polynomial in u_i of lower degree at least $-2n$ and upper degree at most $2n$, from which the required result for u_i follows.

The result for u_{n+1} can be proved similarly. \square

Proposition 12. *The odd-order DASASM partition function $Z(u_1, \dots, u_{n+1})$ is symmetric in u_1, \dots, u_n .*

Proof. The proof is analogous to that used by Kuperberg [32, Lem. 11 & Fig. 13] to show that the partition function for $4n \times 4n$ OOSASMs is symmetric in all of its parameters.

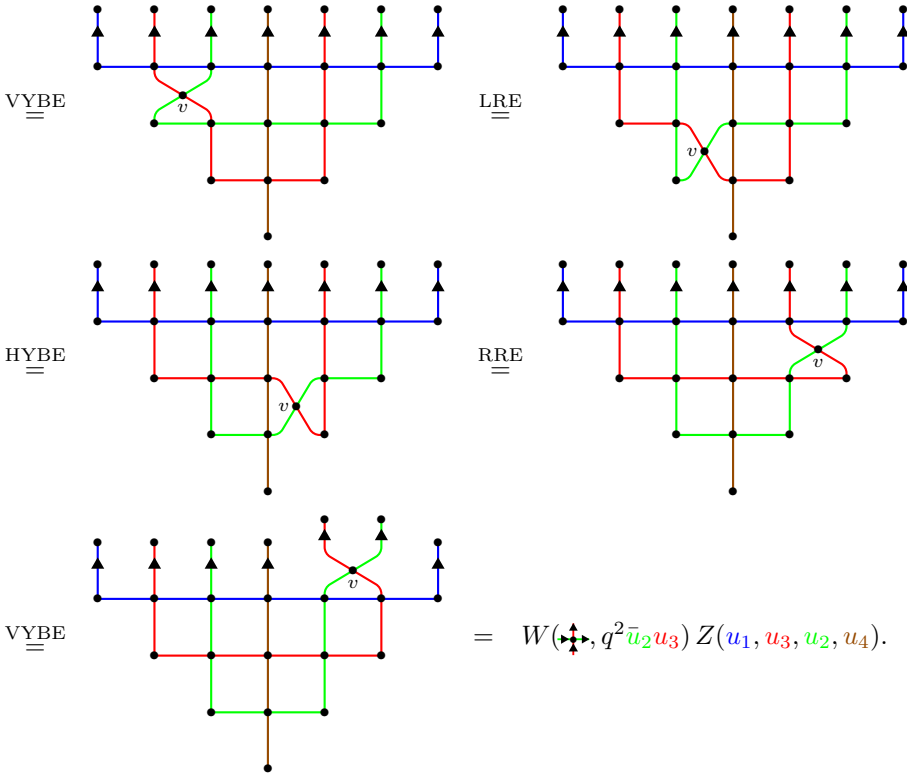
First note that it is sufficient to show that $Z(u_1, \dots, u_{n+1})$ is symmetric in u_i and u_{i+1} , for $i = 1, \dots, n - 1$. The proof of this will be outlined briefly for arbitrary n and i , and then illustrated in more detail for the case $n = 3$ and $i = 2$.

Let \mathcal{T}'_n be a modification of the graph \mathcal{T}_n , in which an additional degree 4 vertex x has been introduced, and the two edges incident with $(0, i)$ and $(0, i + 1)$ are replaced by four edges connecting x to $(0, i)$, $(0, i + 1)$, $(1, i)$ and $(1, i + 1)$. It follows, using (17) and the conditions on the configurations C in (17), that $W(\blacklozenge, q^2 \bar{u}_i u_{i+1}) Z(u_1, \dots, u_{n+1})$ can be expressed as a sum of products of bulk and boundary weights over all orientations of the edges of \mathcal{T}'_n , such that each edge incident with a top vertex $(0, j)$ is directed into that vertex and the six-vertex rule is satisfied at each degree 4 vertex, where the vertex x (whose incident edges necessarily have fixed orientations) is assigned a weight $W(\blacklozenge, q^2 \bar{u}_i u_{i+1})$ and the assignment of weights to other vertices is the same as for \mathcal{T}_n .

It can now be seen that it is possible to apply to $W(\blacklozenge, q^2 \bar{u}_i u_{i+1}) Z(u_1, \dots, u_{n+1})$, in succession, the vertical form of the Yang–Baxter equation (47) $i - 1$ times, the left form of the reflection equation (48) once, the horizontal form of the Yang–Baxter equation (47) $2(n - i) - 1$ times, the right form of the reflection equation (48) once, and the vertical form of the Yang–Baxter equation (47) $i - 1$ times, where each of these equations involves a bulk weight with parameter $q^2 \bar{u}_i u_{i+1}$. The result of applying this sequence of relations is $W(\blacklozenge, q^2 \bar{u}_i u_{i+1}) Z(u_1, \dots, u_{i-1}, u_{i+1}, u_i, u_{i+2}, \dots, u_{n+1})$, as required.

For the case $n = 3$ and $i = 2$, the proof can be illustrated as follows, where the notation will be explained at the end:

$$W(\blacklozenge, q^2 \bar{u}_2 u_3) Z(u_1, u_2, u_3, u_4) = \text{Diagram} \quad [\text{with } v = q^2 \bar{u}_2 u_3]$$



In these diagrams, each graph denotes a sum, over all orientations of the unlabeled edges which satisfy the six-vertex rule at each degree 4 vertex, of products of weights for each degree 2 and degree 4 vertex. The vertex weights are obtained using the same conventions as in Section 3.2, with the assignment of parameters for vertices whose incident edges appear horizontally and vertically being determined by the colors of the incident vertices as in the example in (16). The abbreviations above the = signs are those given in Section 3.2, and indicate the local relations which give the associated equalities. □

It can easily be checked that Propositions 10–12 are also satisfied if the odd-order DASASM partition function $Z(u_1, \dots, u_{n+1})$ is replaced by one of the odd-order DASASM $_{\pm}$ partition functions $Z_{\pm}(u_1, \dots, u_{n+1})$, and that the latter are even or odd in u_{n+1} , with $Z_{\pm}(u_1, \dots, u_n, -u_{n+1}) = \pm(-1)^n Z_{\pm}(u_1, \dots, u_n, u_{n+1})$. However, these additional results will not be needed.

3.4. Specializations of the odd-order DASASM partition function

It will now be shown, in Propositions 13–15, that for certain specializations of the parameters, the DASASM partition function of order $2n + 1$ reduces, up to a factor, to a DASASM partition function of order $2n - 1$ or $2n - 3$.

Only the specialization in Proposition 15 will be used in the proof of Theorem 1 in Section 3.5, while the specializations in Propositions 13, 14 and 15 will be used in the alternative proof of Theorem 1 in Section 3.6.

A further specialization, for which the DASASM partition function is zero, will be given in Proposition 16. This, together with Propositions 13 and 14, will be used in the alternative proof of Corollary 2 in Section 3.6.

Proposition 13. *If $u_1 = q$, then the odd-order DASASM partition function satisfies*

$$Z(q, u_2, \dots, u_{n+1}) = \frac{(q + \bar{q}) \prod_{i=2}^n \sigma(q^3 u_i)^2 \sigma(q^3 u_{n+1})}{\sigma(q^4)^{2n-1}} Z(u_2, \dots, u_{n+1}). \tag{51}$$

Note that Proposition 13 can be combined with Propositions 10–12 to obtain specializations of $Z(u_1, \dots, u_{n+1})$ at $u_i = q^{\pm 1}$ and $u_i = -q^{\pm 1}$, for $i = 1, \dots, n$. These specializations will be discussed in Section 3.6.

Proof. Let $u_1 = q$, consider $Z(u_1, \dots, u_{n+1})$, and apply the formula in (46) for the reduction of right boundary weights at q to $W(C_{1,2n+1}, u_1)$ in (17). It then follows, using the six-vertex rule and the upward orientation of the top edges in each configuration, that the contribution from configuration C in (17) is zero unless the local configurations in the first row of \mathcal{T}_n are fixed, with $C_{11} = \begin{matrix} \uparrow \\ \blacktriangleright \end{matrix}$ and $C_{12} = \dots = C_{1,2n} = \begin{matrix} \uparrow \\ \blacktriangleright \end{matrix}$. This now leads to the RHS of (51). \square

Proposition 14. *If $u_1 u_2 = q^2$, then the odd-order DASASM partition function satisfies*

$$Z(u_1, u_2, \dots, u_{n+1}) = \frac{\sigma(u_1)\sigma(qu_1)\sigma(u_2)\sigma(qu_2) \prod_{i=3}^n (\sigma(q^2 u_1 u_i)\sigma(q^2 u_2 u_i))^2 \sigma(q^2 u_1 u_{n+1})\sigma(q^2 u_2 u_{n+1})}{\sigma(q)^4 \sigma(q^4)^{2(2n-3)}} \times Z(u_3, \dots, u_{n+1}). \tag{52}$$

Note that Proposition 14 can be combined with Propositions 10–12 to obtain specializations of $Z(u_1, \dots, u_{n+1})$ at $u_i u_j = q^{\pm 2}$ and $u_i u_j = -q^{\pm 2}$, for distinct $i, j \in \{1, \dots, n\}$. These specializations will be discussed in Section 3.6.

