1. Let i_n be the number of involutions of $[n]$ (i.e. the number of permutations with cycles of length 1 or 2).

   (a) Show that $i_0 = i_1 = 1$ and for $n \geq 2$

   $$i_n = i_{n-1} + (n-1)i_{n-2}$$

   (b) Show that

   $$\sum_{n\geq 0} i_n \frac{x^n}{n!} = exp(x + x^2/2)$$

   in two ways: using the recursion and using the Exponential Formula.

   (c) Let A be a subset of the positive integers. Compute the exponential generating function

   $$\sum_{n\geq 0} i_n(A) \frac{x^n}{n!}$$

   where $i_n(A)$ is the number of permutations of $[n]$ all of whose cycles have lengths which are elements of A. For example if $A = \{1, 2\}$

   $$\sum_{n\geq 0} i_n(A) \frac{x^n}{n!} = exp(x + x^2/2).$$

2. Let a_n be the number of permutations of $[n]$ that have an even number of cycles, all of them of odd length. Use a parity argument to show that if n is odd then $a_n = 0$ Use exponential generating functions to show that if n is even then

   $$a_n = \binom{n}{n/2} \frac{n!}{2^n}$$

3. Suppose $\mathcal{S}$ is a labeled structure satisfying

   $$\mathcal{S}(L) \cap \mathcal{S}(M) = \emptyset \text{ if } L \cap M = \emptyset$$

   and $\mathcal{T}$ is any labeled structure. Their composition, $\mathcal{T} \circ \mathcal{S}$ is the structure such that

   $$\mathcal{T} \circ \mathcal{S}(L) = \{(\{S_1, S_2, \ldots\}, T) \mid L_1/L_2/\ldots \vdash [n],\ S_i \in \mathcal{S}(L_i)\ \forall i,\ T \in \mathcal{T}(\{S_1, S_2, \ldots\})\}.$$

   (a) Prove that

   $$F_{\mathcal{T}\circ\mathcal{S}} = F_{\mathcal{T}}(F_{\overline{\mathcal{S}}}(x)).$$

   (b) Use this to reprove the Exponential Formula.

4. Use Lagrange inversion to count the number of rooted forests on n vertices with k trees.

5. Show that the number of odd hook lengths minus the number of even hook lengths of a partition is a triangular number (a number of the form $(k-1)k/2$ for $k \in \mathbb{Z}$.

6. Show that the number of SYT of shape (n, n) is the Catalan number C_n. Give two proofs.

7. Given two partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$ of n, we say that λ dominates μ if

   $$\lambda_1 + \ldots + \lambda_i \geq \mu_1 + \ldots + \mu_i$$

   for all $i > 0$ and we write $\lambda \trianglerighteq \mu$.

   (a) Prove that there exists $\alpha \in P(n)$ the set of partitions of n such that $\lambda \trianglerighteq \alpha$ for all $\lambda \in P(n)$.

(b) Prove that if $\lambda \triangleright \mu$ and there is no ν such that $\lambda \triangleright \nu \triangleright \mu$ then there exists $j < k$ such that $\lambda_j = \mu_j + 1$ and $\lambda_k = \mu_k - 1$ and $\lambda_i = \mu_i$ otherwise.

(c) Given λ and μ, compute the partition ν such that $\nu \trianglerighteq \lambda$ and $\nu \trianglerighteq \mu$ and ν is the smallest in the dominance order with these properties.

(d) Prove that if $\lambda \trianglerighteq \mu$ then $\mu' \trianglerighteq \lambda'$. Here λ' is the conjugate of λ.

(e) Let $K_{\lambda,\mu}$ be the number of semi standard Young tableaux of shape λ and content μ (i.e. μ_i entries equal to i for all i). Prove that if $K_{\lambda,\mu} > 0$ then $\lambda \trianglerighteq \mu$.

8. Show that Sym is an algebra, that is, it is a vector space which is closed under multiplication of symmetric functions.

9. Let τ be an unlabelled rooted tree with n vertices. A descendent of a vertex v is a vertex u such that the path from u to the root goes through v. A standard labelling of a tree is a bijective labelling of the vertices with $1, 2, \ldots, n$ such that the label of a vertex v is smaller than the label of its descendants. Given a vertex v of τ let $\tau(v)$ be the number of descendants of v. Show that the number of standard labelling of τ is

$$\frac{n!}{\prod_{v \in \tau}(\tau(v) + 1)}.$$