1. Let $\beta(\ell, \lambda)$ be the number of walks of length ℓ on Young's lattice starting at $\emptyset$ and ending at λ using up and down steps.

   (a) Prove that $\beta(\ell, \lambda) = 0$ if $\ell - \lambda$ is odd.

   (b) Prove that $\beta(|\lambda|, \lambda) = f^\lambda$ where f^λ is the number of standard Young tableaux of shape λ.

   (c) Prove that $\beta(\ell, \lambda) = (\lambda|(U + D)^\ell \cdot \emptyset)$.

   (d) Use the fact that $DU = UD + I$, to prove that
$$(U + D)^\ell = \sum_{i,j} b_{i,j}(\ell) U^i D^j$$
   where $b_{i,j}(\ell)$ is such that $b_{i,j}(\ell) = 0$ if $i < 0$ or $j < 0$ or $i > \ell$ or $j > \ell$
$$b_{i,j}(\ell) = b_{i,j-1}(\ell - 1) + (i + 1)b_{i+1,j}(\ell - 1) + b_{i-1,j}(\ell - 1)$$
   if $\ell > 0$ and $b_{0,0}(0) = 1$.

   (e) Prove that $U^i D^j \cdot \emptyset = 0$ if $j > 0$.

   (f) Show that if $|\lambda| = i$ and $\ell - i$ is even then
$$\beta(\ell, \lambda) = f^\lambda \binom{\ell}{i} (1 \cdot 3 \cdot 5 \cdot \ldots \cdot (\ell - i - 1)).$$

2. A function $f : \{1, \ldots, n\} \to \{0, \ldots, m\}$ is m-compatible with a permutation $\pi = \pi_1 \ldots \pi_n$ if
   - $f(\pi_1) \geq f(\pi_2) \geq \ldots \geq f(\pi_n)$
   - $f(\pi_i) > f(\pi_{i+1})$ if $i \in Des(\pi)$

   Here $Des(\pi) = \{i \mid \pi_i > \pi_{i+1}\}$.

   Let $C_m(\pi)$ be the number of functions that are m-compatible with π. Prove that
$$\sum_{m \geq 0} C_m(\pi) x^m = \frac{x^{|Des(\pi)|}}{(1 - x)^{n+1}}.$$

3. A plane partition $P = (P_{i,j})$ of shape λ is a filling of the cells of λ with positive integers so that rows and columns weakly decrease, i.e. for all i, j $P_{i,j} \geq P_{i+1,j}$ and $P_{i,j} \geq P_{i,j+1}$.

   (a) Let $pp(n, r, c)$ be the number of plane partitions P (of any shape with at most r rows and c columns) such that $|P| = \sum_{i,j} P_{i,j} = n$. Show that
$$\sum_{n \geq 0} pp(n, r, c) x^n = \prod_{i=1}^{r} \prod_{j=1}^{c} \frac{1}{1 - x^{i+j-1}}.$$
   (For example give a bijection with certain reverse plane partitions).

   (b) Given a plane partition Π and $i \geq 0$ let
$$\lambda^{(i)} = (P_{1,i+1}, P_{2,i+2}, P_{3,i+3}, \ldots)$$
   and
$$\lambda^{(-i)} = (P_{i+1,1}, P_{i+2,2}, P_{i+3,3}, \ldots).$$
   Show that for $i \geq 0$, $\lambda^{(i)} \succeq \lambda^{(i+1)}$ and $\lambda^{(-i)} \succeq \lambda^{(-i-1)}$.

   (c) Use the previous result to show that
$$\sum_{n \geq 0} pp(n, r, c) x^n = (\emptyset | D(x^{r+c}) D(x^{r+c-1}) \ldots D(x^{r+1}) U(x^{-r}) U(x^{-r+1}) \ldots U(x^{-1}) \cdot \emptyset).$$
   Here $U(x) \cdot \lambda = \sum_{\mu \succeq \lambda} x^{|\mu| - |\lambda|} \cdot \mu$ and $D(x) \cdot \lambda = \sum_{\mu \preceq \lambda} x^{|\lambda| - |\mu|} \cdot \mu$.

(d) Prove that

$$\sum_{n\geq 0} pp(n,r,c)x^n = \sum_{\lambda} s_\lambda(1,q,q^2,\dots,q^{c-1})s_\lambda(q,q^2,\dots,q^r).$$

(e) Show that if $pp(n)$ be the number of plane partitions P (of any shape) such that $|P| = \sum_{i,j} P_{i,j} = n$, then

$$\sum_{n\geq 0} pp(n)x^n = \prod_{i\geq 1} \frac{1}{(1-x^i)^i}.$$

4. Given a partition $(\lambda_1 > \lambda_2 > \dots > \lambda_k > 0)$ its shifted diagram is the set of cells (i,j) such that $i \leq j \leq \lambda_i + i - 1$. If $|\lambda| = n$, a shifted Young tableau of shape λ is a bijective filling of the shifted diagram of λ with $\{1,\dots,n\}$ such that the entries increase in rows and columns.

   Given a cell $u = (i,j)$ in the shifted diagram of a partition, we define its shifted hook-length $h^{sh}(u)$ as the number of cells in the same row as u that lie to the right of u, respectively in the same column and weakly below u, plus the number of cells in the $(j+1)$-th row of the shifted Young diagram.

   More specifically

$$h^{sh}(u) = \#\{(i,j') \mid j' \geq j\} \cup \{(i',j) \mid i' > i\} \cup \{(j+1,j') \mid j' > j\}.$$

   Propose an algorithm that generates a shifted Young tableau of shape λ with probability

$$\frac{\prod_u h_u^{sh}}{n!}.$$

5. Given two partitions λ and μ we say that they cointerlace if $\lambda' \succeq \mu'$ and we write $\lambda \succeq' \mu$. Here λ' is the conjugate of λ.

   (a) Prove that if $\lambda' \succeq \mu'$ then $\lambda_i - \mu_i \in \{0,1\}$.

   (b) Let $D'(x)$ be the operator such that

$$D'(x) = \sum_{\mu \preceq' \lambda} x^{|\lambda|-|\mu|} \cdot \mu.$$

   Use the fact that $D'(x)U(y) = (1+xy)U(y)D'(x)$ to show that

$$(\lambda | U(x_n)D'(x_n^{-1})U(x_{n-1})\dots D'(x_3^{-1})U(x_2)D'(x_2^{-1})U(x_1)D'(x_1^{-1})\cdot\emptyset) = \prod_{1\leq i<j\leq n}(1+x_i/x_j)s_\lambda(x_1,\dots,x_n).$$

   (c) Given two partitions μ, ν such that $\mu \succeq \nu$, show that if $\lambda \preceq' \mu$ then for all j we have $\lambda_j + 1 \geq \nu_j \geq \lambda_{j+1}$.