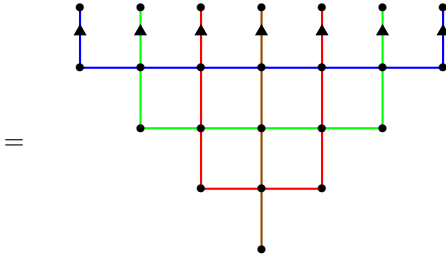
Proof. The proof will be outlined for arbitrary n , and then illustrated in more detail for the case $n = 3$. Let $u_1 u_2 = q^2$, consider $Z(u_1, \dots, u_{n+1})$, and apply the formula in (46) for the reduction of bulk weights at q^2 to $W(C_{12}, u_1 u_2)$ and $W(C_{1,2n}, u_1 u_2)$ in (17). This leads to a fixing of local configurations in the first row of \mathcal{T}_n , specifically $C_{11} = \begin{matrix} \uparrow \\ \blacktriangleright \end{matrix}$ and $C_{13} = \dots = C_{1,2n-1} = \begin{matrix} \uparrow \\ \blacktriangleright \end{matrix}$ for each configuration C in (17) with a nonzero contribution, thereby giving a factor $W(\begin{matrix} \uparrow \\ \blacktriangleright \end{matrix}, u_1) \prod_{i=3}^n W(\begin{matrix} \uparrow \\ \blacktriangleright \end{matrix}, u_1 u_i)^2 W(\begin{matrix} \uparrow \\ \blacktriangleright \end{matrix}, u_1 u_{n+1})$.

Now apply the right boundary unitarity equation (49) to the boundary weights $W(C_{1,2n+1}, u_1)$ and $W(C_{2,2n}, u_2)$ in (17). This gives a factor $\sigma(u_1)\sigma(u_2)/\sigma(q)^2$, and

leads to a fixing of local configurations in the second row of \mathcal{T}_n , specifically $C_{22} = \begin{matrix} \uparrow \\ \bullet \\ \rightarrow \end{matrix}$ and $C_{23} = \dots = C_{2,2n-1} = \begin{matrix} \uparrow \\ \bullet \\ \leftarrow \end{matrix}$ for each configuration C in (17) with a nonzero contribution, thereby giving a further factor $W(\begin{matrix} \uparrow \\ \bullet \\ \rightarrow \end{matrix}, u_2) \prod_{i=3}^n W(\begin{matrix} \uparrow \\ \bullet \\ \leftarrow \end{matrix}, u_2 u_i)^2 W(\begin{matrix} \uparrow \\ \bullet \\ \leftarrow \end{matrix}, u_2 u_{n+1})$, and yielding the RHS of (52).

For $n = 3$, the proof can be illustrated as follows:

$$Z(u_1, u_2, u_3, u_4) \Big|_{u_1 u_2 = q^2}$$



(46) =

$$\frac{\sigma(qu_1)}{\sigma(q)}$$

=

$$\frac{\sigma(qu_1) \sigma(q^2 u_1 u_3)^2 \sigma(q^2 u_1 u_4)}{\sigma(q) \sigma(q^4)^3}$$

RBUE =

$$\frac{\sigma(u_1) \sigma(qu_1) \sigma(u_2) \sigma(q^2 u_1 u_3)^2 \sigma(q^2 u_1 u_4)}{\sigma(q)^3 \sigma(q^4)^3}$$

=

$$\frac{\sigma(u_1) \sigma(qu_1) \sigma(u_2) \sigma(q u_2) (\sigma(q^2 u_1 u_3) \sigma(q^2 u_2 u_3))^2 \sigma(q^2 u_1 u_4) \sigma(q^2 u_2 u_4)}{\sigma(q)^4 \sigma(q^4)^6} Z(u_3, u_4),$$

where the notation is the same as in the example in the proof of Proposition 12. Note that, in the second and third graphs above, the curved lines connecting certain pairs

of vertices denote single edges (which accordingly have a single orientation), but their color changes midway so that the parameters associated with their endpoints are given correctly. □

Proposition 15. *If $u_1 u_{n+1} = q^2$, then the odd-order DASASM partition function satisfies*

$$\begin{aligned}
 & Z(u_1, \dots, u_{n+1}) \\
 &= \frac{\sigma(qu_1) (\sigma(q\bar{u}_1) + \sigma(q)) \prod_{i=2}^n \sigma(q^2 u_1 u_i) \sigma(q^2 u_i u_{n+1})}{\sigma(q)^2 \sigma(q^4)^{2n-2}} Z(u_2, \dots, u_n, u_1). \quad (53)
 \end{aligned}$$

Note that Proposition 15 can be combined with Propositions 10–12 to obtain specializations of $Z(u_1, \dots, u_{n+1})$ at $u_i u_{n+1} = q^{\pm 2}$ and $u_i u_{n+1} = -q^{\pm 2}$, for $i = 1, \dots, n$. These specializations will be discussed in Section 3.5.

Proof. The proof will be outlined briefly for arbitrary n , and then illustrated in more detail for the case $n = 3$.

Let $u_1 u_{n+1} = q^2$, consider $Z(u_1, \dots, u_{n+1})$, and apply the formula in (46) for the reduction of bulk weights at q^2 to the weight $W(C_{1,n+1}, u_1 u_{n+1})$ in (17). This leads to a fixing of local configurations in the left half of the first row of \mathcal{T}_n , specifically $C_{11} = \begin{smallmatrix} \blacktriangle \\ \rightarrow \end{smallmatrix}$ and $C_{12} = \dots = C_{1n} = \begin{smallmatrix} \blacktriangle \\ \rightarrow \end{smallmatrix}$ for each configuration C in (17) with a nonzero contribution, thereby giving a factor $W(\begin{smallmatrix} \blacktriangle \\ \rightarrow \end{smallmatrix}, u_1) \prod_{i=2}^n W(\begin{smallmatrix} \blacktriangle \\ \rightarrow \end{smallmatrix}, u_1 u_i)$.

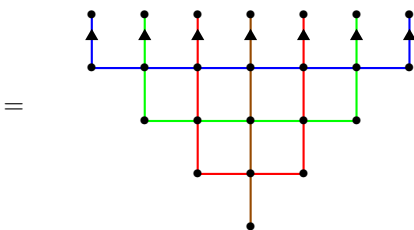
Now, for $i = 2, \dots, n$, apply, in succession, the horizontal form of the Yang–Baxter equation (47) $n - i$ times, the right form of the reflection equation (48) once, and the vertical form of the Yang–Baxter equation (47) $i - 2$ times, where each of these equations involves a bulk weight with parameter $u_i u_{n+1} = q^2 \bar{u}_1 u_i$. This leads to a further fixing of $n - 1$ local configurations, which gives a factor $\prod_{i=2}^n W(\begin{smallmatrix} \blacktriangle \\ \rightarrow \end{smallmatrix}, u_i u_{n+1})$. The overall result is then

$$\begin{aligned}
 & \frac{\sigma(qu_1) \prod_{i=2}^n \sigma(q^2 u_1 u_i) \sigma(q^2 u_i u_{n+1})}{\sigma(q) \sigma(q^4)^{2n-2}} ((W(\begin{smallmatrix} \blacktriangle \\ \rightarrow \end{smallmatrix}, u_1) + W(\begin{smallmatrix} \blacktriangle \\ \leftarrow \end{smallmatrix}, u_1)) Z_-(u_2, \dots, u_n, u_1) \\
 & \qquad \qquad \qquad + (W(\begin{smallmatrix} \blacktriangle \\ \leftarrow \end{smallmatrix}, u_1) + W(\begin{smallmatrix} \blacktriangle \\ \rightarrow \end{smallmatrix}, u_1)) Z_+(u_2, \dots, u_n, u_1)),
 \end{aligned}$$

from which the RHS of (53) follows.

For $n = 3$, the proof can be illustrated as follows:

$$Z(u_1, u_2, u_3, u_4) \Big|_{u_1 u_4 = q^2}$$



$$\begin{aligned}
 & \stackrel{(46)}{=} \text{Diagram 1} \\
 & = \frac{\sigma(q u_1) \sigma(q^2 u_1 u_2) \sigma(q^2 u_1 u_3)}{\sigma(q) \sigma(q^4)^2} \text{Diagram 2} \\
 & = \frac{\sigma(q u_1) \sigma(q^2 u_1 u_2) \sigma(q^2 u_1 u_3)}{\sigma(q) \sigma(q^4)^2} \text{Diagram 3} \quad \left[\text{with } v = u_2 u_4 = q^2 \bar{u}_1 u_2, \right. \\
 & \quad \left. w = u_3 u_4 = q^2 \bar{u}_1 u_3 \right] \\
 & \stackrel{\text{HYBE, RRE}}{=} \frac{\sigma(q u_1) \sigma(q^2 u_1 u_2) \sigma(q^2 u_1 u_3)}{\sigma(q) \sigma(q^4)^2} \text{Diagram 4} \\
 & \stackrel{\text{RRE, VYBE}}{=} \frac{\sigma(q u_1) \sigma(q^2 u_1 u_2) \sigma(q^2 u_1 u_3)}{\sigma(q) \sigma(q^4)^2} \text{Diagram 5} \\
 & = \frac{\sigma(q u_1) \sigma(q^2 u_1 u_2) \sigma(q^2 u_2 u_4) \sigma(q^2 u_1 u_3) \sigma(q^2 u_3 u_4)}{\sigma(q) \sigma(q^4)^4} \text{Diagram 6}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{\sigma(qu_1)\sigma(q^2u_1u_2)\sigma(q^2u_2u_4)\sigma(q^2u_1u_3)\sigma(q^2u_3u_4)}{\sigma(q)\sigma(q^4)^4} \left(\frac{\sigma(q\bar{u}_1)}{\sigma(q)} + 1 \right) \\
 &\qquad \qquad \qquad \times (Z_-(u_2, u_3, u_1) + Z_+(u_2, u_3, u_1)) \\
 &= \frac{\sigma(qu_1)(\sigma(q\bar{u}_1) + \sigma(q))\sigma(q^2u_1u_2)\sigma(q^2u_2u_4)\sigma(q^2u_1u_3)\sigma(q^2u_3u_4)}{\sigma(q)^2\sigma(q^4)^4} Z(u_2, u_3, u_1),
 \end{aligned}$$

where the notation is the same as in the examples in the proofs of Propositions 12 and 14. Note that in proceeding from the fourth to the fifth lines above, the only change is that the graph is redrawn and partly recolored (where the recoloring is justified by the existence of a relation between u_1 and u_4). This is done so that the applications of the horizontal form of the Yang–Baxter equation (47) in the next step can be visualized more easily. □

Proposition 16. *The odd-order DASASM partition function satisfies*

$$Z(u_1, \dots, u_{n-1}, q^2, 1) = 0. \tag{54}$$

Note that Proposition 16 can be combined with Propositions 10–12 to give $Z(u_1, \dots, u_n, 1) = 0$ at $u_i = q^{\pm 2}$ and $u_i = -q^{\pm 2}$, for $i = 1, \dots, n$.

