1. Prove that for all partitions λ with at most n parts and $1 \le k < n$

$$s_\lambda(x_1, \dots, x_n) = \sum_\mu s_\mu(x_1, \dots, x_k) s_{\lambda/\mu}(x_{k+1}, \dots, x_n).$$

where the sum is on all partitions μ with at most k parts.

2. Let μ and ν be two partitions. Prove that given μ and ν

$$\sum_\lambda s_{\lambda/\mu}(x_1, x_2, \dots, x_n) s_{\lambda/\nu}(y_1, y_2, \dots, y_m) = (\mu | D(x_1) \dots D(x_n) U(y_m) \dots U(y_1) \cdot \nu)$$

Prove that

$$\sum_\lambda s_{\mu/\lambda}(x_1, x_2, \dots, x_n) s_{\nu/\lambda}(y_1, y_2, \dots, y_m) = (\mu | U(x_1) \dots U(x_n) D(y_m) \dots D(y_1) \cdot \nu)$$

Use these to prove that

$$\sum_\lambda s_{\lambda/\mu}(x_1, x_2, \dots, x_n) s_{\lambda/\nu}(y_1, y_2, \dots, y_m) = \prod_{i=1}^n \prod_{j=1}^m \frac{1}{1 - x_i y_j} \sum_\lambda s_{\mu/\lambda}(y_1, y_2, \dots, y_m) s_{\nu/\lambda}(x_1, x_2, \dots, x_n).$$

3. Show that if P is a partial standard Young tableau and $x \notin P$ then the result of the insertion of x in P defined by Robinson and Schensted is still a partial standard Young tableau.

4. Define a column insertion algorithm to prove that

$$\sum_\lambda s_\lambda(x_1, x_2, \dots, x_n) s_{\lambda'}(y_1, y_2, \dots, y_m) = \prod_{i=1}^n \prod_{j=1}^m (1 + x_i y_j).$$

Hint: define a weight-preserving bijection from matrices M with all entries zero or one and pairs of tableaux (P, Q) of the same shape such that P' (the conjugate of P) is semi-standard and Q is semi-standard.

5. Given a matrix $M = (m_{ij})$ where m_{ij} is a non negative integer for all i and $j > 0$. Let (P, Q) be the image of M by the Robinson-Schensted-Knuth correspondence and suppose that P is of shape λ. Compute λ_1 in terms of the entries of M.

Hint: if M is a permutation matrix, λ_1 is the longest increasing subsequence of the corresponding permutation, i.e.

$$\lambda = \max_{\ell;\ i_1 < \dots < i_\ell;\ j_1 < \dots < j_\ell} \{ \sum_{k=1}^{\ell} M_{i_k, j_k} \}.$$

6. Recall that λ/μ is a horizontal strip if $\lambda'_i - \mu'_i \in \{0, 1\}$ for all i. Given a horizontal strip λ/μ let

$$I = \{ i \mid \lambda'_i - \mu'_i = 1 \ \&\ \lambda'_{i+1} - \mu'_{i+1} = 0 \}.$$

$$J = \{ i \mid \lambda'_i - \mu'_i = 0 \ \&\ \lambda'_{i+1} - \mu'_{i+1} = 1 \}.$$

Let

$$\psi_{\lambda/\mu}(t) = \prod_{i \in I} (1 - t^{m_i(\lambda)}); \quad \phi_{\lambda/\mu}(t) = \prod_{i \in J} (1 - t^{m_i(\mu)}).$$

where $m_i(\lambda)$ is the multiplicity of i in λ. Prove that

$$\frac{\psi_{\lambda/\mu}(t)}{\phi_{\lambda/\mu}(t)} = \frac{b_\lambda(t)}{b_\mu(t)}.$$

with $b_\lambda(t) = \prod_{i \ge 1} \prod_{j=1}^{m_i(\lambda)} (1 - t^j)$.

Use this to prove that $Q_\lambda(x; t) = b_\lambda(t) P_\lambda(x; t)$. Here $P_\lambda(x; t)$ is the Hall Littlewood functions indexed by λ.

7. Using the combinatorial definition of $P_\lambda(x;t)$, prove that $P_\lambda(x;1) = m_\lambda(x)$.

8. Show that if $G = \langle h \rangle$ is cyclic of order n and ω is an nth root of unity then the map $\rho : G \to GL(\mathbb{C})$ defined by $\rho(h^i) = [\omega^i]$ is a representation of G.

9. Prove that the triple
$$\left(\binom{[n]}{k}, \langle (1, \ldots, n) \rangle, \begin{bmatrix} n \\ k \end{bmatrix}_q \right)$$
exhibits the cyclic sieving phenomenon. Here $\binom{[n]}{k}$ is the set of subsets of $[n]$ of size k.