Proof. This result can be obtained using Proposition 15 (to give $Z(q^2, u_2, \dots, u_n, 1) = 0$), and Proposition 12 (to interchange u_1 and u_n in $Z(u_1, \dots, u_n, 1)$).

Alternatively, it can be obtained directly, so this approach will also be given here. Let v denote the second-last vertex in the central column of the graph \mathcal{T}_n , i.e., $v = (n, n + 1)$. Now observe that the set of configurations of the six-vertex model on \mathcal{T}_n (for $n \geq 1$) can be partitioned into subsets of size three (which, incidentally, implies that $|\text{DASASM}(2n + 1)|$ is divisible by 3, for $n \geq 1$), such that the configurations within each subset have the same orientations on all edges, except for those incident with v from the left, below and the right. In particular, if the edge incident with v from above has orientation a , then the orientations l, b and r of the edges incident with v from the left, below and the right, respectively, are $(l, b, r) = (a, \tilde{a}, \tilde{a}), (\tilde{a}, \tilde{a}, a)$ or $(\tilde{a}, a, \tilde{a})$ (where orientations are taken as in or out, with respect to v , and \tilde{a} is the reversal of a). It can now be seen, using (17) and applying (46) to the bulk weight of v , that the contribution to $Z(u_1, \dots, u_{n-1}, q^2, 1)$ of the case $(a, \tilde{a}, \tilde{a})$ is zero, while the contributions of the cases $(\tilde{a}, \tilde{a}, a)$ and $(\tilde{a}, a, \tilde{a})$ cancel, since the only vertex at which their weights differ is $(n, n + 2)$, with the right boundary weights for that vertex being 1 and $\sigma(qq^2)/\sigma(q) = -1$. The result (54) now follows immediately. □

3.5. Proof of Theorem 1

Theorem 1 will now be proved, using results from Sections 3.3 and 3.4.

Consider a family of functions $X(u_1, \dots, u_{n+1})$ which satisfy the following properties.

- (i) $X(u_1) = 1$.
- (ii) $X(u_1, \dots, u_{n+1})$ is a Laurent polynomial in u_{n+1} of lower degree at least $-n$ and upper degree at most n .
- (iii) If $u_1 u_{n+1} = q^2$, then

$$X(u_1, \dots, u_{n+1}) = \frac{\sigma(qu_1)(\sigma(q\bar{u}_1) + \sigma(q)) \prod_{i=2}^n \sigma(q^2 u_i u_i) \sigma(q^2 u_i u_{n+1})}{\sigma(q)^2 \sigma(q^4)^{2n-2}} \times X(u_2, \dots, u_n, u_1).$$

- (iv) $X(\bar{u}_1, \dots, \bar{u}_{n+1}) = X(u_1, \dots, u_{n+1})$.
- (v) $X(u_1, \dots, u_{n+1})$ is even in u_i , for each $i = 1, \dots, n$.
- (vi) $X(u_1, \dots, u_{n+1})$ is symmetric in u_1, \dots, u_n .

It will first be shown that $X(u_1, \dots, u_{n+1})$ is uniquely determined by these properties.

By (ii), $X(u_1, \dots, u_{n+1})$ is uniquely determined if it is known at $2n + 1$ values of u_{n+1} . It can be seen that combining (iii) with (iv)–(vi) leads to expressions for $X(u_1, \dots, u_{n+1})$ at $4n$ values of u_{n+1} , i.e., at $u_{n+1} = q^{\pm 2} \bar{u}_i$ and $u_{n+1} = -q^{\pm 2} \bar{u}_i$, for each $i = 1, \dots, n$. In particular, (iii) consists of an expression for $X(u_1, \dots, u_n, q^2 \bar{u}_1)$ in terms of $X(u_2, \dots, u_n, u_1)$. Replacing u_1, \dots, u_n by $\bar{u}_1, \dots, \bar{u}_n$, and applying (iv), then gives an expression for $X(u_1, \dots, u_n, q^2 \bar{u}_1)$ in terms of $X(u_2, \dots, u_n, u_1)$. Replacing u_1 by $-u_1$ in the previous two cases, and applying (v), then gives expressions for $X(u_1, \dots, u_n, -q^2 \bar{u}_1)$ in terms of $X(u_2, \dots, u_n, -u_1)$. Finally, interchanging u_1 and u_i for any $i \in \{2, \dots, n\}$ in the previous four cases, and applying (vi), gives expressions for $X(u_1, \dots, u_n, q^{\pm 2} \bar{u}_i)$ in terms of $X(u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_n, u_i)$, and for $X(u_1, \dots, u_n, -q^{\pm 2} \bar{u}_i)$ in terms of $X(u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_n, -u_i)$.

Since $X(u_1)$ is fully known by (i), it follows by induction on n that $X(u_1, \dots, u_{n+1})$ is known at $2n + 1$ (or, in fact, $4n$) values of u_{n+1} . Therefore, $X(u_1, \dots, u_{n+1})$ is uniquely determined by properties (i)–(vi).

The validity of (30) in Theorem 1 will now be verified by showing that both sides satisfy properties (i)–(vi).

If $X(u_1, \dots, u_{n+1})$ is taken to be the odd-order DASASM partition function $Z(u_1, \dots, u_{n+1})$, i.e., the LHS of (30), then the required properties are satisfied, since (i) follows from (17) with $n = 0$, while (ii)–(vi) follow from Propositions 10, 11, 12 and 15.

For the remainder of this section, let $X(u_1, \dots, u_{n+1})$ be the RHS of (30), and define

$$F(u_1, \dots, u_{n+1}) = \frac{\sigma(q^2)^n}{\sigma(q)^{2n} \sigma(q^4)^{n^2}} \prod_{i=1}^n \frac{\sigma(u_i) \sigma(qu_i) \sigma(q\bar{u}_i) \sigma(q^2 u_i u_{n+1}) \sigma(q^2 \bar{u}_i \bar{u}_{n+1})}{\sigma(u_i \bar{u}_{n+1})} \times \prod_{1 \leq i < j \leq n} \left(\frac{\sigma(q^2 u_i u_j) \sigma(q^2 \bar{u}_i \bar{u}_j)}{\sigma(u_i \bar{u}_j)} \right)^2,$$

$$\begin{aligned}
 D(u_1, \dots, u_{n+1}) &= \det_{1 \leq i, j \leq n+1} \left(\begin{cases} \frac{q^2 + \bar{q}^2 + u_i^2 + \bar{u}_j^2}{\sigma(q^2 u_i u_j) \sigma(q^2 \bar{u}_i \bar{u}_j)}, & i \leq n \\ \frac{1}{u_j^2 - 1}, & i = n + 1 \end{cases} \right), \\
 Y(u_1, \dots, u_{n+1}) &= (u_{n+1} - 1) F(u_1, \dots, u_{n+1}) D(u_1, \dots, u_{n+1}), \\
 \widehat{Y}(u_1, \dots, u_{n+1}) &= (u_{n+1}^2 - 1) F(u_1, \dots, u_{n+1}) D(u_1, \dots, u_{n+1}), \tag{55}
 \end{aligned}$$

so that

$$\begin{aligned}
 X(u_1, \dots, u_{n+1}) &= Y(u_1, \dots, u_{n+1}) + Y(\bar{u}_1, \dots, \bar{u}_{n+1}) \\
 &= \frac{\widehat{Y}(u_1, \dots, u_{n+1})}{u_{n+1} + 1} + \frac{\widehat{Y}(\bar{u}_1, \dots, \bar{u}_{n+1})}{\bar{u}_{n+1} + 1}. \tag{56}
 \end{aligned}$$

Then $F(u_1) = 1$, $D(u_1) = \frac{1}{u_1^2 - 1}$ and $Y(u_1) = \frac{1}{u_1 + 1}$, from which it follows that (i) is satisfied.

Proceeding to (ii), it can be seen that $(u_{n+1}^2 - 1) \prod_{i=1}^n \sigma(q^2 u_i u_{n+1}) \sigma(q^2 \bar{u}_i \bar{u}_{n+1}) \times D(u_1, \dots, u_{n+1})$ is an (even) Laurent polynomial in u_{n+1} of lower degree at least $-2n$ and upper degree at most $2n$. Also, $D(u_1, \dots, u_n, \pm u_i) = 0$ for each $i = 1, \dots, n$, since columns i and $n + 1$ of the matrix are then equal. It follows that $(u_{n+1}^2 - 1) \prod_{i=1}^n \sigma(q^2 u_i u_{n+1}) \sigma(q^2 \bar{u}_i \bar{u}_{n+1}) / \sigma(u_i \bar{u}_{n+1}) D(u_1, \dots, u_{n+1})$, and hence also $\widehat{Y}(u_1, \dots, u_{n+1})$, is a Laurent polynomial in u_{n+1} of lower degree at least $-n$ and upper degree at most n . Now note that if a function $f(x)$ is a Laurent polynomial of lower degree at least $-n$ and upper degree at most n , then so is the function $g(x) = f(x)/(x+1) + f(\bar{x})/(\bar{x}+1)$, since $g(x) = (f(x) + xf(\bar{x})) / (x+1)$, where $f(x) + xf(\bar{x})$ is a Laurent polynomial of lower degree at least $-n$ and upper degree at most $n+1$ which vanishes at $x = -1$. Therefore, it follows from the previous observations and the last expression of (56) that (ii) is satisfied.

Proceeding to (iii), consider $Y(u_1^{\pm 1}, \dots, u_{n+1}^{\pm 1})$, and multiply the first row of the matrix in $D(u_1^{\pm 1}, \dots, u_{n+1}^{\pm 1})$ by the factor $\sigma(q^2 \bar{u}_1 \bar{u}_{n+1})$ from $F(u_1, \dots, u_{n+1})$ ($= F(\bar{u}_1, \dots, \bar{u}_{n+1})$). Now set $u_1 u_{n+1} = q^2$, which leads to the first row becoming $(0, \dots, 0, (q^2 + \bar{q}^2 + u_1^{\pm 2} + q^{\mp 4} u_1^{\pm 2}) / \sigma(q^4))$. It then follows, after rearranging and canceling certain products and signs, that for $u_1 u_{n+1} = q^2$,

$$\begin{aligned}
 Y(u_1^{\pm 1}, \dots, u_{n+1}^{\pm 1}) &= \frac{\sigma(q u_1) (\sigma(q \bar{u}_1) + \sigma(q)) \prod_{i=2}^n \sigma(q^2 u_1 u_i) \sigma(q^2 u_i u_{n+1})}{\sigma(q)^2 \sigma(q^4)^{2n-2}} \\
 &\quad \times Y(u_2^{\pm 1}, \dots, u_n^{\pm 1}, u_1^{\pm 1}).
 \end{aligned}$$

Therefore, (iii) is satisfied, due to the first equation of (56).

Proceeding to the remaining properties, (iv) is satisfied due to (56), (v) is satisfied since $F(u_1, \dots, u_{n+1})$ and all entries of the matrix in $D(u_1, \dots, u_{n+1})$ are even in u_i , for each $i = 1, \dots, n$, and (vi) is satisfied since if u_i and u_j are interchanged, then $F(u_1, \dots, u_{n+1})$ is unchanged, while rows i and j , and columns i and j , are interchanged in the matrix in $D(u_1, \dots, u_{n+1})$, for all distinct $i, j \in \{1, \dots, n\}$.

3.6. Alternative proofs of Theorem 1 and Corollary 2

Alternative proofs of Theorem 1 and Corollary 2 will now be sketched. Essentially, these involve regarding the odd-order DASASM partition function $Z(u_1, \dots, u_{n+1})$ as a Laurent polynomial in u_1 , rather than a Laurent polynomial in u_{n+1} , as done in Section 3.5.

Consider a family of functions $X(u_1, \dots, u_{n+1})$ which satisfy properties (i) and (iii)–(vi) of Section 3.5, together with the following properties.

- (vii) $X(u_1, u_2)$ is given by the RHS of (18).
- (viii) $X(u_1, \dots, u_{n+1})$ is a Laurent polynomial in u_1 of lower degree at least $-2n$ and upper degree at most $2n$.
- (ix) $X(q, u_2, \dots, u_{n+1}) = \frac{(q + \bar{q}) \prod_{i=2}^n \sigma(q^3 u_i)^2 \sigma(q^3 u_{n+1})}{\sigma(q^4)^{2n-1}} X(u_2, \dots, u_{n+1})$.
- (x) If $u_1 u_2 = q^2$, then

$$\begin{aligned}
 X(u_1, u_2, \dots, u_{n+1}) = & \\
 & \frac{\sigma(u_1)\sigma(qu_1)\sigma(u_2)\sigma(qu_2)\prod_{i=3}^n (\sigma(q^2 u_1 u_i)\sigma(q^2 u_2 u_i))^2 \sigma(q^2 u_1 u_{n+1})\sigma(q^2 u_2 u_{n+1})}{\sigma(q)^4 \sigma(q^4)^{2(2n-3)}} \\
 & \times X(u_3, \dots, u_{n+1}).
 \end{aligned}$$

It follows, using an argument similar to that in the first part of Section 3.5, that $X(u_1, \dots, u_{n+1})$ is uniquely determined by these properties. In particular, by (v) (with $i = 1$) and (viii), $X(u_1, \dots, u_{n+1})$ is uniquely determined if it is known at $2n + 1$ values of u_1^2 . Combining (iii), (ix) and (x) with (iv) and (vi) gives expressions for $X(u_1, \dots, u_{n+1})$ at $2n + 2$ values of u_1^2 , i.e., at $u_1^2 = q^{\pm 2}$ and $u_1^2 = q^{\pm 4} u_i^2$, for $i = 2, \dots, n + 1$. Using these expressions, and applying induction on n with (i) and (vii), it follows that $X(u_1, \dots, u_{n+1})$ is known at $2n + 1$ (or, in fact, $2n + 2$) values of u_1^2 , as required.

If $X(u_1, \dots, u_{n+1})$ is taken to be the odd-order DASASM partition function $Z(u_1, \dots, u_{n+1})$, then as already found in Section 3.5, (i) and (iii)–(vi) are satisfied. Furthermore, (vii)–(x) are satisfied, since (vii) is given by (18), while (viii)–(x) follow from Propositions 11, 13 and 14.

If $X(u_1, \dots, u_{n+1})$ is now taken to be the RHS of (30), then as already found in Section 3.5, (i) and (iii)–(vi) are satisfied. It can also be shown, using arguments similar to those of Section 3.5, that (vii)–(x) are satisfied, from which the required equality in Theorem 1 then follows.

Note that when showing that (viii) is satisfied by the RHS of (30), it can be verified straightforwardly that the function

$$P(u_1, \dots, u_{n+1}) =$$

$$\frac{\sigma(u_1)\sigma(qu_1)\sigma(q\bar{u}_1)\sigma(q^2u_1u_{n+1})\sigma(q^2\bar{u}_1\bar{u}_{n+1})\prod_{i=2}^n(\sigma(q^2u_1u_i)\sigma(q^2\bar{u}_1\bar{u}_i))^2D(u_1, \dots, u_{n+1})}{\sigma(u_1\bar{u}_{n+1})}$$

is a Laurent polynomial in u_1 . However, it then needs to be shown that $P(u_1, \dots, u_{n+1})/\prod_{i=2}^n \sigma(u_1\bar{u}_i)^2$ is also a Laurent polynomial in u_1 . This can be done by introducing

$$P'(v_1, \dots, v_n; u_1, \dots, u_{n+1}) = \frac{\sigma(u_1)\sigma(q^2v_1u_1)\sigma(q^2\bar{v}_1\bar{u}_1)\prod_{i=2}^{n+1}\sigma(q^2v_1u_i)\sigma(q^2\bar{v}_1\bar{u}_i)\prod_{i=2}^n\sigma(q^2v_iu_1)\sigma(q^2\bar{v}_i\bar{u}_1)}{\sigma(u_1\bar{u}_{n+1})} \times \det_{1 \leq i, j \leq n+1} \left(\begin{cases} \frac{q^2 + \bar{q}^2 + v_i^2 + \bar{u}_i^2}{\sigma(q^2v_iu_j)\sigma(q^2\bar{v}_i\bar{u}_j)}, & i \leq n \\ \frac{1}{u_j^2 - 1}, & i = n + 1 \end{cases} \right).$$

It can then be checked that $P'(v_1, \dots, v_n; u_1, \dots, u_{n+1})$ is a Laurent polynomial in u_1 and v_1 , which vanishes at $u_1 = \pm u_i$ and $v_1 = \pm v_i$ for each $i = 2, \dots, n$, so that $P'(v_1, \dots, v_n; u_1, \dots, u_{n+1})/\prod_{i=2}^n \sigma(u_1\bar{u}_i)\sigma(v_1\bar{v}_i)$ is also a Laurent polynomial in u_1 and v_1 . Furthermore, it can be checked that $P(u_1, \dots, u_{n+1}) = P'(u_1, \dots, u_n; u_1, \dots, u_{n+1})/((qu_1 + \bar{q}\bar{u}_1)(q\bar{u}_1 + \bar{q}u_1))$ and that $P'(u_1, \dots, u_n; u_1, \dots, u_{n+1})$ vanishes at $u_1^2 = -q^{\pm 2}$, from which the required result follows.

Proceeding to the alternative proof of Corollary 2, let $X(u_1, \dots, u_n, 1)$ satisfy properties (i) and (iv)–(vi) of Section 3.5, and properties (vii)–(x) of this section, with u_{n+1} set to 1 in each case, together with a modification of property (iii) of Section 3.5 given by

$$(iii') \quad X(u_1, \dots, u_{n-1}, q^2, 1) = 0.$$

Then, using the same argument as used for the alternative proof of Theorem 1, $X(u_1, \dots, u_n, 1)$ is uniquely determined. In particular, combining (iii'), (ix) and (x) with (iv) and (vi) gives expressions for $X(u_1, \dots, u_n, 1)$ at $2n + 2$ values of u_1^2 , i.e., at $u_1^2 = q^{\pm 2}$, $u_1^2 = q^{\pm 4}$ and $u_1^2 = q^{\pm 4}\bar{u}_i^2$, for $i = 2, \dots, n$.

If $X(u_1, \dots, u_n, 1)$ is taken to be either the LHS or RHS of (31), then, as found previously for the case of arbitrary u_{n+1} , (i) and (iv)–(x) are satisfied. Furthermore, (iii') is satisfied by the LHS due to Proposition 16, and by the RHS due to the presence of the term $\sigma(q^2\bar{u}_n)$ on the RHS. The required equality in Corollary 2 now follows.

Note that this can be regarded as a shorter proof of Corollary 2 than that of Sections 2 and 3.5 since, with regards to the LHS of (31), Proposition 15 has been replaced by Propositions 13, 14 and 16, each of which has a simpler proof than Proposition 15, and with regards to the RHS of (31), computations involving the more complicated RHS of (30) have now been avoided.

3.7. Proof of Theorem 3

Theorem 3 will now be proved, using Theorem 1.

In particular, the following determinant identity of Okada [36, Thm. 4.2] will be applied to (30) at $q = e^{i\pi/6}$. For all $a_1, x_1, b_1, y_1, \dots, a_k, x_k, b_k, y_k$,

$$\det_{1 \leq i, j \leq k} \left(\frac{a_i - b_j}{x_i - y_j} \right) = \frac{(-1)^{k(k+1)/2}}{\prod_{i,j=1}^k (x_i - y_j)} \det \begin{pmatrix} 1 & a_1 & x_1 & a_1 x_1 & \dots & x_1^{k-1} & a_1 x_1^{k-1} \\ 1 & b_1 & y_1 & b_1 y_1 & \dots & y_1^{k-1} & b_1 y_1^{k-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & a_k & x_k & a_k x_k & \dots & x_k^{k-1} & a_k x_k^{k-1} \\ 1 & b_k & y_k & b_k y_k & \dots & y_k^{k-1} & b_k y_k^{k-1} \end{pmatrix}. \tag{57}$$

Let $Y(u_1, \dots, u_{n+1})$ be defined as in (55), so that

$$Z(u_1, \dots, u_{n+1}) = Y(u_1, \dots, u_{n+1}) + Y(\bar{u}_1, \dots, \bar{u}_{n+1}). \tag{58}$$

Setting $q = e^{i\pi/6}$ gives

$$\begin{aligned} & Y(u_1, \dots, u_{n+1}) \Big|_{q=e^{i\pi/6}} \\ &= \frac{(u_{n+1} - 1) u_{n+1}^n}{3^{n(n-1)/2}} \prod_{i=1}^n \frac{(u_i^2 - 1)(u_i^2 - 1 + \bar{u}_i^2)(u_i^2 u_{n+1}^2 + 1 + \bar{u}_i^2 \bar{u}_{n+1}^2)}{u_i^2 - u_{n+1}^2} \\ & \times \prod_{1 \leq i < j \leq n} \frac{(u_i^2 u_j^2 + 1 + \bar{u}_i^2 \bar{u}_j^2)^2}{(u_i^2 - u_j^2)(\bar{u}_i^2 - \bar{u}_j^2)} \det_{1 \leq i, j \leq n+1} \left(\begin{cases} \frac{u_i^2 + 1 + \bar{u}_j^2}{u_i^2 u_j^2 + 1 + \bar{u}_i^2 \bar{u}_j^2}, & i \leq n \\ \frac{1}{u_j^2 - 1}, & i = n + 1 \end{cases} \right). \end{aligned} \tag{59}$$

Now observe that

$$\frac{u_i^2 + 1 + \bar{u}_j^2}{u_i^2 u_j^2 + 1 + \bar{u}_i^2 \bar{u}_j^2} = u_i^2 \bar{u}_j^2 \frac{u_i^2 + u_i^4 - (\bar{u}_j^2 + \bar{u}_j^4)}{u_i^6 - \bar{u}_j^6}, \quad \frac{1}{u_j^2 - 1} = -\bar{u}_j^2 \frac{-1 - (\bar{u}_j^2 + \bar{u}_j^4)}{1 - \bar{u}_j^6},$$

and apply (57) to (59) with

$$a_i = \begin{cases} u_i^2 + u_i^4, & i \leq n, \\ -1, & i = n + 1, \end{cases} \quad x_i = \begin{cases} u_i^6, & i \leq n, \\ 1, & i = n + 1, \end{cases} \quad b_j = \bar{u}_j^2 + \bar{u}_j^4, \quad y_j = \bar{u}_j^6, \quad k = n + 1,$$

to give

$$\begin{aligned} & Y(u_1, \dots, u_{n+1}) \Big|_{q=e^{i\pi/6}} = \frac{(-1)^{n(n-1)/2}}{u_{n+1}^2 \prod_{i=1}^n \prod_{j=1}^{n+1} (u_i^6 - \bar{u}_j^6) \prod_{i=1}^{n+1} (1 - \bar{u}_i^6)} \\ & \times \frac{(u_{n+1} - 1) u_{n+1}^n}{3^{n(n-1)/2}} \prod_{i=1}^n \frac{(u_i^2 - 1)(u_i^2 - 1 + \bar{u}_i^2)(u_i^2 u_{n+1}^2 + 1 + \bar{u}_i^2 \bar{u}_{n+1}^2)}{u_i^2 - u_{n+1}^2} \end{aligned}$$

$$\times \prod_{1 \leq i < j \leq n} \frac{(u_i^2 u_j^2 + 1 + \bar{u}_i^2 \bar{u}_j^2)^2}{(u_i^2 - u_j^2)(\bar{u}_i^2 - \bar{u}_j^2)}$$

$$\times \det \begin{pmatrix} 1 & u_1^2 + u_1^4 & u_1^6 & u_1^8 + u_1^{10} & \dots & u_1^{6n} & u_1^{6n+2} + u_1^{6n+4} \\ 1 & \bar{u}_1^2 + \bar{u}_1^4 & \bar{u}_1^6 & \bar{u}_1^8 + \bar{u}_1^{10} & \dots & \bar{u}_1^{6n} & \bar{u}_1^{6n+2} + \bar{u}_1^{6n+4} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & u_n^2 + u_n^4 & u_n^6 & u_n^8 + u_n^{10} & \dots & u_n^{6n} & u_n^{6n+2} + u_n^{6n+4} \\ 1 & \bar{u}_n^2 + \bar{u}_n^4 & \bar{u}_n^6 & \bar{u}_n^8 + \bar{u}_n^{10} & \dots & \bar{u}_n^{6n} & \bar{u}_n^{6n+2} + \bar{u}_n^{6n+4} \\ 1 & -1 & 1 & -1 & \dots & 1 & -1 \\ 1 & \bar{u}_{n+1}^2 + \bar{u}_{n+1}^4 & \bar{u}_{n+1}^6 & \bar{u}_{n+1}^8 + \bar{u}_{n+1}^{10} & \dots & \bar{u}_{n+1}^{6n} & \bar{u}_{n+1}^{6n+2} + \bar{u}_{n+1}^{6n+4} \end{pmatrix} . \tag{60}$$

The determinant in (60) is equal to

$$\det \begin{pmatrix} 1 & 1 + u_1^2 + u_1^4 & u_1^2 + u_1^4 + u_1^6 & u_1^6 + u_1^8 + u_1^{10} & \dots & u_1^{6n} + u_1^{6n+2} + u_1^{6n+4} \\ 1 & 1 + \bar{u}_1^2 + \bar{u}_1^4 & \bar{u}_1^2 + \bar{u}_1^4 + \bar{u}_1^6 & \bar{u}_1^6 + \bar{u}_1^8 + \bar{u}_1^{10} & \dots & \bar{u}_1^{6n} + \bar{u}_1^{6n+2} + \bar{u}_1^{6n+4} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 + u_n^2 + u_n^4 & u_n^2 + u_n^4 + u_n^6 & u_n^6 + u_n^8 + u_n^{10} & \dots & u_n^{6n} + u_n^{6n+2} + u_n^{6n+4} \\ 1 & 1 + \bar{u}_n^2 + \bar{u}_n^4 & \bar{u}_n^2 + \bar{u}_n^4 + \bar{u}_n^6 & \bar{u}_n^6 + \bar{u}_n^8 + \bar{u}_n^{10} & \dots & \bar{u}_n^{6n} + \bar{u}_n^{6n+2} + \bar{u}_n^{6n+4} \\ 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & 1 + \bar{u}_{n+1}^2 + \bar{u}_{n+1}^4 & \bar{u}_{n+1}^2 + \bar{u}_{n+1}^4 + \bar{u}_{n+1}^6 & \bar{u}_{n+1}^6 + \bar{u}_{n+1}^8 + \bar{u}_{n+1}^{10} & \dots & \bar{u}_{n+1}^{6n} + \bar{u}_{n+1}^{6n+2} + \bar{u}_{n+1}^{6n+4} \end{pmatrix}$$

$$= \prod_{i=1}^n (1 + u_i^2 + u_i^4) \prod_{i=1}^{n+1} (1 + \bar{u}_i^2 + \bar{u}_i^4) \det \begin{pmatrix} 1 & u_1^2 & u_1^6 & u_1^8 & \dots & u_1^{6n-4} & u_1^{6n} \\ 1 & \bar{u}_1^2 & \bar{u}_1^6 & \bar{u}_1^8 & \dots & \bar{u}_1^{6n-4} & \bar{u}_1^{6n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & u_n^2 & u_n^6 & u_n^8 & \dots & u_n^{6n-4} & u_n^{6n} \\ 1 & \bar{u}_n^2 & \bar{u}_n^6 & \bar{u}_n^8 & \dots & \bar{u}_n^{6n-4} & \bar{u}_n^{6n} \\ 1 & \bar{u}_{n+1}^2 & \bar{u}_{n+1}^6 & \bar{u}_{n+1}^8 & \dots & \bar{u}_{n+1}^{6n-4} & \bar{u}_{n+1}^{6n} \end{pmatrix}$$

$$= (-1)^n \prod_{i=1}^n (1 + u_i^2 + u_i^4)(u_i^2 - \bar{u}_{n+1}^2)(\bar{u}_i^2 - \bar{u}_{n+1}^2)(u_i^2 - \bar{u}_i^2) \prod_{i=1}^{n+1} (1 + \bar{u}_i^2 + \bar{u}_i^4)$$

$$\times \prod_{1 \leq i < j \leq n} (u_i^2 - u_j^2)(u_i^2 - \bar{u}_j^2)(\bar{u}_i^2 - u_j^2)(\bar{u}_i^2 - \bar{u}_j^2)$$

$$\times s_{(n, n-1, n-1, \dots, 2, 2, 1, 1)}(u_1^2, \bar{u}_1^2, \dots, u_n^2, \bar{u}_n^2, \bar{u}_{n+1}^2), \tag{61}$$

where the first expression is obtained by adding column $i - 1$ to column i in the matrix in (60), for each $i = 2, \dots, 2n + 2$, and the final expression is obtained by reversing the order of the columns of the matrix and applying (25).

Replacing the determinant in (60) by the final expression in (61), and canceling certain products and signs, now gives

$$Y(u_1, \dots, u_{n+1}) \Big|_{q=e^{i\pi/6}}$$

$$= 3^{-n(n-1)/2} \frac{u_{n+1}^n}{u_{n+1}+1} s_{(n,n-1,n-1,\dots,2,2,1,1)}(u_1^2, \bar{u}_1^2, \dots, u_n^2, \bar{u}_n^2, \bar{u}_{n+1}^2), \tag{62}$$

from which (32) follows using (58).

Finally, note that an alternative proof of Theorem 3 would be to take $X(u_1, \dots, u_{n+1})$ as the RHS of (32), and show that properties (i)–(vi) of Section 3.5, or properties (i) and (iii)–(vi) of Section 3.5 and (vii)–(x) of Section 3.6, with $q = e^{i\pi/6}$ in (iii), (ix) and (x), are then satisfied.

4. Discussion

In this final section, some further matters related to the main results of this paper are discussed.

4.1. Factorization of the Schur function in Corollary 4

As shown in Corollary 4, the odd-order DASASM partition function at $u_{n+1} = 1$ and $q = e^{i\pi/6}$ is, up to a simple factor, given by the Schur function $s_{(n,n-1,n-1,\dots,2,2,1,1)}(u_1^2, \bar{u}_1^2, \dots, u_n^2, \bar{u}_n^2, 1)$.

In [3], it will be shown that this function is a member of a collection of Schur functions which can be factorized in terms of characters of irreducible representations of orthogonal and symplectic groups. This collection also includes certain Schur functions indexed by rectangular shapes, for which the factorizations were obtained by Ciucu and Krattenthaler [18, Thms. 3.1 & 3.2].

Using the same notation for odd orthogonal and symplectic characters as that of Ciucu and Krattenthaler [18, Eqs. (3.7) & (3.9)], the factorization of the Schur function in (33) is explicitly

$$\begin{aligned} & s_{(2n,2n-1,2n-1,\dots,2,2,1,1)}(u_1, \bar{u}_1, \dots, u_{2n}, \bar{u}_{2n}, 1) = \\ & \quad \prod_{i=1}^{2n} (u_i + 1 + \bar{u}_i) sp_{(n-1,n-1,\dots,2,2,1,1)}(u_1, \dots, u_{2n}) \\ & \quad \times so_{(n,n-1,n-1,\dots,2,2,1,1)}(u_1, \dots, u_{2n}), \\ & s_{(2n+1,2n,2n,\dots,2,2,1,1)}(u_1, \bar{u}_1, \dots, u_{2n+1}, \bar{u}_{2n+1}, 1) = \\ & \quad \prod_{i=1}^{2n+1} (u_i + 1 + \bar{u}_i) sp_{(n,n-1,n-1,\dots,2,2,1,1)}(u_1, \dots, u_{2n+1}) \\ & \quad \times so_{(n,n,\dots,2,2,1,1)}(u_1, \dots, u_{2n+1}). \end{aligned} \tag{63}$$

It can now be seen, which was not explicitly apparent in (33), that $\prod_{i=1}^n (u_i^2 + 1 + \bar{u}_i^2)$ is a factor of $Z(u_1, \dots, u_n, 1) \Big|_{q=e^{i\pi/6}}$.

4.2. The odd-order DASASM partition function at special values of q

As shown by, for example, Kuperberg [32] and Okada [37], the partition functions of cases of the six-vertex model associated with symmetry classes of ASMs often simplify if

a particular parameter, which corresponds to the parameter q in this paper, is assigned to certain roots of unity.

For odd-order DASASMs, one such assignment is $q = e^{i\pi/6}$, which has been considered in detail in Sections 2 and 3.7, and which is associated with straight enumeration.

Another such assignment is $q = e^{i\pi/4}$, for which

$$\begin{aligned} & \left((-i\sigma(q^4))^{n^2} Z(u_1, \dots, u_{n+1}) \right) \Big|_{q \rightarrow e^{i\pi/4}} \\ &= \prod_{i=1}^n (u_i + \bar{u}_i)(u_i u_{n+1} + \bar{u}_i \bar{u}_{n+1}) \prod_{1 \leq i < j \leq n} (u_i u_j + \bar{u}_i \bar{u}_j)^2. \end{aligned} \tag{64}$$

This result can be proved directly, as follows. First, observe that multiplying $Z(u_1, \dots, u_{n+1})$ by $(-i\sigma(q^4))^{n^2}$ is equivalent to renormalizing each of the n^2 bulk weights in each term of (17) by a factor of $-i\sigma(q^4)$. Setting $q \rightarrow e^{i\pi/4}$, these renormalized weights are $(-i\sigma(q^4)W(c, u)) \Big|_{q \rightarrow e^{i\pi/4}} = u + \bar{u}$, for $c = \begin{smallmatrix} \blacklozenge \\ \blacklozenge \end{smallmatrix}, \begin{smallmatrix} \blacklozenge \\ \blacklozenge \end{smallmatrix}, \begin{smallmatrix} \blacklozenge \\ \blacklozenge \end{smallmatrix}, \begin{smallmatrix} \blacklozenge \\ \blacklozenge \end{smallmatrix}$, and $(-i\sigma(q^4)W(c, u)) \Big|_{q \rightarrow e^{i\pi/4}} = 0$, for $c = \begin{smallmatrix} \blacklozenge \\ \blacklozenge \end{smallmatrix}, \begin{smallmatrix} \blacklozenge \\ \blacklozenge \end{smallmatrix}$. It now follows from the bijection (11), and the properties of odd DASASM triangles, that the C -dependent term of (17) associated with $((-i\sigma(q^4))^{n^2} Z(u_1, \dots, u_{n+1})) \Big|_{q \rightarrow e^{i\pi/4}}$ is zero unless the odd DASASM triangle which corresponds to C consists of all 0's, except for a single 1 at either the start or end of each row. Observing that a 0 at the start or end of row i , for $i = 1, \dots, n$, is associated with weight $\sigma(qu_i)/\sigma(q)$ or $\sigma(q\bar{u}_i)/\sigma(q)$, respectively, and that 1's are associated with weight 1, it can then be seen that the relevant sum over 2^n configurations is $\prod_{i=1}^n (\sigma(qu_i)/\sigma(q) + \sigma(q\bar{u}_i)/\sigma(q))(u_i u_{n+1} + \bar{u}_i \bar{u}_{n+1}) \prod_{1 \leq i < j \leq n} (u_i u_j + \bar{u}_i \bar{u}_j)^2$, which gives (64). Note that, since the central entry of the relevant DASASMs in this case is always 1, it also follows that $(\sigma(q^4)^{n^2} Z_+(u_1, \dots, u_{n+1})) \Big|_{q \rightarrow e^{i\pi/4}} = (\sigma(q^4)^{n^2} Z(u_1, \dots, u_{n+1})) \Big|_{q \rightarrow e^{i\pi/4}}$ and $(\sigma(q^4)^{n^2} Z_-(u_1, \dots, u_{n+1})) \Big|_{q \rightarrow e^{i\pi/4}} = 0$.

We have also obtained some results and conjectures for the odd-order DASASM partition function at $q = e^{i\pi/3}$. In this case, $W(c, 1) \Big|_{q=e^{i\pi/3}} = -1$, for $c = \begin{smallmatrix} \blacklozenge \\ \blacklozenge \end{smallmatrix}, \begin{smallmatrix} \blacklozenge \\ \blacklozenge \end{smallmatrix}, \begin{smallmatrix} \blacklozenge \\ \blacklozenge \end{smallmatrix}, \begin{smallmatrix} \blacklozenge \\ \blacklozenge \end{smallmatrix}$, and $W(c, 1) \Big|_{q=e^{i\pi/3}} = 1$, for $c = \begin{smallmatrix} \blacklozenge \\ \blacklozenge \end{smallmatrix}, \begin{smallmatrix} \blacklozenge \\ \blacklozenge \end{smallmatrix}$. Hence, due to (11) and the second equation of (15), $Z(1, \dots, 1) \Big|_{q=e^{i\pi/3}}$ or $Z_{\pm}(1, \dots, 1) \Big|_{q=e^{i\pi/3}}$ correspond to enumerations in which $A \in \text{DASASM}(2n+1)$ is weighted by $(-1)^{M(A)}$, where $M(A)$ is the number of 0's among the entries A_{ij} for $i = 1, \dots, n, j = i + 1, \dots, 2n + 1 - i$, i.e., the number of 0's in the odd DASASM triangle associated with A which are not at the start or end of a row. We conjecture that these weighted enumerations are given by

$$\begin{aligned} \sum_{A \in \text{DASASM}(2n+1)} (-1)^{M(A)} &= (-1)^{n(n-1)/2} V_n, \\ \sum_{A \in \text{DASASM}_{\pm}(2n+1)} (-1)^{M(A)} &= \frac{1}{2} (-1)^{n(n-1)/2} ((-1)^n \mp 3) V_n, \end{aligned} \tag{65}$$

where $V_n = \frac{(2n)! |(3n-1)/2|!}{3!^{|(n-1)/2|} (3n)! [(n-1)/2]!} \prod_{i=0}^n \frac{(3i)!}{(n+i)!}$. As shown by Okada [37, Thm. 1.2 (A5) & (A6)], V_n is the number of $(2n + 1) \times (2n + 1)$ VHSASMs.

4.3. HTSASMs and DASASMs

Using the same argument as used for odd-order DASASMs in Section 1.4, it can be seen that the central entry of an odd-order HTSASM must again be ± 1 . Denoting the sets of all $(2n + 1) \times (2n + 1)$ HTSASMs with fixed central entry -1 and 1 as $\text{HTSASM}_-(2n + 1)$ and $\text{HTSASM}_+(2n + 1)$ respectively, it was shown by Razumov and Stroganov [40, p. 1197] that

$$\frac{|\text{HTSASM}_-(2n + 1)|}{|\text{HTSASM}_+(2n + 1)|} = \frac{n}{n + 1}. \tag{66}$$

Therefore, combining (43) and (66) gives

$$\frac{|\text{HTSASM}_-(2n + 1)|}{|\text{HTSASM}_+(2n + 1)|} = \frac{|\text{DASASM}_-(2n + 1)|}{|\text{DASASM}_+(2n + 1)|}. \tag{67}$$

Since $\text{DASASM}(2n + 1)_\pm$ is a subset of $\text{HTSASM}_\pm(2n + 1)$, (67) states that the ratio between the numbers of $(2n + 1) \times (2n + 1)$ HTSASMs with central entry -1 and 1 remains unchanged if the matrices are restricted to those which are also diagonally and antidiagonally symmetric. It would be interesting to obtain a direct proof of this result, without necessarily showing that either of the ratios is $n/(n + 1)$.

4.4. Further work

Some directions in which we are undertaking further work closely related to that of this paper are as follows.

First, as discussed in Section 3.2, the boundary weights used for odd-order DASASMs are a special case of the most general boundary weights which satisfy the reflection equation for the six-vertex model. By using other boundary weights in the odd-order DASASM partition function, properties analogous to those of Sections 3.3 and 3.4 are again satisfied, and it is possible to obtain results, including enumeration formulae, for certain subclasses of odd-order DASASMs. Work on this has been reported in [4].

Second, it is straightforward to define cases of the six-vertex model which are similar to the case considered in this paper, and whose configurations are in bijection with DSASMs or even-order DASASMs. The underlying graphs for these cases have already been introduced by Kuperberg [32, Figs. 12 & 13], in order to study OSASMs and OOSASMs. (More precisely, these are graphs for $2n \times 2n$ DSASMs and $4n \times 4n$ DASASMs, but the generalizations to DSASMs of any order and DASASMs of any even order are trivial.) Replacing the boundary weights which were used for OSASMs and OOSASMs by those used in Table 2 for odd-order DASASMs, leads to partition functions which give the numbers of DSASMs or even-order DASASMs, when $q = e^{i\pi/6}$ and the other parameters are all 1. Alternatively, using yet further boundary weights leads to partition functions associated with subclasses of DSASMs and even-order DASASMs other than

OSASMs and OOSASMs. All of these partition functions satisfy properties analogous to those identified for the odd-order DASASM partition function in Sections 3.3 and 3.4, which enables various results to be obtained for these cases. Work on this will be reported in [8,9].

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References

- [1] J.-C. Aval, P. Duchon, Enumeration of alternating sign matrices of even size (quasi)-invariant under a quarter-turn rotation, in: 21st International Conference on Formal Power Series and Algebraic Combinatorics, FPSAC 2009, in: *Discrete Math. Theor. Comput. Sci. Proc.*, AK, Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2009, pp. 115–126, arXiv:0906.3445, MR2721506.
- [2] J.-C. Aval, P. Duchon, Enumeration of alternating sign matrices of even size (quasi)-invariant under a quarter-turn rotation, *Electron. J. Combin.* 17 (2010) 51 (electronic), arXiv:0910.3047, MR2607337.
- [3] A. Ayyer, R. Behrend, I. Fischer, Factorization theorems for classical group characters, with applications to alternating sign matrices, in preparation.
- [4] A. Ayyer, R. Behrend, I. Fischer, Extreme diagonally and antidiagonally symmetric alternating sign matrices of odd order, preprint, arXiv:1611.03823, 2016.
- [5] A. Ayyer, D. Romik, New enumeration formulas for alternating sign matrices and square ice partition functions, *Adv. Math.* 235 (2013) 161–186, <http://dx.doi.org/10.1016/j.aim.2012.11.006>, arXiv:1202.3651, MR3010055.
- [6] R. Baxter, *Exactly Solved Models in Statistical Mechanics*, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London, 1989, reprint of the 1982 original, MR998375.
- [7] R. Behrend, Multiply-refined enumeration of alternating sign matrices, *Adv. Math.* 245 (2013) 439–499, <http://dx.doi.org/10.1016/j.aim.2013.05.026>, arXiv:1203.3187, MR3084435.
- [8] R. Behrend, I. Fischer, Alternating sign trapezoids and cyclically symmetric lozenge tilings of hexagons with a central triangular hole, in preparation.
- [9] R. Behrend, I. Fischer, Diagonally symmetric alternating sign matrices, in preparation.
- [10] D. Betea, M. Wheeler, Refined Cauchy and Littlewood identities, plane partitions and symmetry classes of alternating sign matrices, *J. Combin. Theory Ser. A* 137 (2016) 126–165, <http://dx.doi.org/10.1016/j.jcta.2015.08.007>, arXiv:1402.0229, MR3403518.
- [11] D. Betea, M. Wheeler, P. Zinn-Justin, Refined Cauchy/Littlewood identities and six-vertex model partition functions: II. Proofs and new conjectures, *J. Algebraic Combin.* 42 (2) (2015) 555–603, <http://dx.doi.org/10.1007/s10801-015-0592-3>, arXiv:1405.7035, MR3369568.
- [12] M. Bousquet-Mélou, L. Habsieger, Sur les matrices à signes alternants, *Discrete Math.* 139 (1–3) (1995) 57–72, [http://dx.doi.org/10.1016/0012-365X\(94\)00125-3](http://dx.doi.org/10.1016/0012-365X(94)00125-3), *Formal Power Series and Algebraic Combinatorics* (Montreal, QC, 1992), MR1336832.
- [13] D. Bressoud, *Proofs and Confirmations. The Story of the Alternating Sign Matrix Conjecture*, MAA Spectrum, Mathematical Association of America and Cambridge University Press, Washington, DC and Cambridge, 1999, MR1718370.
- [14] L. Cantini, Determinantal identities for doubly refined enumerations of alternating sign matrices, Talk given at Workshop on Combinatorial Physics, Cardiff University, 17–19 December 2013.
- [15] L. Cantini, A. Sportiello, Proof of the Razumov–Stroganov conjecture, *J. Combin. Theory Ser. A* 118 (5) (2011) 1549–1574, <http://dx.doi.org/10.1016/j.jcta.2011.01.007>, arXiv:1003.3376, MR2771600.

- [16] L. Cantini, A. Sportiello, A one-parameter refinement of the Razumov–Stroganov correspondence, *J. Combin. Theory Ser. A* 127 (2014) 400–440, <http://dx.doi.org/10.1016/j.jcta.2014.07.003>, arXiv:1202.5253, MR3252669.
- [17] I. Cherednik, Factorizing particles on a half line and root systems, *Theoret. and Math. Phys.* 61 (1) (1984) 977–983, <http://dx.doi.org/10.1007/BF01038545>, MR774205.
- [18] M. Ciucu, C. Krattenthaler, A factorization theorem for classical group characters, with applications to plane partitions and rhombus tilings, in: *Advances in Combinatorial Mathematics*, Springer, Berlin, 2009, pp. 39–59, arXiv:0812.1251, MR2683226.
- [19] F. Colomo, A. Pronko, Square ice, alternating sign matrices, and classical orthogonal polynomials, *J. Stat. Mech. Theory Exp.* 2005 (2005) P01005, <http://dx.doi.org/10.1088/1742-5468/2005/01/P01005> (electronic), arXiv:math-ph/0411076, MR2114554.
- [20] F. Colomo, A. Pronko, The role of orthogonal polynomials in the six-vertex model and its combinatorial applications, *J. Phys. A* 39 (28) (2006) 9015–9033, <http://dx.doi.org/10.1088/0305-4470/39/28/S15>, arXiv:math-ph/0602033, MR2240471.
- [21] J. de Gier, Fully packed loop models on finite geometries, in: *Polygons, Polyominoes and Polycubes*, in: *Lecture Notes in Phys.*, vol. 775, Springer, Dordrecht, 2009, pp. 317–346, arXiv:0901.3963, MR2790336.
- [22] J. de Gier, P. Pyatov, P. Zinn-Justin, Punctured plane partitions and the q -deformed Knizhnik–Zamolodchikov and Hirota equations, *J. Combin. Theory Ser. A* 116 (4) (2009) 772–794, <http://dx.doi.org/10.1016/j.jcta.2008.11.008>, arXiv:0712.3584, MR2513634.
- [23] H. de Vega, A. González-Ruiz, Boundary K -matrices for the six vertex and the $n(2n-1) A_{n-1}$ vertex models, *J. Phys. A* 26 (12) (1993) L519–L524, <http://dx.doi.org/10.1088/0305-4470/26/12/007>, arXiv:hep-th/9211114, MR1236142.
- [24] N. Elkies, G. Kuperberg, M. Larsen, J. Propp, Alternating-sign matrices and domino tilings. II, *J. Algebraic Combin.* 1 (3) (1992) 219–234, <http://dx.doi.org/10.1023/A:1022483817303>, arXiv:math/9201305, MR1194076.
- [25] I. Fischer, A new proof of the refined alternating sign matrix theorem, *J. Combin. Theory Ser. A* 114 (2) (2007) 253–264, <http://dx.doi.org/10.1016/j.jcta.2006.04.004>, arXiv:math/0507270, MR2293090.
- [26] I. Fischer, L. Riegler, Vertically symmetric alternating sign matrices and a multivariate Laurent polynomial identity, *Electron. J. Combin.* 22 (1) (2015) 1.5 (electronic), arXiv:1403.0535, MR3315447.
- [27] S. Ghoshal, A. Zamolodchikov, Boundary S matrix and boundary state in two-dimensional integrable quantum field theory, *Internat. J. Modern Phys. A* 9 (21 & 24) (1994) 3841–3885 & 4353, <http://dx.doi.org/10.1142/S0217751X94001552>, arXiv:hep-th/9306002, MR1285930.
- [28] C. Hagendorf, A. Morin-Duchesne, Symmetry classes of alternating sign matrices in a nineteen-vertex model, *J. Stat. Mech. Theory Exp.* 2016 (2016) 053111, <http://dx.doi.org/10.1088/1742-5468/2016/05/053111> (electronic), arXiv:1601.01859, MR3514161.
- [29] A. Izergin, Partition function of the six-vertex model in a finite volume, *Sov. Phys. Dokl.* 32 (11) (1987) 878–879, MR919260.
- [30] V. Korepin, Calculation of norms of Bethe wave functions, *Comm. Math. Phys.* 86 (3) (1982) 391–418, <http://dx.doi.org/10.1007/BF01212176>, MR677006.
- [31] G. Kuperberg, Another proof of the alternating-sign matrix conjecture, *Int. Math. Res. Not. IMRN* (3) (1996) 139–150, <http://dx.doi.org/10.1155/S1073792896000128>, arXiv:math/9712207, MR1383754.
- [32] G. Kuperberg, Symmetry classes of alternating-sign matrices under one roof, *Ann. of Math.* (2) 156 (3) (2002) 835–866, <http://dx.doi.org/10.2307/3597283>, arXiv:math/0008184, MR1954236.
- [33] W. Mills, D. Robbins, H. Rumsey Jr., Proof of the Macdonald conjecture, *Invent. Math.* 66 (1) (1982) 73–87, <http://dx.doi.org/10.1007/BF01404757>, MR652647.
- [34] W. Mills, D. Robbins, H. Rumsey Jr., Alternating sign matrices and descending plane partitions, *J. Combin. Theory Ser. A* 34 (3) (1983) 340–359, [http://dx.doi.org/10.1016/0097-3165\(83\)90068-7](http://dx.doi.org/10.1016/0097-3165(83)90068-7), MR700040.
- [35] W. Mills, D. Robbins, H. Rumsey Jr., Self-complementary totally symmetric plane partitions, *J. Combin. Theory Ser. A* 42 (2) (1986) 277–292, [http://dx.doi.org/10.1016/0097-3165\(86\)90098-1](http://dx.doi.org/10.1016/0097-3165(86)90098-1), MR847558.
- [36] S. Okada, Applications of minor summation formulas to rectangular-shaped representations of classical groups, *J. Algebra* 205 (2) (1998) 337–367, <http://dx.doi.org/10.1006/jabr.1997.7408>, MR1632816.

- [37] S. Okada, Enumeration of symmetry classes of alternating sign matrices and characters of classical groups, *J. Algebraic Combin.* 23 (1) (2006) 43–69, <http://dx.doi.org/10.1007/s10801-006-6028-3>, arXiv:math/0408234, MR2218849.
- [38] A. Razumov, Y. Stroganov, Refined enumerations of some symmetry classes of alternating-sign matrices, *Theoret. and Math. Phys.* 141 (3) (2004) 1609–1630, <http://dx.doi.org/10.1023/B:TAMP.0000049757.07267.9d>, arXiv:math-ph/0312071, MR2141132.
- [39] A. Razumov, Y. Stroganov, Enumeration of quarter-turn symmetric alternating-sign matrices of odd-order, *Theoret. and Math. Phys.* 149 (3) (2006) 1639–1650, <http://dx.doi.org/10.1007/s11232-006-0148-8>, arXiv:math-ph/0507003, MR2321099.
- [40] A. Razumov, Y. Stroganov, Enumerations of half-turn-symmetric alternating-sign matrices of odd-order, *Theoret. and Math. Phys.* 148 (3) (2006) 1174–1198, <http://dx.doi.org/10.1007/s11232-006-0111-8>, arXiv:math-ph/0504022, MR2283658.
- [41] A. Razumov, Y. Stroganov, A statistical model of three colors with boundary conditions of domain wall type: functional equations, *Theoret. and Math. Phys.* 161 (1) (2009) 1325–1339, <http://dx.doi.org/10.1007/s11232-009-0119-y>, arXiv:0805.0669, MR2664880.
- [42] D. Robbins, The story of 1, 2, 7, 42, 429, 7436, . . . , *Math. Intelligencer* 13 (2) (1991) 12–19, <http://dx.doi.org/10.1007/BF03024081>, MR1098216.
- [43] D. Robbins, Symmetry classes of alternating sign matrices, preprint, arXiv:math/0008045, 2000.
- [44] D. Robbins, H. Rumsey Jr., Determinants and alternating sign matrices, *Adv. Math.* 62 (2) (1986) 169–184, [http://dx.doi.org/10.1016/0001-8708\(86\)90099-X](http://dx.doi.org/10.1016/0001-8708(86)90099-X), MR865837.
- [45] E. Sklyanin, Boundary conditions for integrable quantum systems, *J. Phys. A* 21 (10) (1988) 2375–2389, <http://dx.doi.org/10.1088/0305-4470/21/10/015>, MR953215.
- [46] R. Stanley, A baker’s dozen of conjectures concerning plane partitions, in: *Combinatoire énumérative*, Montreal, QC, 1985, in: *Lecture Notes in Math.*, vol. 1234, Springer, Berlin, 1986, pp. 285–293, MR927770.
- [47] R. Stanley, *Enumerative Combinatorics. Volume 2*, Cambridge Studies in Advanced Mathematics, vol. 62, Cambridge University Press, Cambridge, 1999, MR1676282.
- [48] Y. Stroganov, Izergin–Korepin determinant reloaded, preprint, arXiv:math-ph/0409072, 2004.
- [49] Y. Stroganov, Izergin–Korepin determinant at a third root of unity, *Theoret. and Math. Phys.* 146 (1) (2006) 53–62, <http://dx.doi.org/10.1007/s11232-006-0006-8>, arXiv:math-ph/0204042, MR2243403.
- [50] Y. Stroganov, $1/N$ phenomenon for some symmetry classes of the odd alternating sign matrices, preprint, arXiv:0807.2520, 2008.
- [51] O. Tsuchiya, Determinant formula for the six-vertex model with reflecting end, *J. Math. Phys.* 39 (11) (1998) 5946–5951, <http://dx.doi.org/10.1063/1.532606>, arXiv:solv-int/9804010, MR1653120.
- [52] M. Wheeler, P. Zinn-Justin, Refined Cauchy/Littlewood identities and six-vertex model partition functions: III. Deformed bosons, *Adv. Math.* 299 (2016) 543–600, <http://dx.doi.org/10.1016/j.aim.2016.05.010>, arXiv:1508.02236, MR3519476.
- [53] D. Zeilberger, Proof of the alternating sign matrix conjecture, *Electron. J. Combin.* 3 (2) (1996) 13 (electronic), *The Foata Festschrift*, arXiv:math/9407211, MR1392498.